Instantaneous solitons and fractal solitons for a (2+1)-dimensional nonlinear system*

Pan Zhen-Huan(潘震环), Ma Song-Hua(马松华)†, and Fang Jian-Ping(方建平)

Department of Physics, Zhejiang Lishui University, Lishui 323000, *China*

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By an improved projective equation approach and a linear variable separation approach, a new family of exact solutions of the (2+1)-dimensional Broek–Kaup system is derived. Based on the derived solitary wave solution and by selecting appropriate functions, some novel localized excitations such as instantaneous solitons and fractal solitons are investigated.

Keywords: improved projective equation approach, Broek–Kaup system, exact solutions, instantaneous solitons and fractal solitons

PACC: 0340K, 0200, 0365

1. Introduction

As is well known, many dynamical problems in physics and other natural fields are usually characterized by the nonlinear evolution of partial differential equations known as governing equations. Searching for an analytical exact solution to a nonlinear system has long been an important and interesting topic in nonlinear science both for physicists and mathematicians, and various methods for obtaining exact solutions of a nonlinear system have been proposed, for instance, the bilinear method, the standard Painlevé truncated expansion, the method of "coalescence of eigenvalue" or "wavenumbers", the homogenous balance method, the homotopy-perturbation method, the hyperbolic function method, the Jacobian elliptic method, the variable separation method, the (G'/G)-expansion method,[1−12] and the projective equation approach,[13−18] etc. In this paper, with a new projective equation ($\phi' = \sigma\phi + \phi^2$) and a linear variable separation approach and a new ansatz, we obtain a family of exact solutions for the (2+1)-dimensional Breor–Kaup (BK) system

$$\begin{aligned} &u_{ty} - u_{xxy} + 2(u_x u)_y + 2v_{xx} = 0, \\ &v_t + 2(uv)_x + v_{xx} = 0, \end{aligned} \tag{1}$$

based on the derived solitary wave solution and by selecting appropriate functions, some novel localized excitations such as instantaneous solitons and fractal solitons are investigated.

The BK system is used to model the nonlinear and dispersive long gravity waves travelling in two horizontal directions in shallow water with uniform depth, and it can also be derived from the celebrated Kadomtsev–Petviashvili (KP) equation by the symmetry constraint.[19] When $y = x$, the (2+1)-dimensional BK system is reduced further to a usual (1+1)-dimensional BK system, which can be used to describe the propagation of long waves in shallow water.[20] Using some suitable dependent and independent variable transformations, Chen and Li[21] have proved that the (2+1)-dimensional BK system can be transformed to the (2+1)-dimensional dispersive long wave equation (DLWE) and (2+1)-dimensional Ablowitz–Kaup–Newell–Segur (AKNS) system. The (2+1)-dimensional BK system has been widely investigated in detail by many researchers.[22]

2. New solitary wave solutions to the BK system

In the usual projective equation approach, the basic idea of the algorithm is as follows: Consider a given nonlinear partial differential equation (NPDE) with independent variables $x = (x_0 = t, x_1, x_2, \ldots, x_m)$,

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†Corresponding author. E-mail: msh6209@yahoo.com.cn

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and one dependent variable u, in the form

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \tag{2}$$

where P is in general a polynomial function of its arguments, and the subscripts denote the partial derivatives. One assumes its solution in the form,

$$u = A(x) + \sum_{i=1}^{n} B_i(x)\phi^i(q(x)) \tag{3}$$

with

$$\phi' = \sigma\phi + \phi^2, \tag{4}$$

where σ is a constant and the prime denotes differentiation with respect to q. To determine u explicitly, one may substitute Eqs. (3) and (4) into the given NPDE, collect the coefficients of the polynomials in ϕ, then eliminate each coefficient to derive a set of partial differential equations of A, B_i, and q, thus solve this system of partial differential equations to obtain A, B_i, and q. Finally, equation (4) is known to possess the solutions

$$\phi = \begin{cases} -\dfrac{1}{2}\sigma\left[1 + \tanh\left(\dfrac{1}{2}\sigma q\right)\right], & \sigma \neq 0, \\ -\dfrac{1}{2}\sigma\left[1 + \coth\left(\dfrac{1}{2}\sigma q\right)\right], & \sigma \neq 0, \\ \dfrac{-1}{q}, & \sigma = 0. \end{cases} \tag{5}$$

Substituting A_i, B_i, q and Eqs. (5) into Eq. (3), one obtains the exact solutions to the given NPDE.

In order to find some new solutions of the BK system, we try to improve the above method. We rewrite the ansatz (3) as

$$u = A(x) + \sum_{i=1}^{n} B_i(x)\phi^i(q(x)) + C_i(x)\phi^{i-1}(q(x))\sqrt{\sigma\phi + \phi^2(q(x))}). \tag{6}$$

First, let us make a transformation for Eq. (1): $v = u_y$. Substituting this transformation into Eq. (1), yields

$$u_{yt} + [2u_x u_y + 2uu_{xy} + u_{xxy}] = 0. \tag{7}$$

Now we apply the improved projective equation to Eq. (7). By the balancing procedure, ansatz (6) becomes

$$\begin{aligned} u = {} & f(x,y,t) + g(x,y,t)\phi(q(x,y,t)) \\ & + h(x,y,t)\sqrt{\sigma\phi(q(x,y,t)) + \phi^2(q(x,y,t))}, \end{aligned} \tag{8}$$

where f, g, h and q are functions of (x, y, t) to be determined. Substituting Eqs. (8) and (4) into Eq. (7) and collecting the coefficients of the polynomials of ϕ, then setting each coefficient to zero, we have

$$\begin{aligned} & f = -\frac{1}{4}\frac{2q_t + 2q_{xx} + \sigma q_x^2}{q_x}, \\ & g = -\frac{1}{2}q_x, \quad h = \frac{1}{2}q_x \end{aligned} \tag{9}$$

with

$$q = \chi(x,t) + \varphi(y), \tag{10}$$

where $\chi \equiv \chi(x,t), \quad \varphi \equiv \varphi(y)$ are two arbitrary variable separation functions of (x,t) and of y, respectively. Based on the solutions of Eq. (4), one thus obtains an explicit solution of Eq. (1).

Case 1 For $\sigma \neq 0$, we can derive the following solitary wave solutions of Eq. (1)

$$u_1 = -\frac{1}{2}\frac{\chi_{xx} + \chi_t}{\chi_x} + \frac{1}{4}\chi_x\sigma\left\{\tanh\left(\frac{1}{2}\sigma(\chi+\varphi)\right) + \sqrt{-1 + \tanh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)^2}\right\}, \tag{11}$$

$$v_1 = \frac{1}{8}\sigma^2\chi_x\varphi_y\left\{1 - \tanh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)^2 - \tanh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)\sqrt{-1 + \tanh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)^2}\right\}, \tag{12}$$

$$u_2 = -\frac{1}{2}\frac{\chi_{xx} + \chi_t}{\chi_x} + \frac{1}{4}\chi_x\sigma\left\{\coth\left(\frac{1}{2}\sigma(\chi+\varphi)\right) + \sqrt{-1 + \coth\left(\frac{1}{2}\sigma(\chi+\varphi)\right)^2}\right\}, \tag{13}$$

$$v_2 = \frac{1}{8}\sigma^2\chi_x\varphi_y\left\{1 - \coth\left(\frac{1}{2}\sigma(\chi+\varphi)\right)^2 - \coth\left(\frac{1}{2}\sigma(\chi+\varphi)\right)\sqrt{-1 + \coth\left(\frac{1}{2}\sigma(\chi+\varphi)\right)^2}\right\} \tag{14}$$

with two arbitrary functions being $\chi(x,t)$ and $\varphi(y)$.

Case 2 For $\sigma = 0$, we can derive the following variable separated solution of Eq. (1)

$$u_3 = -\frac{1}{2}\frac{\chi_t + \chi_{xx}}{\chi_x} + \frac{\chi_x}{\chi + \varphi}, \tag{15}$$

$$v_3 = -\frac{\chi_x \varphi_y}{(\chi + \varphi)^2} \tag{16}$$

with two arbitrary functions being $\chi(x,t)$ and $\varphi(y)$.

3. Localized structures in the BK system

Due to the arbitrariness of the functions $\chi(x,t)$ and $\varphi(y)$ included in the above cases, the physical quantities u and v may possess rich structures. For example, when $\chi = ax + ct$ and $\varphi = ky$, all the solutions of the above cases become simple travelling wave excitations. Moreover, based on the derived solutions, we may obtain rich localized structures such as instantaneous solitons and fractal solitons.

3.1. Instantaneous soliton excitations

In this section, we first analyse some instantaneous soliton localized excitations of the solution v_2 given in Eq. (14) in Case 1. For instance, if we choose χ and φ as

$$\chi = 1 + \tanh(x^2 + t), \quad \varphi = 1 + \tanh(y^2), \tag{17}$$

we can obtain an instantaneous soliton excitation for the physical quantity v_2 under the condition (17) presented in Fig. 1 with fixed parameters $\sigma = -1$ at times $t = 0,\ 0.5,\ 1,\ 3$. From Fig. 1, we can see that the corresponding dromion soliton decreases from about 0.05 to zero as the time increases from 0 to 3.

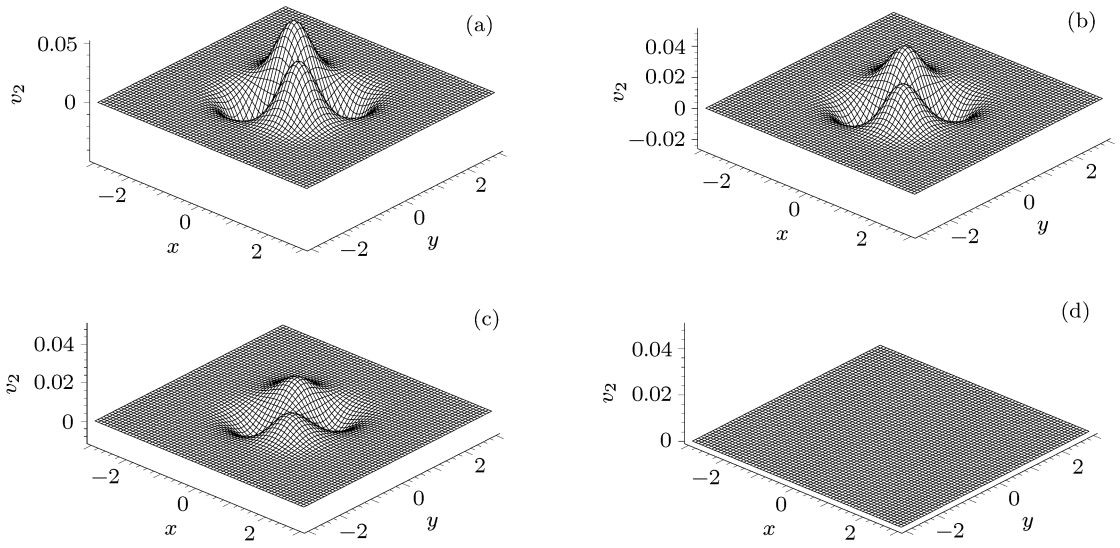


Fig. 1. A plot of a special type of instantaneous soliton structure for the physical quantity v_2 given by the solution (14) with the choice (17) and $\sigma = -1$ at $t = 0$ (a), 0.5 (b), 1.0 (c), and 3.0 (d).

Furthermore, if we choose χ and φ as

$$\chi = 1 + 2\exp(\sqrt{x^2 + t}), \quad \varphi = 1 + 2\exp(\sqrt{y^2}), \tag{18}$$

we can obtain another type of instantaneous soliton excitation for the physical quantity v_2 under the condition (18), as shown in Fig. 2 with fixed parameters $\sigma = -1$ at $t = 0,\ 1,\ 2,\ 5$. From Fig. 2, we notice that this another type instantaneous soliton decreases from about 0.01 to zero as the time increases from 0 to 5.

3.2. Fractal soliton excitations

In the following discussion, we mainly analyse the fractal localized structures of (2+1)-dimensional BK system. We know that it is very difficult to find some appropriate functions which can be used to depict regular fractal patterns possessing self-similar structures. Fortunately, in a recent study, we have found many lower-dimensional piecewise smooth functions with fractal properties, which can be used to construct exact solutions of higher-dimensional soliton systems that also posses fractal structures; this situation also occurs in

the (2+1)-dimensional BK system.

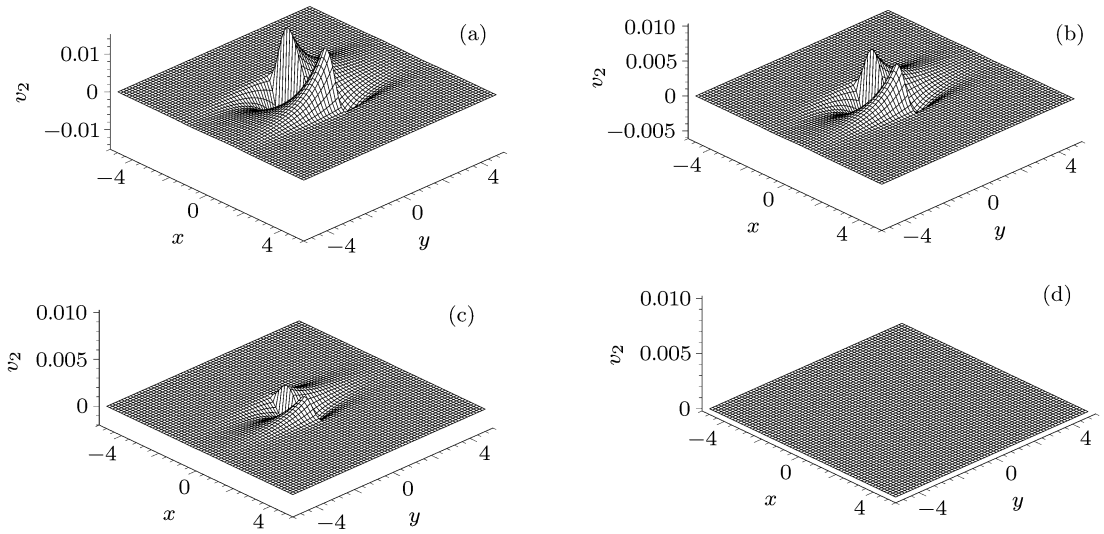


Fig. 2. A plot of another type of instantaneous soliton structure for the physical quantity v_2 given by the solution (14) with the choice of Eq. (18) for $\sigma = -1$ and $t = 0$ (a), 1 (b), 2 (c), and 5 (d).

3.2.1. Fractal structure via the sine function

In this subsubsection, we first discuss the fractal behaviour of the physical quantity v_3 given in Eq. (16) by using the sine function. When $\chi(x,t)$ and $\varphi(y)$ in physical quantity v_3 are chosen to be

$$\begin{aligned}\chi &= 1 + \exp\left[-(x-ct)((x-ct) + 2\sin(\ln(x-ct)^2))\right],\\ \varphi &= 1 + \exp\left[-y(y + 2\sin(\ln(y)^2))\right],\end{aligned} \tag{19}$$

we can derive a fractal dromion. Figure 3(a) shows a plot of this special type of fractal dromion structure for the v_3 given by Eq. (16) with the choice Eq. (19) at $\sigma = -1$, $t = 0$. From Fig. 3, we notice that the dromion structure possesses a self-similar fractal property. Near the centre in Fig. 3(a) there are many peaks distributed in a fractal manner.

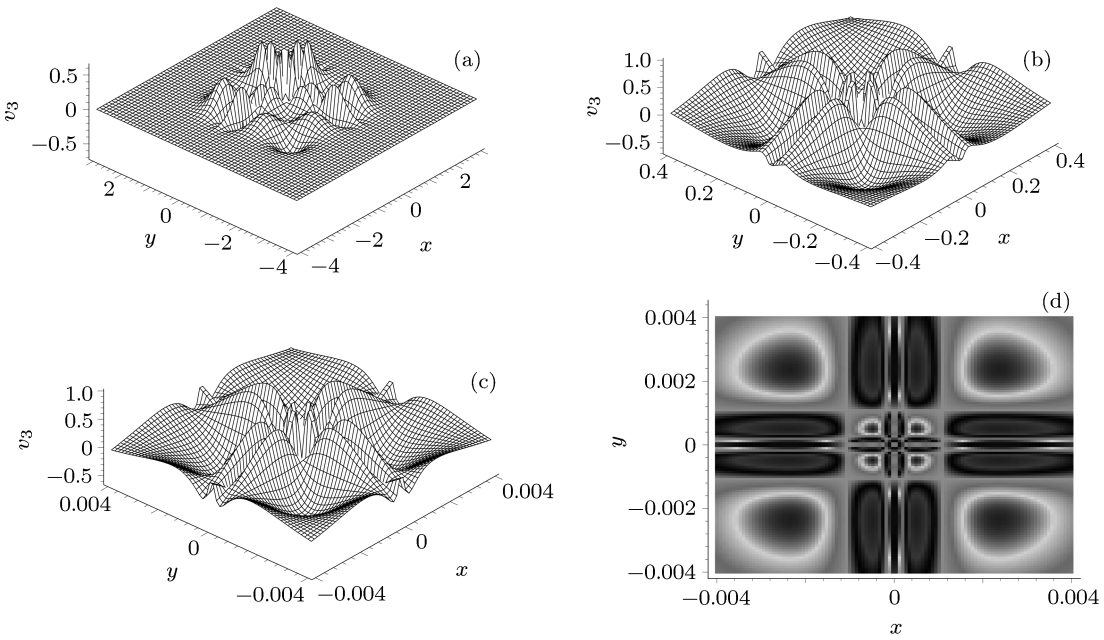


Fig. 3. (a) A fractal dromion structure for the v_3 given by Eq. (16) with the choice (19) and $\sigma = -1$ at time $t = 0$. (b) and (c) Self-similar structure of the fractal dromion related to (a) in the region ($x \in [-0.4, 0.4], y \in [-0.4, 0.4]$) and ($x \in [-0.004, 0.004]$, $y \in [-0.004, 0.004]$) respectively. (d) Density of the fractal dromion related to Fig. 3(b) in the region ($x \in [-0.004, 0.004]$, $y \in [-0.004, 0.004]$).

In order to observe the self-similar structure of the fractal dromion more clearly, one may enlarge a small region near the centre of Fig. 3(a). Figures 3(b) and 3(c) show plots of the self-similar structure of the fractal dromion in the following regions: ($x \in [-0.4, 0.4]$, $y \in [-0.4, 0.4]$) and ($x \in [-0.004, 0.004]$, $y \in [-0.004, 0.004]$). Figure 3(d) shows the density of the fractal structure of the dromion in the region ($x \in [-0.004, 0.004]$, $y \in [-0.004, 0.004]$).

3.2.2. Fractal structure via the Jacobi function

In the following, we discuss the fractal structure of the physical quantity v_2 given by Eq. (14) by using Jacobi function. For example, when choosing $\chi(x,t)$ and $\varphi(y)$ in solution v_2 to

$$\begin{aligned}\chi &= 0.1 + 2\exp\left[-(x-ct)((x-ct) + 4\,\mathrm{sn}(\ln x^2, 0.8))\right],\\ \varphi &= 0.1 + 2\exp\left[-y(y + 4\,\mathrm{sn}(\ln y^2, 0.8))\right],\end{aligned} \tag{20}$$

where sn is the Jacobi sine function, we can derive another fractal soliton excitation. Figure 4 shows a plot of the fractal soliton structure for the physical quantity v_2 given by Eq. (14) with the choice of Eq. (20) at $\sigma = -1$, $t = 0$. From Fig. 4, we can see that near the centre of Fig. 4(a) there are four peaks distributed in a fractal manner. Figures 4(b) and 4(c) show plots of the self-similar structure of the fractal soliton in the regions: ($x \in [-0.031, 0.03]$, $y \in [-0.03, 0.03]$) and ($x \in [-0.003, 0.003]$, $y \in [-0.003, 0.003]$). Figure 4(d) shows the density of the fractal structure in the region ($x \in [-0.03, 0.03]$, $y \in [-0.03, 0.03]$), this fractal structure is obviously different from that in Fig. 3. If we enlarge a smaller region near the centre of Figs. 4(b) and 4(c), we can find an identical structure with that presented as a whole in Figs. 4(b) and 4(c).

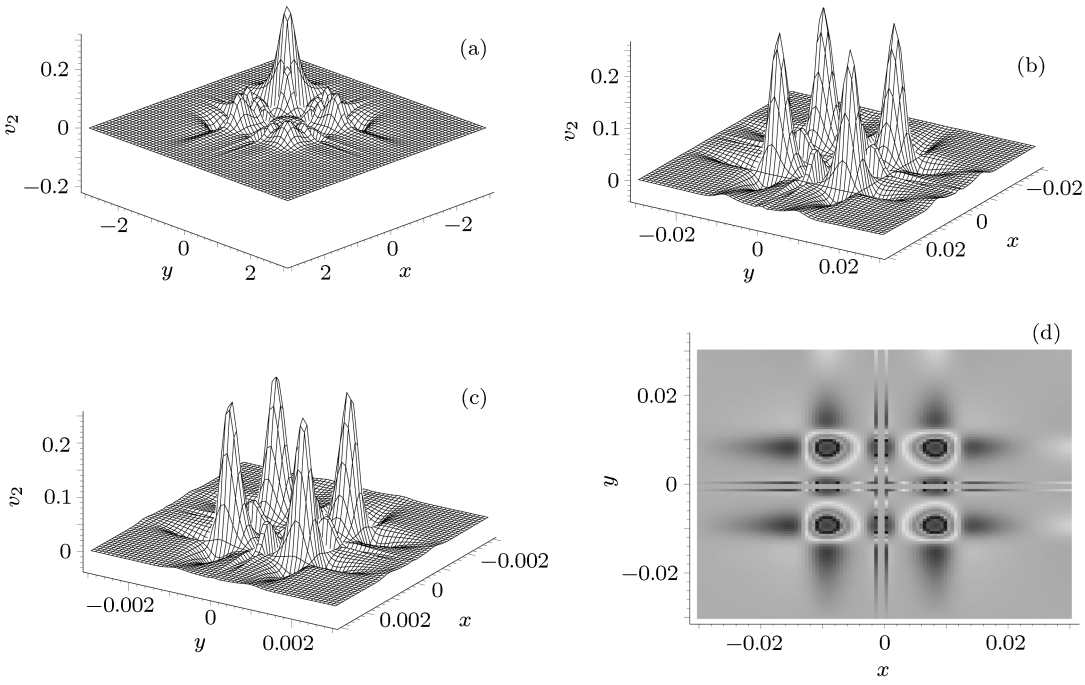


Fig. 4. (a) A fractal dromion structure for v_2 given by Eq. (14) with the choice of Eq. (19) and $\sigma = -1$ at time $t = 0$. (b), (c) Self-similar structure of the fractal dromion related to (a) in the regions ($x \in [-0.03, 0.03]$, $y \in [-0.03, 0.03]$) and ($x \in [-0.003, 0.003]$, $y \in [-0.003, 0.003]$) respectively. (d) Density of the fractal dromion related to Fig. 3(b) in the region ($x \in [-0.03, 0.03]$, $y \in [-0.03, 0.03]$).

4. Summary and discussion

The projective equation approach is a kind of classic, efficient and well-developed method to solve nonlinear evolution equations, the remarkable characteristic of which is that we can have many different ansatzs and therefore, a large number of solutions. In the past, Fang *et al.*[23] have solved the exact solutions of the (2+1)-dimensional BK system via the Ricatti equation ($\phi' = \sigma + \phi^2$) mapping method. In summary, via a new projective equation ($\phi' = \sigma\phi + \phi^2$) and a new ansatz $\{u = A(x) + \sum_{i=1}^{n} B_i(x)\phi^i(q(x)) + C_i(x)\phi^{i-1}(q(x))\sqrt{\sigma + \phi^2(q(x))}\}$,

we found some new exact solutions of the (2+1)-dimensional Broek–Kaup system, in which a square root of $\{\sigma+\phi^2(q(x))\}$ is included. Based on the derived solutions, we obtained some instantaneous soliton excitations. Then, we employed the sine function and Jacobi function to find the fractal structures of the solutions v_2 and v_3 of the (2+1)-dimensional BK system. Especially, we have found the four dromion self-similar fractal structures of v_2 presented in Fig. 4, which are different from the previous work. Fractals are the universal phenomenon in nature, such as curved coastline, the growth of bacterium community and vegetation, the distribution of blood vessel on retina, the formations of galaxy, and so on. From this, the complex world can be depicted by simple rules, and so people can understand better about nature.

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