Wave function for the squeezed atomic coherent state in entangled state representation and some of its applications*

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Based on the Einstein, Podolsky, and Rosen (EPR) entangled state representation, this paper introduces the wave function for the squeezed atomic coherent state (SACS), which turns out to be just proportional to a single-variable ordinary Hermite polynomial of order $2j$. As important applications of the wave function, the Wigner function of the SACS and its marginal distribution are obtained and the eigenproblems of some Hamiltonians for the generalized angular momentum system are solved.

Keywords: squeezed atomic coherent state, entangled state representation, Wigner function, eigenvalue and eigenstate

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1. Introduction

In quantum optics, the atomic coherent state (ACS), which was first defined by Arecchi *et al.*,[1] is recognized as an important type of two-mode correlated state. As a description of the assembly of two-level atoms, the ACS has many attractive properties analogous to those of the fields of coherent states, so the properties[2−4] and applications[5,6] of the ACS have been widely discussed in many domains of quantum optics. For instance, Agarwal[2] investigated the properties of the CAS, such as orthogonal and overcomplete properties, and obtained the intimate relationship between the ACS representation and the idea of state multipoles. Christopher and Mark[4] discussed the spin squeezing and entanglement viaburning in ACS. References [3], [5] and [6] gave the explicit Wigner distribution of the ACS and some of its practical applications.

Operating a two-mode squeezed operator on the ACS, we obtain the squeezed atomic coherent state (SACS). The 'squeezing' of the type of state is of considerable contemporary interest because it is also a nonclassical state which exhibits a spin squeezing property. Nowadays spin squeezing is a subject of vigorous study owing to the desire to reduce projection noise in high precession population spectroscopy relevant to atomic frequency standards[7,8] and it turns out that a collection of squeezed spins is entangled. Also the SACS has been widely applied in many fields of modern physics.[9−11] For instance, Ocak and Altanhan[9] used the SACS, which are defined through the spin operator representation in a differential equation giving quantum mechanical potentials, to study a spin system describing a magnetic system with the spin value $S = 1$. Arvieu and Rozmej[10] emphasized that the SACS can describe the rotation of quantum simple systems like rigid diatomic molecules or rigid nuclei. In addition, the SACS also has some potential applications in spintronics,[12] quantum information,[13] and quantum computation.[14] In principle the SACS can be generated through some nonlinear interaction between a system of effective two-level atoms and a suitable radiation field.[15] Therefore, it is essential and significative to construct the SACS and discuss some of its properties in theory.

In this paper we are motivated to derive the wave function of the SACS in the entangled state $\langle\eta|$ representation[16,17] and present some of its practical applications in quantum optics. Our work is arranged as follows: in Section 2, we recall the properties of the entangled state $\langle\eta|$ representation. In Section 3, we calculate the wave function of the SACS in the $\langle\eta|$

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representation, which is related to the single-variable ordinary Hermite polynomial of order $2j$. In the final section we use the wave function obtained in Section 3 to derive the Wigner function of the SACS and its marginal distributions, and then solve eigenproblems for some Hamiltonians of the generalized angular momentum system.

2. Properties of the entangled state $\langle\eta|$

In order to obtain the wave function of the SACS in the $\langle\eta|$ representation, let us begin with reviewing the entangled state $\langle\eta|$ and some of its properties. The state $\langle\eta|$ is defined as[16,17]

$$\langle\eta| = \langle 00|\exp\left[-|\eta|^2/2 + \eta^* a - \eta b + ab\right], \quad \eta = \eta_1 + \mathrm{i}\eta_2, \tag{1}$$

which obeys the eigenvector equations

$$\langle\eta|\left(a - b^\dagger\right) = \langle\eta|\,\eta,\ \langle\eta|\left(a^\dagger - b\right) = \langle\eta|\,\eta^* \tag{2}$$

and

$$\frac{\partial}{\partial\eta}\langle\eta| = \left(-\frac{\eta^*}{2} - b\right)\langle\eta|, \quad \frac{\partial}{\partial\eta^*}\langle\eta| = \left(-\frac{\eta}{2} + a\right)\langle\eta|. \tag{3}$$

So in the $\langle\eta|$ basis we have the corresponding relations

$$a \to \frac{\partial}{\partial\eta^*} + \frac{\eta}{2}, \quad b \to -\frac{\partial}{\partial\eta} - \frac{\eta^*}{2}, \tag{4}$$

$$a^\dagger \to -\frac{\partial}{\partial\eta} + \frac{\eta^*}{2}, \quad b^\dagger \to \frac{\partial}{\partial\eta^*} - \frac{\eta}{2}. \tag{5}$$

In terms of the relations between the coordinate (momentum) operator and the Bose creation–annihilation operators, we find that the state $|\eta\rangle$ is the common eigenstate of two particles' relative position $(X_1 - X_2)$ and the total momentum $(P_1 + P_2)$ in two-mode Fock space. Using the normal ordering form of the projector operator of two-mode vacuum state $|00\rangle\langle 00| =: \exp\left(a^\dagger a - b^\dagger b\right) :$, where $: \ :$ is the normal ordering symbol, and the technique of integration within an ordered product of operator,[18−20] we can obtain the complete and orthonormal properties

$$\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = 1, \ \ \langle\eta|\,\eta'\rangle = \pi\delta(\eta - \eta')\,\delta(\eta^* - \eta'^*). \tag{6}$$

Moreover, in the state $|\eta\rangle$ representation the two-mode squeezed operator $S_2(r)$ can be converted into

$$S_2(r) = \exp[r(ab - a^\dagger b^\dagger)] = \frac{1}{\mu}\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta/\mu|, \quad \mu = \mathrm{e}^r, \tag{7}$$

which obeys

$$S_2(r)|\eta\rangle = \mu|\mu\eta\rangle. \tag{8}$$

3. Wave function of the SACS in the $\langle\eta|$ representation

The ACS $|\tau\rangle$ is defined as[21]

$$|\tau\rangle = \exp\left(\mu J_+ - \mu^* J_-\right)|j, -j\rangle, \tag{9}$$

where J_+ is the raising operator of $|j, m\rangle$, $|j, -j\rangle$ is the lowest weight state annihilated by J_-, and $\mu = (\theta/2)\mathrm{e}^{-\mathrm{i}\varphi}$, $\tau = \mathrm{e}^{-\mathrm{i}\varphi}\tan(\theta/2)$. Using the Schwinger Bose operator realization of angular momentum[5,22]

$$J_+ = a^\dagger b, \quad J_- = ab^\dagger, \quad J_z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right) \tag{10}$$

and the eigenstates of J_z and J^2 in two-mode Fock space

$$|j, m\rangle = \frac{a^{\dagger j+m} b^{\dagger j-m}}{\sqrt{(j+m)!\,(j-m)!}}|00\rangle = |j+m\rangle \otimes |j-m\rangle, \tag{11}$$

where J is the angular momentum operator of $|j, m\rangle$in two-mode Fock space we can rewrite $|\tau\rangle$ as

$$|\tau\rangle = \exp\left(\mu J_+ - \mu^* J_-\right)|0\rangle \otimes |2j\rangle = \frac{1}{(1+|\tau|^2)^j}\sum_{l=0}^{2j}\left[\frac{(2j)!}{l!\,(2j-l)!}\right]^{1/2} \times \tau^{2j-l}\,|2j-l\rangle \otimes |l\rangle. \tag{12}$$

On the other hand, recall the generating function formula of the two-variable Hermite polynomial $H_{m,n}(\epsilon, \epsilon^*)$[23]

$$\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}H_{m,n}(\epsilon, \epsilon^*) = \exp\left(-tt' + t\epsilon + t'\epsilon^*\right), \tag{13}$$

which yields

$$H^*_{m,n}(\epsilon, \epsilon^*) = H_{n,m}(\epsilon, \epsilon^*). \tag{14}$$

Using Eqs. (1) and (13) we can expand the entangled state $\langle\eta|$ as

$$\langle\eta| = \mathrm{e}^{-|\eta|^2/2}\langle m, n|\sum_{m,n=0}^{\infty}\frac{(-1)^n}{\sqrt{m!n!}}H_{n,m}(\eta, \eta^*), \tag{15}$$

where $\langle m, n| = \langle 00|\dfrac{a^m b^n}{\sqrt{m!n!}}$ is the two-mode number state. It then follows that the overlap is

$$\langle\eta\,|m, n\rangle = \mathrm{e}^{-|\eta|^2/2}\frac{(-1)^n}{\sqrt{m!n!}}H_{n,m}(\eta, \eta^*). \tag{16}$$

So we get the overlap of the state $\langle\eta|$ and the SACS $|\tau'\rangle = S_2(r)|\tau\rangle$,

$$\langle\eta|\ \tau'\rangle = \langle\eta|\ S_2\left(r\right)|\tau\rangle = \frac{1}{\mu}\left\langle\eta/\mu\ |\tau\right\rangle$$
$$= \frac{\sqrt{(2j)!}\mathrm{e}^{-|\eta|^2/2\mu^2}}{\mu(1+|\tau|^2)^j}\sum_{l=0}^{2j}\frac{(-1)^l\,\tau^{2j-l}H_{l,2j-l}(\eta/\mu,\eta^*/\mu)}{l!\,(2j-l)!}. \tag{17}$$

Using the integration expression of $H_{m,n}\left(\zeta,\xi\right)$[23]

$$H_{m,n}\left(\zeta,\xi\right) = (-1)^n\,\mathrm{e}^{\zeta\xi}\int\frac{\mathrm{d}^2z}{\pi}z^n z^{*m}\exp\left\{-|z|^2+\zeta z-\xi z^*\right\}, \tag{18}$$

we rewrite Eq. (17)

$$\langle\eta|\ \tau'\rangle = \frac{\sqrt{(2j)!}\mathrm{e}^{|\eta|^2/2\mu^2}}{\mu(1+|\tau|^2)^j}\sum_{l=0}^{2j}(-1)^{2j}\int\frac{\mathrm{d}^2z}{\pi}\frac{(\tau z)^{2j-l}\,z^{*l}}{l!\,(2j-l)!}\exp\left\{-|z|^2+\frac{\eta}{\mu}z-\frac{\eta^*}{\mu}z^*\right\}$$
$$= \frac{\mathrm{e}^{|\eta|^2/2\mu^2}}{\mu(1+|\tau|^2)^j\sqrt{(2j)!}}\int\frac{\mathrm{d}^2z}{\pi}(-1)^{2j}\left(\tau z+z^*\right)^{2j}\exp\left\{-|z|^2+\frac{\eta}{\mu}z-\frac{\eta^*}{\mu}z^*\right\}. \tag{19}$$

For convenience, making the integration variable transform

$$z = z'^*-\tau^*z',\quad \tau z+z^* = (1-|\tau|^2)z',\quad \mathrm{d}^2z = -(1-|\tau|^2)\mathrm{d}^2z' \tag{20}$$

and setting

$$\kappa = \frac{\eta^*+\eta\tau^*}{\mu}, \tag{21}$$

from Eq. (19) we get

$$\langle\eta|\ \tau'\rangle = \frac{(-1)\mathrm{e}^{|\eta|^2/2\mu^2}}{\mu(1+|\tau|^2)^j\sqrt{(2j)!}}(1-|\tau|^2)^{2j+1}\left(\frac{\partial}{\partial\kappa}\right)^{2j}$$
$$\times\int\frac{\mathrm{d}^2z'}{\pi}\exp\left\{-(1+|\tau|^2)\,|z'|^2+\tau^*z'^2+\tau z'^{*2}-\kappa z'+\kappa^*z'^*\right\}. \tag{22}$$

Using the following mathematical integral formula[23]

$$\int\frac{\mathrm{d}^2z}{\pi}\exp(\zeta\,|z|^2+\xi z+\lambda z^*+fz^2+gz^{*2}) = \frac{1}{\sqrt{\zeta^2-4fg}}\exp\left[\frac{-\zeta\xi\lambda+\xi^2g+\lambda^2f}{\zeta^2-4fg}\right], \tag{23}$$

with the condition of convergence $\mathrm{Re}\left(\zeta\pm f\pm g\right)<0$ and $\mathrm{Re}\left((\zeta^2-4fg)/(\zeta\pm f\pm g)\right)<0$, we have

$$\langle\eta|\ \tau'\rangle = \frac{(-1)\mathrm{e}^{|\eta|^2/2\mu^2}(1-|\tau|^2)^{2j}}{\mu(1+|\tau|^2)^j\sqrt{(2j)!}}\left(\frac{\partial}{\partial\kappa}\right)^{2j}\exp\left[\frac{-(1+|\tau|^2)\,|\kappa|^2+\tau\kappa^2+\tau^*\kappa^{*2}}{(1-|\tau|^2)^2}\right]. \tag{24}$$

Then using the Rodrigues formula for the single-variable Hermite polynomial $H_m\left(x\right)$[23]

$$H_m\left(x\right) = \mathrm{e}^{x^2}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\mathrm{e}^{-x^2} \tag{25}$$

and letting

$$\frac{1+|\tau|^2}{1-|\tau|^2}\frac{\kappa^*}{2\sqrt{\tau}}-\frac{\sqrt{\tau}}{1-|\tau|^2}\kappa = \chi = \frac{\eta-\tau\eta^*}{2\mu\sqrt{\tau}}, \tag{26}$$

we derive

$$\langle\eta|\ \tau'\rangle = \frac{(-1)^{j+1}\tau^jH_{2j}\left(\mathrm{i}\chi\right)}{\mu(1+|\tau|^2)^j\sqrt{(2j)!}}\exp\left[\frac{|\eta|^2}{2\mu^2}+\frac{\tau^*\kappa^{*2}}{(1-|\tau|^2)^2}-\left(\frac{1+|\tau|^2}{1-|\tau|^2}\frac{\kappa^*}{2\sqrt{\tau}}\right)^2+\chi^2\right]. \tag{27}$$

Substituting Eq. (26) into Eq. (27) leads to the wave function of the SACS in $\langle\eta|$ representation

$$\langle\eta|\ \tau'\rangle = \frac{(-1)^{j+1}\tau^j\mathrm{e}^{-|\eta|^2/2\mu^2}}{\mu(1+|\tau|^2)^j\sqrt{(2j)!}}H_{2j}\left(\mathrm{i}\frac{\eta-\tau\eta^*}{2\mu\sqrt{\tau}}\right), \tag{28}$$

which is just proportional to a single-variable ordinary Hermite polynomial of order $2j$.

4. Some applications of $\langle\eta|\,\tau'\rangle$

4.1. Wigner function of the SACS and its marginal distributions

As an application of the wave function of the SACS, in the present section we want to derive the Wigner function of the SACS and its marginal distributions. In the entangled state $|\eta\rangle$ representation a two-mode Wigner operator is expressed as[23]

$$\Delta\left(\rho,\gamma\right)=\int\frac{\mathrm{d}^2\eta}{\pi^3}\left|\gamma-\eta\right\rangle\left\langle\gamma+\eta\right|\exp\left(\eta\rho^*-\eta^*\rho\right), \tag{29}$$

where $\gamma=\alpha-\beta^*$, $\rho=\alpha+\beta^*$, $\alpha=(x_1+\mathrm{i}\,p_1)/\sqrt{2}$, $\beta=(x_2+\mathrm{i}\,p_2)/\sqrt{2}$. Using Eqs. (28) and (29), we obtain the Wigner function of the SACS $|\tau'\rangle$

$$\begin{aligned}W(\rho,\gamma)&=\left\langle\tau'\right|\Delta\left(\rho,\gamma\right)\left|\tau'\right\rangle\\&=\frac{(-1)^{2(j+1)}\left|\tau\right|^{2j}}{\mu^2\,(2j)!(1+\left|\tau\right|^2)^{2j}}\int\frac{\mathrm{d}^2\eta}{\pi^3}H_{2j}\left[\mathrm{i}\frac{(\gamma+\eta)-\tau\left(\gamma+\eta\right)^*}{2\mu\sqrt{\tau}}\right]\\&\quad\times H_{2j}^*\left[\mathrm{i}\frac{(\gamma-\eta)-\tau\left(\gamma-\eta\right)^*}{2\mu\sqrt{\tau}}\right]\exp\left(-\frac{\left|\gamma\right|^2}{\mu^2}-\frac{\left|\eta\right|^2}{\mu^2}+\eta\rho^*-\eta^*\rho\right).\end{aligned} \tag{30}$$

Making an integration variable transform

$$\eta'=\frac{\eta-\tau\eta^*}{\mu\sqrt{\tau}},\quad\gamma'=\frac{\gamma-\tau\gamma^*}{\mu\sqrt{\tau}}, \tag{31}$$

we can convert Eq. (30) into

$$\begin{aligned}W(\rho,\gamma)=&\frac{(-1)^{2(j+1)}\left|\tau\right|^{2j}\mathrm{e}^{-\left|\gamma\right|^2/\mu^2}}{(2j)!(1+\left|\tau\right|^2)^{2j}}\frac{\left|\tau\right|}{1-\left|\tau\right|^2}\int\frac{\mathrm{d}^2\eta'}{\pi^3}H_{2j}\left(\mathrm{i}\frac{\eta'+\gamma'}{2}\right)H_{2j}\left(\mathrm{i}\frac{\eta'^*-\gamma'^*}{2}\right)\\&\times\exp\left(-\left|\sqrt{\tau}\frac{\eta'+\left|\tau\right|\eta'^*}{1-\left|\tau\right|^2}\right|^2+\mu\sqrt{\tau}\frac{\eta'+\left|\tau\right|\eta'^*}{1-\left|\tau\right|^2}\rho^*-\mu\sqrt{\tau^*}\frac{\eta'^*+\left|\tau\right|\eta'}{1-\left|\tau\right|^2}\rho\right).\end{aligned} \tag{32}$$

Using the generating function formula of $H_n\left(x\right)$[23]

$$H_n\left(x\right)=\left(\frac{\partial}{\partial t}\right)^n\exp\left(2tx-t^2\right)\big|_{t=0}, \tag{33}$$

we can obtain

$$\begin{aligned}W(\rho,\gamma)=&\frac{(-1)^{2(j+1)}\left|\tau\right|^{2j}\mathrm{e}^{-\left|\gamma\right|^2/\mu^2}}{(2j)!(1+\left|\tau\right|^2)^{2j}}\frac{\left|\tau\right|}{1-\left|\tau\right|^2}\left(\frac{\partial}{\partial t}\right)^{2j}\left(\frac{\partial}{\partial t'}\right)^{2j}\\&\times\int\frac{\mathrm{d}^2\eta'}{\pi^3}\exp\left\{\mathrm{i}t\left(\eta'+\gamma'\right)+\mathrm{i}t'\left(\eta'^*-\gamma'^*\right)-t^2-t'^2\right.\\&\left.-\left|\sqrt{\tau}\frac{\eta'+\left|\tau\right|\eta'^*}{1-\left|\tau\right|^2}\right|^2+\mu\sqrt{\tau}\frac{\eta'+\left|\tau\right|\eta'^*}{1-\left|\tau\right|^2}\rho^*-\mu\sqrt{\tau^*}\frac{\eta'^*+\left|\tau\right|\eta'}{1-\left|\tau\right|^2}\rho\right\}\Big|_{t=t'=0}.\end{aligned} \tag{34}$$

Using Eq. (23) to perform the integration over $\mathrm{d}^2\eta'$ leads to the following result:

$$\begin{aligned}&\int\frac{\mathrm{d}^2\eta'}{\pi^3}\left\{-\frac{(1+\left|\tau\right|^2)\left|\eta'\right|^2+\left|\tau\right|\left(\eta'^2+\eta'^{*2}\right)}{(1-\left|\tau\right|^2)^2}\left|\tau\right|\right.\\&\left.+\eta'\left(\mathrm{i}t+\frac{\mu\sqrt{\tau}\rho^*}{1-\left|\tau\right|^2}-\frac{\mu\sqrt{\tau^*}\left|\tau\right|\rho}{1-\left|\tau\right|^2}\right)+\eta'^*\left(\mathrm{i}t'+\frac{\mu\sqrt{\tau}\left|\tau\right|\rho^*}{1-\left|\tau\right|^2}-\frac{\mu\sqrt{\tau^*}\rho}{1-\left|\tau\right|^2}\right)\right\}\\&=\frac{1-\left|\tau\right|^2}{\pi^2\left|\tau\right|}\exp\left\{t^2+t'^2-\frac{1+\left|\tau\right|^2}{\left|\tau\right|}tt'-wt+w^*t'-\frac{1+\left|\tau\right|^2}{\left|\tau\right|}\left|g\right|^2-g^2-g^{*2}\right\},\end{aligned} \tag{35}$$

where we set

$$w=-\mathrm{i}\mu\frac{\rho+\tau\rho^*}{\sqrt{\tau}},\ g=\frac{\mu\left|\tau\right|}{1-\left|\tau\right|^2}\frac{\tau\rho^*-\rho}{\sqrt{\tau}}. \tag{36}$$

Using Eqs. (13), (34) and (35) we can see that the Wigner function of the SACS is characteristic of the

two-variable Hermite polynomial of order $(2j, 2j)$, i.e.,

$$W(\rho,\gamma)=\frac{(-1)^{2(j+1)}}{\pi^2\,(2j)!}H_{2j,2j}[\mathrm{i}\lambda(\rho'+\gamma'),-\mathrm{i}\lambda(\rho'^*+\gamma'^*)] \times\exp(-\mu^2\left|\rho\right|^2-\left|\gamma\right|^2/\mu^2). \tag{37}$$

where

$$\lambda=\left(\frac{|\tau|}{1+|\tau|^2}\right)^{1/2},\quad \rho'=-\mu\frac{\rho+\tau\rho^*}{\sqrt{\tau}}. \tag{38}$$

Now we want to obtain the marginal distributions of the Wigner function $W(\rho,\gamma)$ in Eq. (37). However, we find that it will be very difficult to evaluate the integration directly owing to the existence of the two-variable Hermite polynomial $H_{2j,2j}$ in Eq. (37). Fortunately, we can use the following integral over $\mathrm{d}^2\rho$ for $\Delta\left(\rho,\gamma\right)$,

$$\int\mathrm{d}^2\rho\Delta\left(\rho,\gamma\right)=\frac{1}{\pi}\left|\eta\right\rangle\left\langle\eta\right|_{\eta=\gamma}, \tag{39}$$

so we obtain a marginal distribution of the Wigner function of $|\tau'\rangle$ in the γ variable

$$\int\mathrm{d}^2\rho W\left(\rho,\gamma\right)=\frac{1}{\pi}\left|\langle\eta|\,\tau'\rangle\right|^2_{\eta=\gamma} =\frac{|\tau|^{2j}\,\mathrm{e}^{-|\gamma|^2/\mu^2}}{\pi\mu^2(1+|\tau|^2)^{2j}\,(2j)!} \times\left|H_{2j}\left(\mathrm{i}\frac{\gamma-\tau\gamma^*}{2\mu\sqrt{\tau}}\right)\right|^2. \tag{40}$$

Similarly, performing the integration of $\Delta\left(\rho,\gamma\right)$ in Eq. (29) over $\mathrm{d}^2\gamma$ leads to another projection operator, i.e.,

$$\int\mathrm{d}^2\gamma\Delta\left(\rho,\gamma\right)=\frac{1}{\pi}\left|\xi\right\rangle\left\langle\xi\right|_{\xi=\rho}, \tag{41}$$

where $|\xi\rangle$ is the common eigenstate of two particles' centre-of-mass coordinate (X_1+X_2) and the relative momentum (P_1-P_2) in two-mode Fock space. The state $\langle\xi|$ is the canonical conjugate state to $\langle\eta|$ and its expression is given by

$$\langle\xi|=\langle00|\exp[-\left|\xi\right|^2/2+\xi a+\xi^*b-ab], \quad\xi=\xi_1+\mathrm{i}\xi_2, \tag{42}$$

which is also capable of making up the entangled state representation. Also we prove that the operator $S_2(r)$ is converted into

$$S_2(r)=\mu\int\frac{\mathrm{d}^2\xi}{\pi}\left|\xi\right\rangle\left\langle\mu\xi\right|, \tag{43}$$

which satisfies the following relation

$$S_2(r)\left|\xi\right\rangle=\frac{1}{\mu}\left|\xi/\mu\right\rangle. \tag{44}$$

Then using Eq. (43) and the expansion of the entangled state $\langle\xi|$ in the Fock space we obtain

$$\int\mathrm{d}^2\gamma W\left(\rho,\gamma\right)=\frac{1}{\pi}\left|\langle\xi|\,\tau'\rangle\right|^2_{\xi=\rho} =\frac{\mu^2\left|\tau\right|^{2j}\mathrm{e}^{-\mu^2|\rho|^2}}{\pi(1+|\tau|^2)^{2j}\,(2j)!} \times\left|H_{2j}\left(\mu\frac{\rho^*+\tau\rho}{2\sqrt{\tau}}\right)\right|^2, \tag{45}$$

which is another marginal distribution of the Wigner function of the SACS in the ρ variable. Equation (40) (or Eq. (45)) is proportional to the probability for finding the two particles, which have total momentum $\sqrt{2}\gamma_2$ (or relative momentum $\sqrt{2}\rho_2$) and simultaneously relative position $\sqrt{2}\gamma_1$ (or centre-of-mass position $\sqrt{2}\rho_1$), under the SACS in an entangled way in the $\rho-\gamma$ phase space.

4.2. Eigenproblems for some Hamiltonians of the generalized angular momentum system

As another application of the wave function of the SACS, we consider how to solve the eigenproblems for some Hamiltonians of the generalized angular momentum system. Now we first construct a set of new Bose operator realization of angular momentum, i.e.,

$$J'_+=c^\dagger d,\quad J'_-=cd^\dagger,\quad J'_z=\frac{1}{2}\left(c^\dagger c-d^\dagger d\right), \tag{46}$$

where

$$c=a\cosh r+b^\dagger\sinh r,\quad d=b\cosh r+a^\dagger\sinh r, \quad\left[a,a^\dagger\right]=\left[b,b^\dagger\right]=1. \tag{47}$$

Using $\left[J'_+,J'_-\right]=2J'_z$, $\left[J'_\pm,J'_z\right]=\mp J'_\pm$, we can prove that the state $|\tau'\rangle$ obeys the following eigenvector equations

$$\begin{aligned}&\left(J'_-+\tau^2J'_+\right)\left|\tau'\right\rangle=2j\tau\left|\tau'\right\rangle,\\ &\left(J'_-+\tau J'_z\right)\left|\tau'\right\rangle=j\tau\left|\tau'\right\rangle,\\ &\left(\tau J'_+-J'_z\right)\left|\tau'\right\rangle=j\left|\tau'\right\rangle.\end{aligned} \tag{48}$$

Considering the Hermite Hamiltonian of the generalized angular momentum system

$$H=AJ'_-+A^*J'_+BJ'_z, \tag{49}$$

and supposing that the state $|\tau'\rangle$ is an eigenstate of H with energy eigenvalue E, i.e.,

$$\left\langle\eta\right|H\left|\tau'\right\rangle=E\left\langle\eta\right|\left.\tau'\right\rangle, \tag{50}$$

then we want to obtain the energy E and the value τ. In terms of Eqs. (4), (5) and (46)–(50) we have

$$\begin{aligned}\langle\eta|\, H\,|\tau'\rangle &= \langle\eta|\left[A\left(2j\tau-\tau^2J'_+\right)+A^*J'_+ + B\left(\tau J'_+ - j\right)\right]|\tau'\rangle \\ &= \left(2j\tau A - jB\right)\langle\eta\,|\tau'\rangle + \left(A^*-\tau^2A+\tau B\right)\langle\eta|\, J'_+\,|\tau'\rangle \\ &= \left\{\left(2j\tau A - jB\right) - \left(A^*-\tau^2A+\tau B\right)\frac{\eta^{*2}}{4}\left(\cosh r-\sinh r\right)^2\right. \\ &\quad \left. + \left(A^*-\tau^2A+\tau B\right)\left(\cosh r+\sinh r\right)^2\frac{\partial^2}{\partial\eta^2}\right\}\langle\eta\,|\tau'\rangle\,. \end{aligned} \tag{51}$$

Using Eqs. (26) and (51) we get

$$\frac{\partial}{\partial\eta} = \frac{2\mu}{\sqrt{\tau}}\frac{\mathrm{d}}{\mathrm{d}\chi} \tag{52}$$

and

$$\begin{aligned} E\,\langle\eta\,|\tau'\rangle &= \left\{\left(2j\tau A - jB\right) - \left(A^*-\tau^2A+\tau B\right)\frac{\eta^{*2}}{4}\left(\cosh r-\sinh r\right)^2\right. \\ &\quad \left. + \left(A^*-\tau^2A+\tau B\right)\left(\cosh r+\sinh r\right)^2\frac{4\mu^2}{\tau}\frac{\mathrm{d}^2}{\mathrm{d}\chi^2}\right\}\langle\eta\,|\tau'\rangle\,. \end{aligned} \tag{53}$$

Since Eq. (28) and the Hermite polynomials of different orders are mutually orthogonal, we see that if and only if

$$A^*-\tau^2A+\tau B = 0, \tag{54}$$

then Eq. (53) can be true, thus we get two different values of τ,

$$\tau_\pm = \frac{B\pm\sqrt{B^2+4\left|A\right|^2}}{2A}. \tag{55}$$

Substituting Eq. (55) into Eq. (53), we can get two different energy values

$$E_\pm = 2j\tau_\pm A - jB = \pm j\sqrt{B^2+4\left|A\right|^2}. \tag{56}$$

So the Hamiltonian possesses two eigenstates $\left|\tau'_+\right\rangle$ and $\left|\tau'_-\right\rangle$ with the eigenvalues E_+ and E_-, respectively.

In summary, in this paper we have obtained the wave function of the SACS in the entangled state $\langle\eta|$ representation, which is just proportional to a single-variable ordinary Hermite polynomial of order $2j$. Using the wave function of the SACS we have also obtained the Wigner function of the SACS and its marginal distributions, as well as eigenvalues and eigenstates for some Hamiltonians of the generalized angular momentum system. We hope that the results in this paper will enrich the theory about the SACS in quantum optics. Also we expect that the results can be widely applied in quantum optics and quantum information.

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