New representation of the multimode phase shifting operator and its application*

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Based on the rotation transformation in phase space and the technique of integration within an ordered product of operators, the coherent state representation of the multimode phase shifting operator and one of its new applications in quantum mechanics are given. It is proved that the coherent state is a natural language for describing the phase shifting operator or multimode phase shifting operator. The multimode phase shifting operator is also a useful tool to solve the dynamic problems of the multimode coordinate–momentum coupled harmonic oscillators. The exact energy spectra and eigenstates of such multimode coupled harmonic oscillators can be easily obtained by using the multimode phase shifting operator.

Keywords: phase shifting operator, coupled harmonic oscillators, energy spectrums, energy eigenstates
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1. Introduction

The coherent state of the harmonic oscillator is an important concept and a useful tool in quantum physics.[1−5] By using the coherent state, it is possible to use more transparent classical language to describe the quantum phenomena. It is well known that unitary transformation plays a significant role in solving many dynamic problems, while by the technique of integration within an ordered product of operators (IWOP),[6−9] the coherent state has more applications in unitary transformations (for example, the Wigner transformation and squeezing transformation) and the corresponding coherent state representations of the Wigner operator and the squeezing operators were obtained, separately.[7−10] Recently, the coherent state was applied to the Fresnel transformations in Fourier optics, then the corresponding coherent state representations of both one-mode and two-mode quantum optics Fresnel operators were derived, separately.[11−13] Therefore, some useful operators of the unitary transformations in quantum physics can be expressed by the coherent state.

In quantum optics, the phase shifting operator corresponds to the unitary rotation transformation in phase space and has important applications in a single mode radiation field, especially in the practical homodyne detection of squeezed light.[14−16] In the present work, we search for the coherent state representations of the phase shifting operator and multimode phase shifting operator, then we prove that the coherent state is a natural language for describing the phase shifting operator or multimode phase shifting operator. It is more important that we first give its new application to the dynamic problems of multimode coordinate–momentum coupled harmonic oscillators. In quantum mechanics many physical problems are related to coupled harmonic oscillators. Therefore, some methods have been put forward, and great attention has been paid to research on the dynamic problems of the coupled harmonic oscillators.[17−23] But the application of the phase shifting operator to the dynamic problems of multimode coupled harmonic oscillators has not been reported. In the present work, we prove that the multimode phase shifting operator (or phase shifting operator) is also a more suitable tool for solving the dynamic problems of such multimode coupled harmonic oscillators. The exact energy spectra and eigenstates of multimode coordinate–momentum coupled harmonic oscillators can be easily obtained by the multimode phase shifting operator method.

The remainder of the present paper is organized as follows. In Section 2, based on the rotation transformation in phase space and the IWOP technique,

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we introduce the natural representations of the phase shifting operator and the multimode phase shifting operator in a coherent state. In Section 3, we show that the Hamiltonian of multimode coordinate–momentum coupled harmonic oscillators can be easily diagonalized by the multimode phase shifting operator. Its exact energy spectra and eigenstates are then easily obtained. Therefore, this is a new application of the phase shifting operator to quantum mechanics. Finally, some conclusions drawn from the present study are given in Section 4.

2. New representations of phase shifting operator and multimode phase shifting operator

To start with, we briefly recall the phase shifting operator. In quantum optics, one actually is to measure the rotated quadrature operators in practical homodyne detection,[24] while the rotated quadrature operators can be obtained when the phase shifting operator acts on the coordinator or momentum operators.[14] Therefore, the phase shifting operator is an important concept in quantum optics. It is explicitly expressed as[14]

$$\hat{U}(\theta) = \mathrm{e}^{-\mathrm{i}\theta\hat{a}^{\dagger}\hat{a}} \tag{1}$$

with which the rotation transformation in phase space can be realized. Now, we show that this phase shifting operator can be naturally expressed by coherent state $|z\rangle$. The coherent state is an eigenstate of a bosonic annihilation operator $\hat{a}$, i.e. $\hat{a}\,|z\rangle = z\,|z\rangle$. The normalized coherent state of the harmonic oscillator in Fock space is expressed as [4]

$$|z\rangle = \exp\left(-\frac{1}{2}|z|^2 + z\hat{a}^{\dagger}\right)|0\rangle\,, \tag{2}$$

where $|0\rangle$ is the ground state in Fock space, $\hat{a}^{\dagger}$ is the bosonic creation operator and z represents a complex number, i.e. $z = (m\omega/(2\hbar))^{1/2}x + \mathrm{i}(2\hbar m\omega)^{-1/2}p$, with x and p denoting the coordinate and the momentum of a harmonic oscillator, respectively. In quantum-mechanical phase space, the coherent state has also been proved to be useful.[1−5] Following a method similar to that in Refs. [7] and [25], we introduce a nonsymmetric integration projective operator into the coherent state to realize the rotation transformation in phase space

$$\hat{R}_1(\theta) = \int \frac{\mathrm{d}^2 z}{\pi}\left|\mathrm{e}^{-\mathrm{i}\theta}z\right\rangle\langle z|\,, \tag{3}$$

where the parameter θ denotes the rotation angle of phase space. Equation (3) appeared first in Ref. [7], but its detailed physical explanation in quantum-mechanical phase space was not given in detail. By using the IWOP technique and the following results[9]

$$|0\rangle\langle 0| =: \mathrm{e}^{-\hat{a}^{\dagger}\hat{a}}:, \tag{4}$$

$$\exp[\lambda\hat{a}^{\dagger}\hat{a}] =: \exp[(\mathrm{e}^{\lambda} - 1)\hat{a}^{\dagger}\hat{a}]:,$$

$$\int \frac{\mathrm{d}^2 z}{\pi}\exp[\lambda|z|^2 + fz + gz^*] = -\frac{1}{\lambda}\exp[-fg/\lambda], \quad \mathrm{Re}(\lambda) < 0, \tag{5}$$

where :: denotes the normal product, and after performing integration of Eq. (3) over all the phase space, one can obtain

$$\hat{R}_1(\theta) = \int \frac{\mathrm{d}^2 z}{\pi}\left|\mathrm{e}^{-\mathrm{i}\theta}z\right\rangle\langle z| = \mathrm{e}^{-\mathrm{i}\theta\hat{a}^{\dagger}\hat{a}}. \tag{6}$$

We can easily prove that $\hat{R}_1(\theta)$ is a unitary operator. In the same way, we introduce the following nonsymmetric projective operator

$$\hat{R}_1^{\dagger}(\theta) = \int \frac{\mathrm{d}^2 z}{\pi}|z\rangle\left\langle \mathrm{e}^{-\mathrm{i}\theta}z\right| = \mathrm{e}^{\mathrm{i}\theta\hat{a}^{\dagger}\hat{a}} \equiv \hat{R}_1^{-1}(\theta). \tag{7}$$

Therefore, the operator $\hat{R}_1(\theta)$ is just a unitary operator and obeys $\hat{R}_1(\theta)\hat{R}_1^{\dagger}(\theta) = \hat{R}_1^{\dagger}(\theta)\hat{R}_1(\theta) = 1$. Comparing Eq. (6) with Eq. (1) yields

$$\hat{R}_1(\theta) = \hat{U}(\theta) = \int \frac{\mathrm{d}^2 z}{\pi}\left|\mathrm{e}^{-\mathrm{i}\theta}z\right\rangle\langle z|\,. \tag{8}$$

Therefore, this unitary operator $\hat{R}_1(\theta)$ is just the phase shifting operator in quantum optics.[14] This is a new representation of $\hat{R}_1(\theta)$ (or $\hat{U}(\theta)$). It can be seen that the phase shifting operator can be naturally expressed in the coherent state. In other words, the coherent state is a natural language for describing the phase shifting operator.

Naturally, we can introduce the following nonsymmetric integration projective operator into the multimode coherent state

$$\hat{R}_n(\theta) = \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2 \ldots \mathrm{d}^2 z_n}{\pi^n} \times \left|\mathrm{e}^{-\mathrm{i}\theta}z_1, \mathrm{e}^{-\mathrm{i}\theta}z_2, \ldots, \mathrm{e}^{-\mathrm{i}\theta}z_n\right\rangle \times \langle z_1, z_2, \ldots, z_n|\,, \tag{9}$$

where the $|z_1, z_2, \ldots, z_n\rangle$ is the multimode coherent state, and $|z_1, z_2, \ldots, z_n\rangle =$

$\prod_i^n \exp(-\frac{1}{2}|z_i|^2 + z_i\hat{a}_i^\dagger)|0\rangle$.[1] By the IWOP technique, Eqs. (4) and (5), after calculating, we can obtain

$$\begin{aligned}\hat{R}_n(\theta) &= \prod_{i=1}^{n} \exp(-\mathrm{i}\theta\hat{a}_i^\dagger\hat{a}_i) \\ &= \prod_{i}^{n} \hat{R}_i, \quad i = 1, 2 \ldots, n, \end{aligned} \tag{10}$$

where $\hat{R}_i = \exp(-\mathrm{i}\theta\hat{a}_i^\dagger\hat{a}_i)$ is the one-mode phase shifting operator. By employing the IWOP technique, we can immediately prove that the operator $\hat{R}_n(\theta)$ is a unitary operator and obeys $\hat{R}_n(\theta)\hat{R}_n^\dagger(\theta) = \hat{R}_n^\dagger(\theta)\hat{R}_n(\theta) = 1$. Compared with Eq. (1), Eq. (10) can be just called the multimode phase shifting operator. Thus we can identify from Eqs. (9) and (10) that the coherent state is also a natural language for describing the multimode phase shifting operator. Equation (9) is just the coherent state representation of the multimode phase shifting operator. Since the phase shifting operator can realize the rotation transformation in phase space, next, with the help of the multimode phase shifting operator, we focus our research on the dynamic problems of multimode coordinate–momentum coupled harmonic oscillators.

3. New application of multimode phase shifting operator

It is well known that many physical problems in various fields (such as quantum mechanics, quantum optics, quantum chemistry and so on) are related to the coupled harmonic oscillators. So, it is important to study the exact solutions for such problems. Some methods have been put forward to solve the dynamic problems of coupled harmonic oscillators in previous studies.[17−23] For multimode coordinate–momentum coupled harmonic oscillators, its Hamiltonian is[22]

$$\begin{aligned}\hat{H} &= \sum_{i=1}^{n}\left[\frac{1}{2m}\hat{p}_i^2 + \frac{1}{2}m\omega_i^2\hat{x}_i^2 + \gamma_i(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i)\right] \\ &= \sum_{i=1}^{n}\hat{H}_i, \quad i = 1, 2 \ldots, n, \end{aligned} \tag{11}$$

where γ_i is the real number, $[\hat{q}_i, \hat{p}_i] = \mathrm{i}\hbar$ and $\hat{H}_i$ is the Hamiltonian of a one-mode harmonic oscillator with coordinate–momentum coupling. In addition, the one-mode squeezing state can be produced by this Hamiltonian $\hat{H}_i$ in quantum optics. The expression of $\hat{H}_i$ is

$$\hat{H}_i = \frac{1}{2m}\hat{p}_i^2 + \frac{1}{2}m\omega_i^2\hat{x}_i^2 + \gamma_i(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i). \tag{12}$$

In this work, we prove that the multimode phase shifting operator is also a useful tool to solve the dynamic problems of such multimode coupled harmonic oscillators. Then its exact energy spectra and eigenstates can be easily derived.

Firstly, we introduce the following bosonic annihilation and creation operators:

$$\begin{aligned}\hat{a}_i &= \sqrt{\frac{m\omega_i}{2\hbar}}\left(\hat{x}_i + \frac{\mathrm{i}\hat{p}_i}{m\omega_i}\right), \\ \hat{a}_i^\dagger &= \sqrt{\frac{m\omega_i}{2\hbar}}\left(\hat{x}_i - \frac{\mathrm{i}\hat{p}_i}{m\omega_i}\right), \ i = 1, 2, \ldots, n. \end{aligned} \tag{13}$$

When the multimode phase shifting operator acts on both sides of Eq. (11), we can easily obtain

$$\begin{aligned}\hat{R}_n(\theta)\hat{H}\hat{R}_n^{-1}(\theta) &= \sum_{i=1}^{n}\hat{R}_n(\theta)\hat{H}_i\hat{R}_n^{-1}(\theta) \\ &= \sum_{i=1}^{n}\hat{R}_i(\theta)\hat{H}_i\hat{R}_i^{-1}(\theta), \end{aligned} \tag{14}$$

where $\hat{R}_i = \exp(-\mathrm{i}\theta\hat{a}_i^\dagger\hat{a}_i)$ is the one-mode phase shifting operator and the bosonic commutative $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ has been used. It is convenient to calculation the item $\hat{R}_i(\theta)\hat{H}_i\hat{R}_i^{-1}(\theta)$ in Eq. (14). When the phase shifting operator $\hat{R}_i(\theta)$ acts on $\hat{a}_i$ and $\hat{a}_i^\dagger$, one can derive[14]

$$\begin{aligned}\hat{R}_i(\theta)\hat{a}_i\hat{R}_i^{-1}(\theta) &= \hat{a}_i\,\mathrm{e}^{\mathrm{i}\theta}, \\ \hat{R}_i(\theta)\hat{a}_i^\dagger\hat{R}_i^{-1}(\theta) &= \hat{a}_i^\dagger\,\mathrm{e}^{-\mathrm{i}\theta}. \end{aligned} \tag{15}$$

Combining Eq. (13) and Eq. (15), one can immediately obtain the rotated quadrature operators[14,15]

$$\hat{R}_i(\theta)\hat{x}_i\hat{R}_i^{-1}(\theta) = \sqrt{\frac{\hbar}{2m\omega_i}}(\hat{a}_i\,\mathrm{e}^{\mathrm{i}\theta} + \hat{a}_i^\dagger\,\mathrm{e}^{-\mathrm{i}\theta}) = \hat{x}_i\cos\theta - \frac{\hat{p}_i}{m\omega_i}\sin\theta, \tag{16}$$

$$\hat{R}_i(\theta)\hat{p}_i\hat{R}_i^{-1}(\theta) = \mathrm{i}\sqrt{\frac{\hbar m\omega_i}{2}}(\hat{a}_i^\dagger\,\mathrm{e}^{\mathrm{i}\theta} - \hat{a}_i\,\mathrm{e}^{-\mathrm{i}\theta}) = \hat{p}_i\cos\theta + m\omega_i\hat{x}_i\sin\theta. \tag{17}$$

It can be seen that the phase shifting operator can realize the rotated transformation in quantum phase space.

Combined Eq. (16) and Eq. (17), the item $\hat{R}_i(\theta)\hat{H}_i\hat{R}_i^{-1}(\theta)$ can be rewritten as

$$\hat{R}_i(\theta)\hat{H}_i\hat{R}_i^{-1}(\theta) = \frac{1}{2m}(\hat{p}_i\cos\theta + m\omega_i\hat{x}_i\sin\theta)^2 + \frac{1}{2}m\omega_i^2\left(\hat{x}_i\cos\theta - \frac{\hat{p}_i}{m\omega_i}\sin\theta\right)^2 + \gamma_i\left[\mathrm{i}\hbar + 2(\hat{p}_i\cos\theta + m\omega_i\hat{x}_i\sin\theta)\left(\hat{x}_i\cos\theta - \frac{\hat{p}_i}{m\omega_i}\sin\theta\right)\right], \quad (18)$$

where we have used the commutation relation $[\hat{q}_i, \hat{p}_i] = \mathrm{i}\hbar$. It is evident that when $\theta = \pi/4$, Eq. (18) can be simply written as

$$\hat{R}_i\left(\frac{\pi}{4}\right)\hat{H}_i\hat{R}_i^{-1}\left(\frac{\pi}{4}\right) = \frac{\hat{p}_i^2}{2\mu_i} + \frac{1}{2}\mu_i\Omega_i^2\hat{x}_i^2. \quad (19)$$

Therefore, substituting Eq. (19) into Eq. (14), we can immediately obtain

$$\hat{R}_n\left(\frac{\pi}{4}\right)\hat{H}\hat{R}_n^{-1}\left(\frac{\pi}{4}\right) = \sum_{i=1}^{n}\left(\frac{\hat{p}_i^2}{2\mu_i} + \frac{1}{2}\mu_i\Omega_i^2\hat{x}_i^2\right), \quad (20)$$

where

$$\mu_i = \frac{m\omega_i}{\omega_i - 2\gamma_i}, \quad \Omega_i^2 = (\omega_i^2 - 4\gamma_i^2), \quad i = 1, 2, 3\ldots, n. \quad (21)$$

It is evident that the Hamiltonian of multimode coordinate–momentum coupled harmonic oscillators has been diagonalized. The coordinate–momentum coupled items can be easily eliminated by the rotation transformation in phase space. Therefore, compared with the quadratic form theory,[21,22] the multimode phase shifting operator is a more suitable tool for diagonalizing the Hamiltonian of such multimode coupled harmonic oscillators.

It is well known that the energy eigenvalues of quantum systems do not change under unitary transformation. Therefore, from Eq. (20), we can immediately obtain the exact solution of the energy spectra of the multimode coordinate–momentum coupled harmonic oscillators

$$E_{j_1j_2\ldots j_n} = \sum_{i=1}^{n}\left(j_i + \frac{1}{2}\right)\hbar\Omega_i, \quad j_i = 0, 1, 2\ldots\ . \quad (22)$$

By employing the multimode phase shifting operator, we can easily derive the energy eigenstates of the multimode coordinate–momentum coupled harmonic oscillators. Based on Eq. (20), for multimode harmonic oscillators without coupling, their energy eigenstates are just the multimode Fock states

$$|j_1, j_2, \ldots, j_n\rangle = \prod_{i=1}^{n}|j_i\rangle = \prod_{i=1}^{n}\frac{\hat{a}_i^{\dagger j_i}}{\sqrt{j_i!}}|0\rangle, \quad j_i = 0, 1, 2\ldots\ , \quad (23)$$

where $|j_i\rangle$is the energy eigenstate of an individual simple harmonic oscillator with frequency Ω_i. In order to derive the energy eigenstates of multimode coordinate–momentum coupled harmonic oscillators, we write the Schrödinger equation of Hamiltonian (11)

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|E_{j_1j_2\ldots j_n}\rangle = \hat{H}|E_{j_1j_2\ldots j_n}\rangle, \quad (24)$$

where $|E_{j_1j_2\ldots j_n}\rangle$ is just the energy eigenstate of multimode coordinate–momentum coupled harmonic oscillators. Then based on Eqs. (20) and (24), one can confirm that the energy eigenstate $|E_{j_1j_2\ldots j_n}\rangle$ is connected with the multimode Fock states $|j_1, j_2, \ldots, j_n\rangle$ by the unitary transformation operator $\hat{R}_n^{-1}(\pi/4)$

$$|E_{j_1j_2\ldots j_n}\rangle = \hat{R}_n^{-1}\left(\frac{\pi}{4}\right)|j_1, j_2, \ldots, j_n\rangle. \quad (25)$$

From Eq. (25) it follows that the energy eigenstates of multimode coordinate–momentum coupled harmonic oscillators can be obtained just by applying a phase space rotation of an angle $\theta = \pi/4$ to the multimode Fock states. This explicit solution of Eq. (25) is more simple and convenient.

4. Conclusions

In this paper, new representations of the phase shifting operator and multimode phase shifting operator in a coherent state are obtained by using the IWOP technique. It is found that the coherent state is a natural language for describing the phase shifting operator or multimode phase shifting operator. Because the phase shifting operator can realize the rotation transformation in phase space, based on this quantum property its new application to the dynamic problems of multimode coordinate–momentum coupled harmonic oscillators is given. It is found that the Hamiltonian of the multimode coordinate–momentum coupled harmonic oscillators can be easily diagonalized by the multimode phase shifting operator. Then,

its exact energy spectra and eigenstates are easily obtained. In addition, the energy eigenstates of multimode coordinate–momentum coupled harmonic oscillators are just multimode Fock states with a rotating angle $\theta = \pi/4$ in phase space. Therefore, in this paper, a new application of the multimode phase shifting operator to the dynamic problems of multimode coupled harmonic oscillators is given. It is proved that the multimode phase shifting operator is also a useful tool to solve the dynamic problems of multimode harmonic oscillators with the coupling form $\gamma_i(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i)$.

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