Homotopic mapping method of solitary wave solutions for generalized complex Burgers equation*

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A class of generalized complex Burgers equation is considered. First, a set of equations of the complex value functions are solved by using the homotopic mapping method. The approximate solution for the original generalized complex Burgers equation is obtained. This method can find the approximation of arbitrary order of precision simply and reliably.

Keywords: homotopic mapping, solitary wave, Burgers equation

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1. Introduction

The importance of the study of the Burgers equation has now been realized. Solutions and integrability of the Burgers equation play such an important role in solitary wave that many natural sciences such as chemistry and biology, as well as in mathematics, communications and particularly in almost all branches of physics like the fluid dynamics, plasma physics, field theory, optics and condensed state physics, etc.[1−8] Recently, more new methods appeared, for example, the method of hyperbolic tangent function and homogeneous equilibrium method, the method of Jacobi elliptic function and method of auxiliary equation.[9] Many scholars have done a great deal of work, e.g. the shock wave, scattering of light wave, quantum mechanics, atmospheric physics, network of neurons and so on have been studied for the theory of solitary wave.[1,2,10] The approximate method for nonlinear theory is newly studied. The main essence of this method is the study of nonlinear problems dealt with linear problems by using the approximate expansion. The homotopic mapping method[11,12] is namely such a new method. During the past decade many approximate methods for the nonlinear problem have been developed and refined, including the method of averaging, the boundary layer method, and the methods of matched asymptotic expansion and the multiple scales. Using the method of differential inequalities and others, Mo *et al.* considered also a class of nonlinear problems.[13−24] In this paper, we consider a generalized complex Burgers equation and obtain approximate solution by using a simple and valid technique - the homotopic mapping method.

2. Generalized complex Burgers equation and homotopic mapping

Consider the following generalized complex Burgers equation:[8]

$$u_t + 2uu_x + \mathrm{i}\, u_{xx} = f(u), \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$ and f is a disturbing term, which is a sufficiently smooth function with respect to its variable in the corresponding domain.

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We first consider the typical complex Burgers equation for $f = 0$:

$$u_t + 2uu_x + \mathrm{i}\, u_{xx} = 0. \tag{2}$$

It is a standard example of a type II quadratic flux model. We can obtain a periodic travelling wave solution for Eq. (2):[8]

$$u(t,x) = \frac{1}{2}\frac{(\omega - \sqrt{\omega^2 - 4C_0})\exp[\sqrt{\omega^2 - 4C_0}(\mathrm{i}\,(\xi - \omega\tau) + \chi_0)] - (\omega + \sqrt{\omega^2 - 4C_0})}{\exp[\sqrt{\omega^2 - 4C_0}(\mathrm{i}\,(\xi - \omega\tau) + \chi_0)] - 1}, \quad \omega^2 > 4C_0, \tag{3}$$

where

$$\xi = x\exp\varepsilon(-t + \mathrm{i}C), \quad \tau = \exp 2\varepsilon(-t + \mathrm{i}C),$$

ω and χ_0 are the velocity and initial position of teavelling wave, respectively, and C and C_0 are arbitrary real constants under the condition $\omega^2 > 4C_0$.

In order to obtain the solution of Eq. (1), let

$$u = \sum_{i=0}^{\infty} u_i(t,x)r^i, \tag{4}$$

where r is the parameter. Then we introduce the following homotopic mapping $H(u,r)$:[11,12]

$$H(u,r) = L[u] - L[u_0] + r(L[u_0] + 2uu_x - f(u)), \tag{5}$$

where u_0 is an initial approximation of solution for Eq. (1), which will be determined below, and the linear operator L is

$$L[u] = u_t + \mathrm{i}\, u_{xx}. \tag{6}$$

We know that equation (4) is obtained by using the homotopic mapping (5) and it is a convergent series[11,25] on $[0,1]$.

Obviously, from mapping (5), $H(u,1) = 0$ is the same as Eq. (1). Thus the solution of Eq. (1) is the same as the solution of $H(u,r) = 0$ as $r \to 1$.

In order to obtain quickly the approximate solution of Eq. (1), we consider the following procedures.

3. Approximate solution

We assume that the initial approximation u_0 of solution for Eq. (1) is a periodic travelling wave solution for Eq. (2). That is,

$$u_0(t,x) = \frac{1}{2}\frac{(\omega - \sqrt{\omega^2 - 4C_0})\exp[\sqrt{\omega^2 - 4C_0}(\mathrm{i}\,(\xi - \omega\tau) + \chi_0)] - (\omega + \sqrt{\omega^2 - 4C_0})}{\exp[\sqrt{\omega^2 - 4C_0}(\mathrm{i}\,(\xi - \omega\tau) + \chi_0)] - 1}, \quad \omega^2 > 4C_0, \tag{7}$$

where

$$\xi = x\exp\varepsilon(-t + \mathrm{i}C), \quad \tau = \exp 2\varepsilon(-t + \mathrm{i}C).$$

Substituting Eq. (7) into Eq. (1), expanding nonlinear terms into the power series in r, we compare the coefficients of the same power in r on both sides of the equation. From the coefficients of the first power r^1, we have

$$L(u_1) - f(u_0) = 0. \tag{8}$$

Using the Fourier's transform method, it is easy to see that the solution u_1 of Eq. (8) with zero initial value is

$$\begin{aligned} &\frac{\mathrm{d}F[u_1]}{\mathrm{d}t} - \mathrm{i}\lambda^2 F[u_1] - F[\,f(u_0)] = 0, \\ &F[u_1]\,|_{t=0} = 0\,, \end{aligned} \tag{9}$$

where $F[\cdot]$ denotes the operator of Fourier's transform.

The solution of linear Eq. (9) is

$$F[u_1] = \int_0^t [F(\,f(u_0(\tau,x)))]\exp \mathrm{i}\lambda^2(t-\tau)\,\mathrm{d}\tau.$$

Thus we have

$$u_1(t,x)=\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\,f(u_0(\tau,\xi))\exp\mathrm{i}(\lambda^2(t-\tau)-\lambda\xi+\lambda x)]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau. \tag{10}$$

Substituting Eq. (4) into Eq. (5), we compare the coefficients of the same power in r on both sides of equations. From the coefficients of second power r^2, we have

$$L(u_2)+2\left(u_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_0}{\partial x}\right)-u_1\frac{\mathrm{d}f(u_0)}{\mathrm{d}u_0}=0.$$

Analogously, we can obtain the solution $u_2(t,x)$ for the above linear equation as follows:

$$\begin{aligned}u_2(t,x)=\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Bigg[&-2\left(u_0(\tau,\xi)\frac{\partial u_1}{\partial\xi}(\tau,\xi)+u_1(\tau,\xi)\frac{\partial u_0}{\partial\xi}(\tau,\xi)\right)\\&+u_1(\tau,\xi)\frac{\mathrm{d}f_0(u_0)}{\mathrm{d}u_0}(\tau,\xi))\exp\mathrm{i}(\lambda^2(t-\tau)-\lambda\xi+\lambda x)\Bigg]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau,\end{aligned} \tag{11}$$

where u_0 and u_1 are denoted by Eqs. (7) and (10), respectively.

Thus from Eqs. (7), (10), (11) and (4), we have the first and second order approximations $U_1(t,x)$ and $U_2(t,x)$ of a solitary wave solution for the generalized complex Burgers equation (1) as

$$\begin{aligned}U_1(t,x)=&\frac{1}{2}\frac{(\omega-\sqrt{\omega^2-4C_0})\exp[\sqrt{\omega^2-4C_0}(\mathrm{i}\,(\xi-\omega\tau)+\chi_0)]-(\omega+\sqrt{\omega^2-4C_0})}{\exp[\sqrt{\omega^2-4C_0}(\mathrm{i}\,(\xi-\omega\tau)+\chi_0)]-1}\\&+\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\,f(u_0(\tau,\xi))\exp\mathrm{i}(\lambda^2(t-\tau)-\lambda\xi+\lambda x)]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau,\end{aligned} \tag{12}$$

$$\begin{aligned}U_2(t,x)=&\frac{1}{2}\frac{(\omega-\sqrt{\omega^2-4C_0})\exp[\sqrt{\omega^2-4C_0}(\mathrm{i}\,(\xi-\omega\tau)+\chi_0)]-(\omega+\sqrt{\omega^2-4C_0})}{\exp[\sqrt{\omega^2-4C_0}(\mathrm{i}\,(\xi-\omega\tau)+\chi_0)]-1}\\&+\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Bigg[-2\left(u_0(\tau,\xi)\frac{\partial u_1}{\partial\xi}(\tau,\xi)+u_1(\tau,\xi)\frac{\partial u_0(\tau,\xi)}{\partial\xi}\right)\\&+f(u_0(\tau,\xi))+u_1(\tau,\xi)\frac{\mathrm{d}f(u_0(\tau,\xi))}{\mathrm{d}u_0}\exp\mathrm{i}(\lambda^2(t-\tau)-\lambda\xi+\lambda x)\Bigg]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau,\end{aligned} \tag{13}$$

where $\xi=x\exp\varepsilon(-t+\mathrm{i}C)$, $\tau=\exp 2\varepsilon(-t+\mathrm{i}C)$ and u_0 and u_1 are denoted by Eqs. (7) and (11), respectively.

Similarly, we can also obtain the arbitrary n-th order approximations $U_n(t,x)$ $(n\geq 3)$ of a solitary wave solutions for the generalized complex Burgers equation (1).

4. Disturbed Burgers equation

As a special example, we study a disturbed Burgers equation. That is, let the disturbed term $f(u)=\varepsilon\exp u$ for Eq. (1), where $0<\varepsilon\ll 1$. Then we consider the following disturbed complex Burgers equation:

$$u_t+2uu_x+\mathrm{i}\,u_{xx}=\varepsilon\,\exp u,\quad 0<\varepsilon\ll 1. \tag{14}$$

Using the above homotopic mapping method, let $u=\sum_{i=0}^{\infty}u_i(t,x)r^i$. From Eq. (7), we first select $u_0(t,x)$:

$$u_0(t,x)=\frac{1}{2}\frac{(\omega-\sqrt{\omega^2-4C_0})\exp[\sqrt{\omega^2-4C_0}(\mathrm{i}\,(\xi-\omega\tau)+\chi_0)]-(\omega+\sqrt{\omega^2-4C_0})}{\exp[\sqrt{\omega^2-4C_0}(\mathrm{i}\,(\xi-\omega\tau)+\chi_0)]-1},\quad \omega^2>4C_0, \tag{15}$$

where $\xi=x\exp\varepsilon(-t+\mathrm{i}C)$, $\quad\tau=\exp 2\varepsilon(-t+\mathrm{i}C)$. Thus from Eq. (10), we have

$$u_1(t,x)=\frac{\varepsilon}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp u_0(\tau,\xi)[\exp\mathrm{i}(\lambda^2(t-\tau)-\lambda\xi+\lambda x)]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau. \tag{16}$$

Then from Eq. (13), the second order approximation $U_2(t,x)$ of a solitary wave solution for the disturbed complex Burgers equation (14) is

$$U_2(t,x)=\frac{1}{2}\frac{(\omega-\sqrt{\omega^2-4C_0})\exp[\sqrt{\omega^2-4C_0}(\mathrm{i}\,(\xi-\omega\tau)+\chi_0)]-(\omega+\sqrt{\omega^2-4C_0})}{\exp[\sqrt{\omega^2-4C_0}(\mathrm{i}\,(\xi-\omega\tau)+\chi_0)]-1}$$

$$-\frac{1}{\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(u_0(\tau,\xi)\frac{\partial u_1(\tau,\xi)}{\partial\xi}+u_1(\tau,\xi)\frac{\partial u_0(\tau,\xi)}{\partial\xi}\right)\exp\mathrm{i}(\lambda^2(t-\tau)-\lambda\xi+\lambda x)\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau$$
$$+\frac{\varepsilon}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\exp(u_0(\tau,\xi)(1-u_1(\tau,\xi))\exp\mathrm{i}(\lambda^2(t-\tau)-\lambda\xi+\lambda x)]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau, \quad (17)$$

where $\xi = x\exp\varepsilon(-t+\mathrm{i}C)$, $\tau = \exp 2\varepsilon(-t+\mathrm{i}C)$ and u_0 and u_1 are denoted by Eqs. (15) and (16), respectively.

From the homotopic mapping (10), by in the same method, we can obtain high order approximations $U_n(t,x)$, $n = 3,4,\ldots$ of a solitary wave solution for the disturbed complex Burgers equation (14).

5. Discussion

The solitary wave solution for the complex Burgers equation describes a class of complicated natural phenomenon. Hence we need to introduce the basic models, and then we solve them by using the approximate method. The homotopic mapping method is a simple and valid method.

Naturally, we select a periodic travelling wave solution for Eq. (2) as the initial approximation u_0 of the homotopic mapping Eq. (5), which can give readily the approximate travelling wave solution of Eq. (1).

The homotopic mapping method is an approximate analytic method, which differs from general numerical method. The expansions of solution using the homotopic mapping method can be a continuous analytic operation. Thus from Eq. (12) or Eq. (13), we can study further the qualitative and quantitative behaviours of the solitary waves.

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