Novel uncertainty relations associated with fractional Fourier transform*

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In this paper the relations between two spreads, between two group delays, and between one spread and one group delay in fractional Fourier transform (FRFT) domains, are presented and three theorems on the uncertainty principle in FRFT domains are also developed. Theorem 1 gives the bounds of two spreads in two FRFT domains. Theorem 2 shows the uncertainty relation between two group delays in two FRFT domains. Theorem 3 presents the crossed uncertainty relation between one group delay and one spread in two FRFT domains. The novelty of their results lies in connecting the products of different physical measures and giving their physical interpretations. The existing uncertainty principle in the FRFT domain is only a special case of theorem 1, and the conventional uncertainty principle in time-frequency domains is a special case of their results. Therefore, three theorems develop the relations of two spreads in time-frequency domains into the relations between two spreads, between two group delays, and between one spread and one group delay in FRFT domains.

Keywords: fractional Fourier transform (FRFT), uncertainty principle, time-frequency spreads, group delay

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1. Introduction

The uncertainty principle is an elementary principle in signal processing[1−14] and physics.[15−20] In conventional time-frequency domains, the product of time spread and frequency spread cannot be unlimitedly small, which is called the uncertainty principle, i.e. for a given real absolutely integrable signal $f(t)$ (without loss of generalization, assuming that $\int_{-\infty}^{+\infty}|f(t)|^2\,\mathrm{d}t = 1$ and $f(t)$ is a real absolutely integrable function unless otherwise stated) and its Fourier transform $F(u)$, the product has the lowest bound

$$\begin{aligned}\Delta t^2\cdot\Delta u_{\mathrm{cl}}^2 &= \int_{-\infty}^{+\infty}|(t-t_0)\,f(t)|^2\,\mathrm{d}t\\ &\quad\times\int_{-\infty}^{+\infty}|(u-u_0)\,F(u)|^2\,\mathrm{d}u\\ &\geq 1/4,\end{aligned}\tag{1}$$

where $t_0=\int_{-\infty}^{+\infty}t\,|f(t)|^2\,\mathrm{d}t$, $u_0=\int_{-\infty}^{+\infty}u\,|F(u)|^2\,\mathrm{d}u$, Δt^2 is the time spread, and Δu_{cl}^2 is the frequency spread. Let $t_0=0$ (for a given $f(t)$ we assume $t_0\equiv 0$) and $u_0=0$, and the essence of the uncertainty principle will not be changed. Inequality (1) can be written as

$$\begin{aligned}&\Delta t^2\cdot\Delta u_{\mathrm{cl}}^2\\ &=\int_{-\infty}^{+\infty}|tf(t)|^2\,\mathrm{d}t\cdot\int_{-\infty}^{+\infty}|uF(u)|^2\,\mathrm{d}u\\ &\geq 1/4.\end{aligned}\tag{2}$$

For inequalities (1) and (2) one often acquires higher frequency resolution from lower time resolution or vice versa.[1,2] Fractional Fourier transform (FRFT) is a more generalized time-frequency tool,[2,10,11] which is the generalization of Fourier transform etc. The FRFT has been applied to physical optics and signal processing. With the development of time-frequency analysis, the uncertainty principle has been proved to be true by using varieties of transforms: from conventional Fourier transform and short-time Fourier transform to generalized Fourier transform[2−9] (fractional Fourier transform). However, in existing research,[8,9,12−14] only spreads of time and frequency are involved. Since the FRFT has one additional parameter, but the conventional Fourier transform does not, then do other new uncertainty relations hold in the new domain—the FRFT

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domain? In the present paper, the uncertainty principle is extended from the time-frequency domain to the FRFT domain, and new uncertainty relations in the FRFT domain are presented. The rest of the present paper is organized as follows. The basic conception of FRFT and its properties are reviewed in Section 2. Three results of the uncertainty principle in the FRFT domain are presented in Section 3. An example of their results is given in Section 4. Finally, some conclusions are drawn from the present study in Section 5.

2. Fractional Fourier transform

For a given function $f(t)$, its FRFT[2] is defined as

$$F_\alpha(u)=F_\alpha\left(f(t)\right)=\int_{-\infty}^{+\infty}f(t)K_\alpha(u,t)\mathrm{d}t=\begin{cases}\sqrt{\frac{1-\mathrm{i}\cot\alpha}{2\pi}}\cdot\mathrm{e}^{\frac{\mathrm{i}u^2\cot\alpha}{2}}\int_{-\infty}^{+\infty}\mathrm{e}^{\frac{-\mathrm{i}ut}{\sin\alpha}}\mathrm{e}^{\frac{\mathrm{i}t^2\cot\alpha}{2}}f(t)\mathrm{d}t, & \alpha\neq n\pi\\ f(t), & \alpha=2n\pi\\ f(-t), & \alpha=(2n\pm1)\pi.\end{cases}\tag{3}$$

In the following, we assume that $\alpha\neq n\pi$. FRFT has some properties:[1,2,14]

1) Additivity property:

$$F_\alpha\left\{F_\beta\left(f(t)\right)\right\}=F_{(\alpha+\beta)}\left(f(t)\right),\tag{4}$$

2) Reversibility property:

$$F_\alpha\left\{F_{-\alpha}\left(f(t)\right)\right\}=f(t),\tag{5}$$

3)Product property:

$$F_\alpha\left\{tf(t)\right\}=u\cos\alpha\cdot F_\alpha(u)+\mathrm{i}\sin\alpha F'_\alpha(u),\tag{6}$$

4)Differential property:

$$F_\alpha\left\{f'(t)\right\}=\cos\alpha F'_\alpha(u)+\mathrm{i}u\sin\alpha F_\alpha(u),\tag{7}$$

5) Parseval's principle:

$$\int_{-\infty}^{+\infty}f(t)g^*(t)\mathrm{d}t=\int_{-\infty}^{+\infty}F_{(a,b,c,d)}(u)G^*_{(a,b,c,d)}(u)\mathrm{d}u.\tag{8}$$

Since $f(t)$ is real absolutely integrable, i.e. $\int_{-\infty}^{+\infty}|f(t)|\,\mathrm{d}t<+\infty$, then $\lim_{t\to\pm\infty}f(t)=0$, $\lim_{t\to\pm\infty}|f(t)|^2=0$, $\lim_{t\to\pm\infty}t\,|f(t)|^2=0$. We obtain

$$\int_{-\infty}^{+\infty}tf(t)f'(t)\mathrm{d}t=-1/2,\tag{9}$$

$$\int_{-\infty}^{+\infty}f(t)f'(t)\mathrm{d}t=0.\tag{10}$$

3. Uncertainty principle in the FRFT domain

In the time and frequency domain, $\int_{-\infty}^{+\infty}|tf(t)|^2\mathrm{d}t=\int_{-\infty}^{+\infty}|F'(u)|^2\mathrm{d}u$ and $\int_{-\infty}^{+\infty}|uF(u)|^2\mathrm{d}u=\int_{-\infty}^{+\infty}|f'(t)|^2\mathrm{d}t$ hold, and the conventional uncertainty principle (2) can be written as:

$$\begin{aligned}&\int_{-\infty}^{+\infty}|tf(t)|^2\mathrm{d}t\cdot\int_{-\infty}^{+\infty}|uF(u)|^2\mathrm{d}u\\ =&\int_{-\infty}^{+\infty}|F'(u)|^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty}|f'(t)|^2\mathrm{d}t\\ =&\int_{-\infty}^{+\infty}|F'(u)|^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty}|uF(u)|^2\mathrm{d}u\\ \geq&\ 1/4.\end{aligned}\tag{11}$$

The obvious observation is that the uncertainty principle has different representations, i.e. inequality (11), but in the conventional time-frequency domain these different representations are equivalent. However, in the FRFT domain we have found that different representations have different bounds and physical meanings.

Before discussing the following three theorems, we introduce a lemma as:

Lemma 1 For any given $f(t)$, if $F_\alpha(u)$ and $F_\beta(v)$ are two FRFTs of $f(t)$ respectively for different parameters α and β, and $\int_{-\infty}^{+\infty}u\,|F_\alpha(u)|^2\mathrm{d}u\equiv0$, then $\int_{-\infty}^{+\infty}v\,|F_\beta(v)|^2\mathrm{d}v\equiv0$.

Proof From Parseval's principle (8) and product

property (6), we obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty} u\left|F_\alpha(u)\right|^2 \mathrm{d}u \\
&= \int_{-\infty}^{+\infty} uF_\alpha(u)\cdot\left(F_\alpha(u)\right)^* \mathrm{d}u \\
&= \int_{-\infty}^{+\infty} \left(\cos\alpha\cdot tf(t)+\mathrm{i}\sin\alpha f'(t)\right) f^*(t)\mathrm{d}t \\
&= \cos\alpha\int_{-\infty}^{+\infty} tf^2(t)\mathrm{d}t+\mathrm{i}\sin\alpha\int_{-\infty}^{+\infty} f(t)f'(t)\mathrm{d}t \\
&= \cos\alpha\cdot t_0 \\
&\equiv 0. \qquad (12)
\end{aligned}$$

Therefore, we have

$$\int_{-\infty}^{+\infty} v\left|F_\beta(v)\right|^2\mathrm{d}v=\cos\beta\cdot t_0\equiv 0. \qquad (13)$$

The proof is completed.

Lemma 1 shows that if the mean value of a variable in an FRFT domain is zero, then the mean value of the valuable in any other FRFT domain must be zero. So in this paper we can assume that the mean value of a valuable in an FRFT domain is zero. Thus the effect of time shifting on our results will not be involved in this paper.

3.1. Uncertainty relation for two spreads

Theorem 1 For a given function $f(t)$, if $F_\alpha(u)$ and $F_\beta(v)$ are two FRFTs of $f(t)$ respectively for different parameters α and β, then

$$\begin{aligned}
&\int_{-\infty}^{+\infty} u^2\left|F_\alpha(u)\right|^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty} v^2\left|F_\beta(v)\right|^2\mathrm{d}v \\
&\geq \frac{\left(\cos\alpha\sin\beta-\cos\beta\sin\alpha\right)^2}{4} \\
&\quad+\left(\cos\alpha\cos\beta\Delta t^2+\frac{\sin\alpha\sin\beta}{4\Delta t^2}\right)^2. \qquad (14)
\end{aligned}$$

Furthermore

$$\begin{aligned}
&\int_{-\infty}^{+\infty} u^2\left|F_\alpha(u)\right|^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty} v^2\left|F_\beta(v)\right|^2\mathrm{d}v \\
&\geq \frac{\left(\cos\alpha\sin\beta+\mathrm{sgn}(W)\cos\beta\sin\alpha\right)^2}{4}, \qquad (15)
\end{aligned}$$

where $\mathrm{sgn}(s)=\begin{cases}+1, & s\geq 0,\\ -1, & s<0,\end{cases}$, $W=\sin(2\alpha)\cdot\sin(2\beta)/4$.

Proof From product property (6) and Parseval's principle (8), we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} u^2\left|F_\alpha(u)\right|^2\mathrm{d}u \\
&=\int_{-\infty}^{+\infty}\left|\cos\alpha\cdot tf(t)+\mathrm{i}\sin\alpha\cdot f'(t)\right|^2\mathrm{d}t, \qquad (16)\\
&\int_{-\infty}^{+\infty} v^2\left|F_\beta(v)\right|^2\mathrm{d}v \\
&=\int_{-\infty}^{+\infty}\left|\cos\beta\cdot tf(t)+\mathrm{i}\sin\beta\cdot f'(t)\right|^2\mathrm{d}t. \qquad (17)
\end{aligned}$$

Using the Cauchy–Schwartz inequality, (16) and (17), we obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty} u^2\left|F_\alpha(u)\right|^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty} v^2\left|F_\beta(v)\right|^2\mathrm{d}v \\
&\geq\left|\int_{-\infty}^{+\infty}\left(\cos\alpha\cdot t\cdot f(t)+\mathrm{i}\cdot\sin\alpha\cdot\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)\left(\cos\beta\cdot t\cdot f(t)+\mathrm{i}\cdot\sin\beta\cdot\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)^*\mathrm{d}t\right|^2. \qquad (18)
\end{aligned}$$

Now for real $f(t)$

$$\begin{aligned}
&\left|\int_{-\infty}^{+\infty}\left(\cos\alpha\cdot t\cdot f(t)+\mathrm{i}\cdot\sin\alpha\cdot\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)\left(\cos\beta\cdot t\cdot f(t)+\mathrm{i}\cdot\sin\beta\cdot\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)^*\mathrm{d}t\right|^2 \\
&=\left|\int_{-\infty}^{+\infty}\left(\cos\alpha\cos\beta\cdot t^2f^2(t)+\sin\alpha\sin\beta\left(f'(t)\right)^2+\mathrm{i}\left(\cos\alpha\sin\beta-\cos\beta\sin\alpha\right)tf(t)f'(t)\right)\mathrm{d}t\right|^2. \qquad (19)
\end{aligned}$$

In a conventional Fourier transform domain, the following equality holds:

$$\int_{-\infty}^{+\infty}\left|f'(t)\right|^2\mathrm{d}t=\Delta u_{\mathrm{cl}}^2. \qquad (20)$$

From expressions (18), (19), (20), (9) and (10), we obtain

$$\int_{-\infty}^{+\infty} u^2 \left|F_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} v^2 \left|F_\beta(v)\right|^2 \mathrm{d}v \geq \frac{\left(\cos\alpha\sin\beta - \cos\beta\sin\alpha\right)^2}{4} + \left(\cos\alpha\cos\beta\Delta t^2 + \Delta u_{\mathrm{cl}}^2 \sin\alpha\sin\beta\right)^2. \tag{21}$$

Furthermore, inserting inequality (2) into inequality (21) yields

$$\int_{-\infty}^{+\infty} u^2 \left|F_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} v^2 \left|F_\beta(v)\right|^2 \mathrm{d}v \geq \frac{\left(\cos\alpha\sin\beta - \cos\beta\sin\alpha\right)^2}{4} + \left(\cos\alpha\cos\beta\Delta t^2 + \frac{\sin\alpha\sin\beta}{4\Delta t^2}\right)^2. \tag{22}$$

From the left-hand side of inequality (22), we have the following relations:

If $\mathrm{sgn}(\cos\alpha\cos\beta) = \mathrm{sgn}(\sin\alpha\sin\beta)$, then

$$\left(\cos\alpha\cos\beta\Delta t^2 + \frac{\sin\alpha\sin\beta}{4\Delta t^2}\right)^2 \geq \left(2\sqrt{\cos\alpha\cos\beta\Delta t^2 \cdot \frac{\sin\alpha\sin\beta}{4\Delta t^2}}\right)^2 = \cos\alpha\cos\beta\sin\alpha\sin\beta; \tag{23}$$

If $\mathrm{sgn}(\cos\alpha\cos\beta) = -\mathrm{sgn}(\sin\alpha\sin\beta)$, then

$$\left(\cos\alpha\cos\beta\Delta t^2 + \frac{\sin\alpha\sin\beta}{4\Delta t^2}\right)^2 \geq 0. \tag{24}$$

Thus, combining inequality (22) with inequalities (23) and (24) leads to

$$\int_{-\infty}^{+\infty} u^2 \left|F_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} v^2 \left|F_\beta(v)\right|^2 \mathrm{d}v \geq \frac{\left(\cos\alpha\sin\beta + \mathrm{sgn}(W)\cos\beta\sin\alpha\right)^2}{4}, \tag{25}$$

where $W = \sin(2\alpha)\cdot\sin(2\beta)/4$.

The proof of theorem 1 is completed.

Theorem 1 shows that for a given function $f(t)$, the product of its two spreads in any two different FRFT domains cannot be unlimitedly small, and its lowest bound is associated with parameters α and β. If their mean values are not zero, through appropriate shifting we can make them equal zero like conventional counterparts.[8,12] In the following, we will study extended relations for the given $f(t)$ in different domains.

3.2. Uncertainty relation for two group delays

Theorem 2 For a given function $f(t)$, if $F_\alpha(u)$ and $F_\beta(v)$ are two FRFTs of $f(t)$ respectively for different parameters α and β, then

$$\int_{-\infty}^{+\infty} \left|F'_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \geq \frac{\left(\sin\alpha\cos\beta - \cos\alpha\sin\beta\right)^2}{4} + \left(\sin\alpha\sin\beta\Delta t^2 + \frac{\cos\alpha\cos\beta}{4\Delta t^2}\right)^2. \tag{26}$$

Furthermore

$$\int_{-\infty}^{+\infty} \left|F'_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \geq \frac{\left(\sin\alpha\cos\beta + \mathrm{sgn}(W)\cos\alpha\sin\beta\right)^2}{4}, \tag{27}$$

where $W = \sin(2\alpha)\cdot\sin(2\beta)/4$.

Proof From the differential property (7) and the generalized Parseval's principle (10), we have

$$\int_{-\infty}^{+\infty} \left|F'_\alpha(u)\right|^2 \mathrm{d}u = \int_{-\infty}^{+\infty} \left|\cos\alpha\cdot f'(t) + \mathrm{i}\sin\alpha\cdot t f(t)\right|^2 \mathrm{d}t, \tag{28}$$

$$\int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v = \int_{-\infty}^{+\infty} \left|\cos\beta\cdot f'(t) + \mathrm{i}\sin\beta\cdot t f(t)\right|^2 \mathrm{d}t. \tag{29}$$

From the Cauchy–Schwartz inequality, then we have

$$\int_{-\infty}^{+\infty} \left|F'_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v$$

$$\geq \left| \int_{-\infty}^{+\infty} (\cos\alpha \cdot f'(t) + \mathrm{i}\sin\alpha \cdot tf(t)) (\cos\beta \cdot f'(t) + \mathrm{i}\sin\beta \cdot tf(t))^* \, \mathrm{d}t \right|^2. \tag{30}$$

Now for a real function $f(t)$, from expressions (2), (9), (10), (20) and (30) we have

$$\int_{-\infty}^{+\infty} |F'_\alpha(u)|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \geq \frac{(\sin\alpha\cos\beta - \cos\alpha\sin\beta)^2}{4} + \left(\sin\alpha\sin\beta\Delta t^2 + \frac{\cos\alpha\cos\beta}{4\Delta t^2}\right)^2. \tag{31}$$

From the left side of inequality (31), we have the following relations:
If $\mathrm{sgn}(\cos\alpha\cos\beta) = \mathrm{sgn}(\sin\alpha\sin\beta)$, then

$$\left(\sin\alpha\sin\beta\Delta t^2 + \frac{\cos\alpha\cos\beta}{4\Delta t^2}\right)^2 \geq \left(2\sqrt{\sin\alpha\sin\beta\Delta t^2 \cdot \frac{\cos\alpha\cos\beta}{4\Delta t^2}}\right)^2 = \sin\alpha\sin\beta\cos\alpha\cos\beta; \tag{32}$$

If $\mathrm{sgn}(\cos\alpha\cos\beta) = -\mathrm{sgn}(\sin\alpha\sin\beta)$, then

$$\left(\sin\alpha\sin\beta\Delta t^2 + \frac{\cos\alpha\cos\beta}{4\Delta t^2}\right)^2 \geq 0. \tag{33}$$

Therefore

$$\int_{-\infty}^{+\infty} |F'_\alpha(u)|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \geq \frac{(\sin\alpha\cos\beta + \mathrm{sgn}(W)\cos\alpha\sin\beta)^2}{4}. \tag{34}$$

Then the proof of theorem 2 is completed.

Theorem 2 shows that for a given function $f(t)$, the product of its two 'differential energies' in any two different domains cannot be unlimitedly small.

Furthermore, set $F_\alpha(u) = A\,\mathrm{e}^{\mathrm{i}\phi(u)}$, $F_\beta(v) = B\,\mathrm{e}^{\mathrm{i}\varphi(v)}$, where A and B are constants. From inequality (27), we have

$$\int_{-\infty}^{+\infty} |\phi'(u)|^2 |F_\alpha(u)|^2 \, \mathrm{d}u \cdot \int_{-\infty}^{+\infty} |\varphi'(v)|^2 |F_\beta(v)|^2 \mathrm{d}v \geq \frac{(\sin\alpha\cos\beta + \mathrm{sgn}(W)\cos\alpha\sin\beta)^2}{4}, \tag{35}$$

where $\phi'(u)$ and $\varphi'(v)$ are generalized group delays in FRFT domains. From inequality (35) we know that the product of group delay covariances (without loss of generalization, assuming that the mean value of group delay is zero) cannot be unlimitedly small.

If $\phi'(u) \equiv \tau_1$, $\varphi'(v) \equiv \tau_2$, i.e. they are constants simultaneously, then inequality (35) reduces into

$$\int_{-\infty}^{+\infty} |\phi'(u)|^2 |F_\alpha(u)|^2 \, \mathrm{d}u \cdot \int_{-\infty}^{+\infty} |\varphi'(v)|^2 |F_\beta(v)|^2 \mathrm{d}v = (\tau_1\tau_2)^2 \geq \frac{(\sin\alpha\cos\beta + \mathrm{sgn}(W)\cos\alpha\sin\beta)^2}{4}. \tag{36}$$

That is to say, the product of group delays in two FRFT domains for different parameters cannot be unlimitedly small, and the smallest product is

$$\tau_1\tau_2 \geq \frac{|\sin\alpha\cos\beta + \mathrm{sgn}(W)\cos\alpha\sin\beta|}{2}. \tag{37}$$

In fact, inequality (37) represents an ideal case, and it discloses the physical meaning to a great extent.

From inequalities (35)–(37), theorem 2 discloses the mutual relations of group delays in two FRFT domains for different parameters.

The above two cases are concerned with the relations between one spread and one spread, and between one group delay and one group delay. What about the crossed relation between one spread and one group delay?

3.3. Uncertainty relation for spread and group delay

Theorem 3 For a given function $f(t)$, if $F_\alpha(u)$ and $F_\beta(v)$ are two FRFTs of $f(t)$ respectively for different parameters α and β, then

$$\int_{-\infty}^{+\infty} |uF_\alpha(u)|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 dv \geq \frac{(\cos\alpha\cos\beta - \sin\alpha\sin\beta)^2}{4} + \left(\cos\alpha\sin\beta\Delta t^2 + \frac{\sin\alpha\cos\beta}{4\Delta t^2}\right)^2. \tag{38}$$

Furthermore

$$\int_{-\infty}^{+\infty} |uF_\alpha(u)|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \geq \frac{(\cos\alpha\cos\beta + \mathrm{sgn}(W)\sin\alpha\sin\beta)^2}{4}, \tag{39}$$

where $W = \sin(2\alpha)\cdot\sin(2\beta)/4$.

Proof From the product property (6), the differential property (7) and the generalized Parseval's principle (8), we have

$$\int_{-\infty}^{+\infty} u^2 |F_\alpha(u)| \,\mathrm{d}u = \int_{-\infty}^{+\infty} |\cos\alpha\cdot tf(t) + \mathrm{i}\sin\alpha f'(t)|^2 \mathrm{d}t, \tag{40}$$

$$\int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v = \int_{-\infty}^{+\infty} |\cos\beta\cdot f'(t) + \mathrm{i}\sin\beta\cdot tf(t)|^2 \mathrm{d}t. \tag{41}$$

From the Cauchy–Schwartz inequality and expressions (40) and (41), we have

$$\begin{aligned}&\int_{-\infty}^{+\infty} |uF_\alpha(u)|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \\ &\geq \left|\int_{-\infty}^{+\infty} (\cos\alpha\cdot tf(t) + \mathrm{i}\sin\alpha f'(t))\,(\cos\beta f'(t) + \mathrm{i}\sin\beta\cdot tf(t))^*\,\mathrm{d}t\right|^2.\end{aligned} \tag{42}$$

Now for a real function $f(t)$, from inequalities (2), (11), (24) and (42) we have

$$\int_{-\infty}^{+\infty} |uF_\alpha(u)|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \geq \frac{(\cos\alpha\cos\beta - \sin\alpha\sin\beta)^2}{4} + \left(\cos\alpha\sin\beta\Delta t^2 + \frac{\sin\alpha\cos\beta}{4\Delta t^2}\right)^2. \tag{43}$$

From the left side of inequality (43), we have the following relations:
If $\mathrm{sgn}(\cos\alpha\sin\beta) = \mathrm{sgn}(\sin\alpha\cos\beta)$, then

$$\left(\cos\alpha\sin\beta\Delta t^2 + \frac{\sin\alpha\cos\beta}{4\Delta t^2}\right)^2 \geq \left(2\sqrt{\cos\alpha\sin\beta\Delta t^2\cdot\frac{\sin\alpha\cos\beta}{4\Delta t^2}}\right)^2 = \cos\alpha\sin\beta\sin\alpha\cos\beta; \tag{44}$$

If $\mathrm{sgn}(\cos\alpha\sin\beta) = -\mathrm{sgn}(\sin\alpha\cos\beta)$, then

$$\left(\cos\alpha\sin\beta\Delta t^2 + \frac{\sin\alpha\cos\beta}{4\Delta t^2}\right)^2 \geq 0. \tag{45}$$

We have

$$\int_{-\infty}^{+\infty} |uF_\alpha(u)|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \geq \frac{(\cos\alpha\cos\beta + \mathrm{sgn}(W)\sin\alpha\sin\beta)^2}{4}. \tag{46}$$

The proof of theorem 3 is completed.

Theorem 3 shows that the product of one 'differential energy' and spread in two different FRFT domains cannot be unlimitedly small. To some extent we can say that theorem 3 has the mixed physical meaning of theorem 1 and theorem 2. Theorem 1 shows the relation between spreads of two variables in two FRFT domains, and theorem 2 indicates the relation between two group delay covariances in two FRFT domains. Instead, theorem 3 reveals the crossed relation between one group delay covariance and one variable's spread.

One special case is that $\alpha = 2n\pi + \beta$, $F_\alpha(u) = A\,\mathrm{e}^{\mathrm{i}\phi(u)}$, and A is a constant, then inequality (46) becomes

$$\int_{-\infty}^{+\infty} |uF_\alpha(u)|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} |F'_\alpha(u)|^2 \mathrm{d}u \geq \frac{1}{4}. \tag{47}$$

Inequality (47) shows that the group delay and the spread have a crossed relation in an FRFT domain, and they have the lowest bound 1/4.

The conditions under which equalities in three theorems hold can be found easily through solving some differential equations, therefore they will not be involved in this paper.

3.4. Special cases

If $\alpha = 2n\pi$ and $\beta = 2n\pi + \pi/2$, three theorems can reduce into the conventional uncertainty principle (2) in a time-frequency domain.

Theorem 1 reduces into

$$\begin{aligned}&\int_{-\infty}^{+\infty} u^2 \left|F_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} v^2 \left|F_\beta(v)\right|^2 \mathrm{d}v \\ &= \int_{-\infty}^{+\infty} t^2 \left|f(t)\right|^2 \mathrm{d}t \cdot \int_{-\infty}^{+\infty} v^2 \left|F(v)\right|^2 \mathrm{d}v \\ &\geq \frac{1}{4}. \end{aligned} \tag{48}$$

Theorem 2 reduces into

$$\begin{aligned}&\int_{-\infty}^{+\infty} \left|F'_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \\ &= \int_{-\infty}^{+\infty} \left|f'(t)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'(v)\right|^2 \mathrm{d}v \\ &\geq \frac{1}{4}. \end{aligned} \tag{49}$$

Theorem 3 reduces into

$$\begin{aligned}&\int_{-\infty}^{+\infty} \left|uF_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \\ &= \int_{-\infty}^{+\infty} \left|tf(t)\right|^2 \mathrm{d}t \cdot \int_{-\infty}^{+\infty} \left|F'(v)\right|^2 \mathrm{d}v \\ &\geq \frac{1}{4}. \end{aligned} \tag{50}$$

Thus any of the three theorems is the generalization of the conventional uncertainty principle.

In FRFT domains the uncertainty principle has two types: the product of time spread and the spread in the FRFT domain, and the product of spreads in two FRFT domains for different rotation parameters.[1,2] The former is a special case of the latter.

Obviously, we have found that the uncertainty principle by Shinde is a special case of our theorem 1. Therefore, the cases[9,13,14] involved with the product of the time spread and the spread in the FRFT domain are our special case of theorem 1. Furthermore, our theorem 3 can reduce into cases in Refs. [9] and [14]:

$$\begin{aligned}&\int_{-\infty}^{+\infty} \left|uF_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_{\pi/2}(v)\right|^2 \mathrm{d}v \\ &= \int_{-\infty}^{+\infty} \left|uF_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|tf(t)\right|^2 \mathrm{d}t \\ &\geq \frac{\sin^2\alpha}{4}. \end{aligned} \tag{51}$$

As shown above, any of the three theorems is the generalization of the conventional uncertainty principle in time-frequency domains, and they are different in both bound and physical meaning. These theorems develop conventional relations between two spreads in time-frequency domains into the relations between different spreads, between different group delays or between the crossed relation of spread and group delay in the FRFT domain.

4. An experiment

Let $f(t) = H_n(t)\,\mathrm{e}^{-t^2/2}$, and the fractional Fourier transform $F_\alpha(u)$ for $f(t)$ can be easily found:[2,4] $F_\alpha(u) = \mathrm{e}^{-jn\alpha} H_n(t)\,\mathrm{e}^{-u^2/2}$, where $H_n(t)$ is the Hermite polynomial[2,4,5] of degree n. For simplicity, let $n = 1$, $\alpha = \pi/4$, and $\beta = 3\pi/4$, we have

$$\Delta t^2 = \int_{-\infty}^{+\infty} \left|tf(t)\right|^2 \mathrm{d}t = \int_{-\infty}^{+\infty} t^2 \left| \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{1}{2}} t\,\mathrm{e}^{-\frac{t^2}{2}} \right|^2 \mathrm{d}t = \frac{3}{2}, \tag{52}$$

$$\Delta u_\alpha^2 = \int_{-\infty}^{+\infty} \left|uF_\alpha(u)\right|^2 \mathrm{d}u = \int_{-\infty}^{+\infty} u^2 \left| \mathrm{e}^{-j\alpha} \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{1}{2}} t\,\mathrm{e}^{-\frac{u^2}{2}} \right|^2 \mathrm{d}u = \frac{3}{2}, \tag{53}$$

$$\Delta u_\beta^2 = \int_{-\infty}^{+\infty} \left|vF_\beta(v)\right|^2 \mathrm{d}v = \int_{-\infty}^{+\infty} v^2 \left| \mathrm{e}^{-j\beta} \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{1}{2}} t\,\mathrm{e}^{-\frac{v^2}{2}} \right|^2 \mathrm{d}v = \frac{3}{2}. \tag{54}$$

Thus

$$\begin{aligned}&\int_{-\infty}^{+\infty} u^2 \left|F_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} v^2 \left|F_\beta(v)\right|^2 \mathrm{d}v \\ &= \frac{9}{4} \geq \frac{\left(\cos\alpha\sin\beta - \cos\beta\sin\alpha\right)^2}{4} + \left(\cos\alpha\cos\beta\Delta t^2 + \frac{\sin\alpha\sin\beta}{4\Delta t^2}\right)^2 = \frac{25}{36}, \end{aligned} \tag{55}$$

$$\int_{-\infty}^{+\infty} u^2 \left|F_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} v^2 \left|F_\beta(v)\right|^2 \mathrm{d}v = \frac{9}{4} \geq \frac{\left(\cos\alpha\sin\beta + \mathrm{sgn}(W)\cos\beta\sin\alpha\right)^2}{4} = \frac{1}{4}. \tag{56}$$

Thus, theorem 1 defined by inequalities (14) and (15) holds. $\int_{-\infty}^{+\infty} u^2 \left|F_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} v^2 \left|F_\beta(v)\right|^2 \mathrm{d}v$ is indeed greater than the bound defined by theorem 1.

From properties (6) and (7) and $\alpha = \pi/4$, $\beta = 3\pi/4$, we have the following relations:

$$\int_{-\infty}^{+\infty} \left|F'_\alpha(u)\right|^2 \mathrm{d}u = \int_{-\infty}^{+\infty} u^2 \left|F_\alpha(u)\right|^2 \mathrm{d}u = \frac{3}{2}; \tag{57}$$

$$\int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v = \int_{-\infty}^{+\infty} v^2 \left|F_\beta(v)\right|^2 \mathrm{d}v = \frac{3}{2}. \tag{58}$$

Thus we have

$$\begin{aligned} &\int_{-\infty}^{+\infty} \left|F'_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \\ &= \frac{9}{4} \geq \frac{\left(\sin\alpha\cos\beta - \cos\alpha\sin\beta\right)^2}{4} + \left(\sin\alpha\sin\beta\Delta t^2 + \frac{\cos\alpha\cos\beta}{4\Delta t^2}\right)^2 = \frac{25}{36}; \end{aligned} \tag{59}$$

$$\int_{-\infty}^{+\infty} \left|F'_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v = \frac{9}{4} \geq \frac{\left(\sin\alpha\cos\beta + \mathrm{sgn}(W)\cos\alpha\sin\beta\right)^2}{4} = \frac{1}{4}. \tag{60}$$

Thus, theorem 2 defined by inequalities (26) and (27) holds. $\int_{-\infty}^{+\infty} \left|F'_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v$ is indeed greater than the bound defined by theorem 2.

In the same manner, we obtain

$$\begin{aligned} &\int_{-\infty}^{+\infty} \left|uF_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v \\ &= \frac{9}{4} \geq \frac{\left(\cos\alpha\cos\beta - \sin\alpha\sin\beta\right)^2}{4} + \left(\cos\alpha\sin\beta\Delta t^2 + \frac{\sin\alpha\cos\beta}{4\Delta t^2}\right)^2 = \frac{25}{36}; \end{aligned} \tag{61}$$

$$\int_{-\infty}^{+\infty} \left|uF_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v = \frac{9}{4} \geq \frac{\left(\cos\alpha\cos\beta + \mathrm{sgn}(W)\sin\alpha\sin\beta\right)^2}{4} = \frac{1}{4}. \tag{62}$$

Thus, theorem 3 defined by inequalities (38) and (39) holds. $\int_{-\infty}^{+\infty} \left|uF_\alpha(u)\right|^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} \left|F'_\beta(v)\right|^2 \mathrm{d}v$ is indeed greater than the bound defined by theorem 3. This example proves that the bounds introduced by us are effective.

5. Conclusions

In the present paper, the uncertainty relations between two spreads, between two group delays, and between one spread and one group delay, which are unlike the conventional uncertainty principle in time-frequency domains, are investigated and three theorems on the uncertainty principle in FRFT domains are presented. The novelty of these theorems lies in connecting the products of different physical measures and giving their physical interpretations, which has a great potential usage in quantum mechanics.[21−23] Theorem 1 gives the bound of two spreads in two FRFT domains. Theorem 2 gives the relation between two group delays in two FRFT domains. Theorem 3 shows the crossed relation between one group delay and one spread in two FRFT domains. The existing uncertainty principle in the FRFT domain is only a special case of our first theorem, and the conventional uncertainty principle in time-frequency domains is a special case of the uncertainty principle in FRFT domains. We give an example to show the efficiency of our results.

Future work includes finding out how to develop these results from FRFT domains to the linear canonical transform domain and the multidimensional case.

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