Stochastic period-doubling bifurcation analysis of a Rössler system with a bounded random parameter*

Ni Fei(倪 菲)[a)†], Xu Wei(徐 伟)[a)], Fang Tong(方 同)[b)], and Yue Xiao-Le(岳晓乐)[a)]

[a)] *Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an* 710072, *China*
[b)] *Department of Engineering Mechanics, Northwestern Polytechnical University, Xi'an* 710072, *China*

(Received 26 May 2009; revised manuscript received 20 July 2009)

This paper aims to study the stochastic period-doubling bifurcation of the three-dimensional Rössler system with an arch-like bounded random parameter. First, we transform the stochastic Rössler system into its equivalent deterministic one in the sense of minimal residual error by the Chebyshev polynomial approximation method. Then, we explore the dynamical behaviour of the stochastic Rössler system through its equivalent deterministic system by numerical simulations. The numerical results show that some stochastic period-doubling bifurcation, akin to the conventional one in the deterministic case, may also appear in the stochastic Rössler system. In addition, we also examine the influence of the random parameter intensity on bifurcation phenomena in the stochastic Rössler system.

Keywords: Chebyshev polynomial approximation, stochastic Rössler system, stochastic period-doubling bifurcation, bounded random parameter

PACC: 0547

1. Introduction

Due to the randomness of material characteristics, geometric dimensioning, boundary conditions and physical properties, structural models in reality often possess uncertainty, which is usually described by a random parameter. Structures modelled with random parameters are known as stochastic structures. For analysing their dynamic responses, there are three basic mathematical methods: the Monte Carlo method, the stochastic finite element method and the orthogonal polynomial approximation method. The third one exhibits remarkable effectiveness.[1−5] In recent years, orthogonal polynomial approximation methods, such as the Chebyshev polynomial approximation method[6−14] and the Laguerre polynomial approximation method[15] have been successfully applied to explore stochastic bifurcation and stochastic chaos of many typical dynamical systems with bounded random parameters.

Consider the well-known deterministic Rössler system described as[16]

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - r), \end{cases}$$

where x, y, z are state variables and a, b, r are positive system parameters. This system first introduced by Rössler[16] in the 1970s is a basic prototype exhibiting complex dynamics and chaos related to processes of chemical kinetics.[15] A number of researchers have studied the dynamical behaviours of the deterministic Rössler system with regular or random excitations.[17−21] However, so far people have not yet explored the stochastic Rössler system with bounded random parameters. Therefore, we try to study period-doubling bifurcation of the stochastic Rössler system with a bounded random parameter by the Chebyshev polynomial approximation method in this paper.

2. Chebyshev polynomial

As most of the random parameters in the real world are bounded, so we consider a bounded random variable with an arch-like distribution, as shown in Fig. 1, of which the probability distribution function (PDF) can be expressed as follows:[4]

$$p(\xi) = \begin{cases} (2/\pi)\sqrt{1-\xi^2}, & |\xi| \le 1, \\ 0, & |\xi| > 1. \end{cases} \tag{1}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 10872165).
†Corresponding author. E-mail: nifei@mail.nwpu.edu.cn
© 2010 Chinese Physical Society and IOP Publishing Ltd

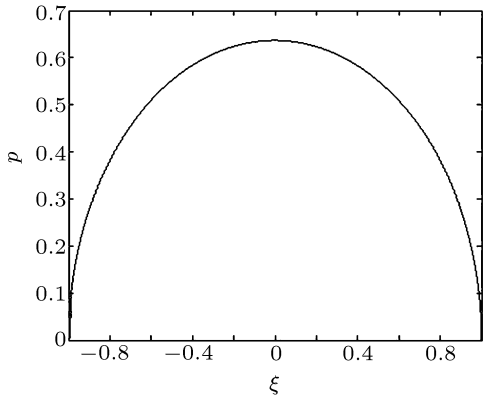


Fig. 1. The arch-like PDF $p(\xi)$ curve for random variable ξ.

Choice of the orthogonal polynomial basis depends on the PDF of the random variable in the orthogonal polynomial approximation method. For the arch-like PDF, the second Chebyshev polynomial is the right choice, given as follows:[4]

$$H_n(\xi) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-k)!}{k!(n-2k)!}(2\xi)^{n-2k}. \quad (2)$$

The recurrent formula for the second Chebyshev polynomials is

$$\xi H_n(\xi) = \frac{1}{2}[H_{n-1}(\xi) + H_{n+1}(\xi)]. \quad (3)$$

The orthogonality of the second Chebyshev polynomials can be expressed as

$$\int_{-1}^{1} \frac{2}{\pi}\sqrt{1-\xi^2} H_i(\xi) H_j(\xi) \mathrm{d}\xi = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \quad (4)$$

Owing to the orthogonality of Chebyshev polynomials, we can expand any measurable function $f(\xi) \subset L^2$ into the following form:

$$f(\xi) = \sum_{i=0}^{\infty} c_i H_i(\xi), \quad (5)$$

where

$$c_i = \int_{-1}^{1} p(\xi) f(\xi) H_i(\xi) \mathrm{d}\xi. \quad (6)$$

Practically, we can only take finite terms in Eq. (5), hence the sum $f(\xi) = \sum_{i=0}^{N} c_i H_i(\xi)$ is merely an approximation of $f(\xi)$ with a minimum mean square residual.[4]

3. Chebyshev polynomial approximation for the stochastic Rössler system

Consider the stochastic Rössler system, whose differential equation is expressed as

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - r), \end{cases} \quad (7)$$

where a, b are constants, and r is a random parameter expressed as

$$r = \bar{r} + \delta\xi, \quad (8)$$

in which $\bar{r}$ is the mean value, and δ is the standard deviation, also regarded as the intensity of r. ξ is a random variable defined on $[-1, 1]$ with an arch-like PDF. Since the system (7) itself is stochastic, its response must be a function of time t and random variable ξ, namely

$$x = x(t, \xi), \quad y = y(t, \xi), \quad z = z(t, \xi).$$

According to the orthogonal polynomial approximation method, the responses of Eq. (7) can take the following series:

$$\begin{aligned} x(t, \xi) &= \sum_{i=0}^{N} x_i(t) H_i(\xi), \\ y(t, \xi) &= \sum_{i=0}^{N} y_i(t) H_i(\xi), \\ z(t, \xi) &= \sum_{i=0}^{N} z_i(t) H_i(\xi), \end{aligned} \quad (9)$$

where the subscript i stands for the sequential number of Chebyshev polynomials and N represents the largest order of the polynomials we have taken. It is worth noting again that $\sum_{i=0}^{N} x_i(t) H_i(\xi)$, $\sum_{i=0}^{N} y_i(t) H_i(\xi)$ and $\sum_{i=0}^{N} z_i(t) H_i(\xi)$ are strictly equivalent to the responses of the stochastic Rössler system only if $N \to \infty$.

Substituting Eqs. (8) and (9) into Eq. (7), we have

$$\begin{cases} \sum\limits_{i=0}^{N} \dot{x}_i(t)H_i(\xi) = -\sum\limits_{i=0}^{N} y_i(t)H_i(\xi) - \sum\limits_{i=0}^{N} z_i(t)H_i(\xi), \\ \sum\limits_{i=0}^{N} \dot{y}_i(t)H_i(\xi) = \sum\limits_{i=0}^{N} x_i(t)H_i(\xi) + a\sum\limits_{i=0}^{N} y_i(t)H_i(\xi), \\ \sum\limits_{i=0}^{N} \dot{z}_i(t)H_i(\xi) = b + \Big(\sum\limits_{i=0}^{N} x_i(t)H_i(\xi)\Big)\Big(\sum\limits_{i=0}^{N} z_i(t)H_i(\xi)\Big) - (\bar{r} + \delta\xi)\sum\limits_{i=0}^{N} z_i(t)H_i(\xi). \end{cases} \tag{10}$$

From the definition of the Chebyshev polynomial, we have

$$\begin{aligned} &H_0(\xi) = 1, \quad H_1(\xi)H_0(\xi) = H_1(\xi), \\ &H_1(\xi)H_1(\xi) = H_2(\xi) + H_0(\xi), \\ &\qquad \cdots . \end{aligned} \tag{11}$$

Since we can reduce any product of two Chebyshev polynomials into a linear combination of individual Chebyshev polynomials, and might as well put the nonlinear term $\Big(\sum\limits_{i=0}^{N} x_i(t)H_i(\xi)\Big)\Big(\sum\limits_{i=0}^{N} z_i(t)H_i(\xi)\Big)$ in Eq. (10) as follows:

$$\Big(\sum_{i=0}^{N} x_i(t)H_i(\xi)\Big)\Big(\sum_{i=0}^{N} z_i(t)H_i(\xi)\Big) = \sum_{j=0}^{2N} m_j(t)H_j(\xi), \tag{12}$$

where $m_i(t)$ are the coefficients of $H_i(t)$ ($i = 0,\ 1,\ \ldots, 2N$). By using the recurrent formula (3) of Chebyshev polynomials, we can reduce the nonlinear term $\xi\sum\limits_{i=0}^{N} z_i(t)H_i(\xi)$ in Eq. (10) into

$$\begin{aligned} \xi\sum_{i=0}^{N} z_i(t)H_i(\xi) &= \frac{1}{2}\sum_{i=0}^{N} z_i(t)[H_{i-1}(\xi) + H_{i+1}(\xi)] \\ &= \frac{1}{2}\sum_{i=0}^{N} [z_{i-1}(t) + z_{i+1}(t)]H_i(\xi), \end{aligned} \tag{13}$$

where $z_{-1} = 0$ and $(z_{i-1}(t) + z_{i+1}(t))/2$ are the coefficients of $H_i(t)$ ($i = 0,\ 1,\ \ldots, N$). Substituting Eqs. (12) and (13) into Eq. (10), we have

$$\begin{cases} \sum\limits_{i=0}^{N} \dot{x}_i(t)H_i(\xi) = -\sum\limits_{i=0}^{N} y_i(t)H_i(\xi) - \sum\limits_{i=0}^{N} z_i(t)H_i(\xi), \\ \sum\limits_{i=0}^{N} \dot{y}_i(t)H_i(\xi) = \sum\limits_{i=0}^{N} x_i(t)H_i(\xi) + a\sum\limits_{i=0}^{N} y_i(t)H_i(\xi), \\ \sum\limits_{i=0}^{N} \dot{z}_i(t)H_i(\xi) = b + \sum\limits_{i=0}^{2N} m_i(t)H_i(\xi) - \bar{r}\sum\limits_{i=0}^{N} z_i(t)H_i(\xi) - \dfrac{\delta}{2}\sum\limits_{i=0}^{N} (z_{i-1}(t) + z_{i+1}(t))H_i(\xi). \end{cases} \tag{14}$$

Multiplying both sides of Eq. (14) by $H_i(\xi)$ ($i = 0\ ,1,\ \ldots, N$) in sequence and taking expectation with respect to ξ, we finally obtain the equivalent deterministic system. According to the accuracy requirement in the following numerical analysis, we take $N = 6$, namely, $x_i(t) = y_i(t) = z_i(t) = 0$, ($i = 7,\ \ldots, N$) and $m_i(t) = 0$ ($i = 7, \ldots, 2N$), and then obtain the corresponding equivalent deterministic system as follows:

$$\begin{cases} \dot{x}_0(t) = -y_0(t) - z_0(t), \quad \dot{y}_0(t) = x_0(t) + ay_0(t), \quad \dot{z}_0(t) = b + m_0(t) - rz_0(t) - \dfrac{\delta}{2}z_1(t), \\ \dot{x}_1(t) = -y_1(t) - z_1(t), \quad \dot{y}_1(t) = x_1(t) + ay_1(t), \quad \dot{z}_1(t) = m_1(t) - rz_1(t) - \dfrac{\delta}{2}(z_0(t) + z_2(t)), \\ \qquad\qquad \vdots \\ \dot{x}_5(t) = -y_5(t) - z_5(t), \quad \dot{y}_5(t) = x_5(t) + ay_5(t), \quad \dot{z}_5(t) = m_5(t) - rz_5(t) - \dfrac{\delta}{2}(z_4(t) + z_6(t)), \\ \dot{x}_6(t) = -y_6(t) - z_6(t), \quad \dot{y}_6(t) = x_6(t) + ay_6(t), \quad \dot{z}_6(t) = m_6(t) - rz_6(t) - \dfrac{\delta}{2}z_5(t). \end{cases} \tag{15}$$

This is just the equivalent deterministic system of Eq. (7), which plays an important role in this paper. As we have taken some expectation with respect to ξ in the process of derivation, the obtained equivalent deterministic system is a kind of weighted average system. Therefore, the response of the equivalent deterministic system must reflect some averaging characteristics of the original stochastic system. By taking averaging with respect to ξ we obtain the ensemble mean response of the equivalent deterministic system (15)

$$\begin{cases} E\left[x(t,\xi)\right] \approx \sum_{i=0}^{6} x_i(t) E\left[H_i(\xi)\right] = x_0(t), \\ E\left[y(t,\xi)\right] \approx \sum_{i=0}^{6} y_i(t) E\left[H_i(\xi)\right] = y_0(t), \\ E\left[z(t,\xi)\right] \approx \sum_{i=0}^{6} z_i(t) E\left[H_i(\xi)\right] = z_0(t). \end{cases} \tag{16}$$

If we sample the random parameter in the equivalent deterministic system (15) as $\xi \equiv 0$, namely, $r = \bar{r}$, then the sample responses of the equivalent deterministic system (16) can be expressed as

$$\begin{cases} x(t,0) \approx \sum_{i=0}^{6} x_i(t) H_i(0) = x_0(t) - x_2(t) + x_4(t) - x_6(t), \\ y(t,0) \approx \sum_{i=0}^{6} y_i(t) H_i(0) = y_0(t) - y_2(t) + y_4(t) - y_6(t), \\ z(t,0) \approx \sum_{i=0}^{6} z_i(t) H_i(0) = z_0(t) - z_2(t) + z_4(t) - z_6(t). \end{cases} \tag{17}$$

When $\xi \equiv 0$ or $\delta = 0$, the stochastic Rössler system (7) is reduced to the deterministic system as follows:

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = b + z\,(x - \bar{r}). \end{cases} \tag{18}$$

4. Stochastic period-doubling bifurcation in the stochastic Rössler system

In this section, we are going to study the stochastic period-doubling bifurcation phenomena in the stochastic Rössler system (7) comparatively, through four different kinds of responses. They are the deterministic response of the deterministic system (DRM) (18), the ensemble mean response of the equivalent deterministic system (EMR) (15), the sample response of the equivalent deterministic system (SRM) (15) and the sample response of the stochastic system (SSR) (7). Having found the responses of system (15) by the fourth-order Runge–Kutta method, we can easily obtain any of DRM or EMR or SRM. Furthermore, we are able to analyse SSR by the Monte Carlo simulation method too.

The initial condition (IC) for the equivalent deterministic system (15) is expressed as

$$\boldsymbol{x}(0) = [x_0(0), \ldots, x_6(0)]^{\mathrm{T}},\ \boldsymbol{y}(0) = [y_0(0), \ldots, y_6(0)]^{\mathrm{T}},$$
$$\boldsymbol{z}(0) = [z_0(0), \ldots, z_6(0)]^{\mathrm{T}},$$

and the IC for systems (7) and (18) as

$$x(0) = x_0(0), \quad y(0) = y_0(0), \quad z(0) = z_0(0).$$

In this paper, the parameter δ is taken to be a small quantity, so we take the initial conditions of the equivalent deterministic system (15) to be the same as the initial conditions of the stochastic system (7) and the deterministic system (18). In the following, the ICs are taken to be

$$\begin{cases} \boldsymbol{x}(0) = [0,0,0,0,0,0,0]^{\mathrm{T}}, \\ \boldsymbol{y}(0) = [-0.7,0,0,0,0,0,0]^{\mathrm{T}}, \\ \boldsymbol{z}(0) = [0.7,0,0,0,0,0,0]^{\mathrm{T}}, \end{cases} \tag{19}$$

and

$$\begin{cases} x(0) = 0, \\ y(0) = -0.7, \\ z(0) = 0.7. \end{cases} \tag{20}$$

4.1. Stochastic period-doubling bifurcation phenomenon

It is well known that there are abundant period-doubling bifurcation and saddle-node bifurcation phenomena in the deterministic Rössler system (18). Similarly, period-doubling bifurcation may also occur in the stochastic Rössler system (7) with $\delta = 0.01$ when a takes different values from 0.08 to 0.19, if we take Eq. (19) as the IC and choose the system parameters to be $b = 0.6$ and $r = 6$. Numerical simulation results are shown in Fig. 2.

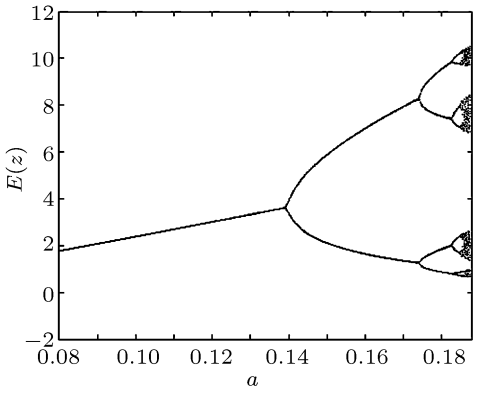


Fig. 2. Bifurcation diagram of EMR against a for the equivalent deterministic system (15).

When system parameters are chosen to be $b = 0.6$, $\bar{r} = 6$, $\delta = 0.01$, and a is taken as the bifurcation parameter. For $a = 0.1$, the numerical results show that there is a steady state 1T (T: period) periodic EMR, together with SSR, whose phase trajectories and time history diagrams are shown in Figs. 3(a) and 3(b) respectively, while for $a = 0.15$, there is another steady state 2T periodic EMR, together with SSR, whose phase trajectories and time history diagrams are shown in Figs. 4(a) and 4(b) respectively. As we take $a = 0.18$, 4T periodic EMR and SSR replace the former 2T periodic ones, whose phase trajectories and time history diagrams are shown in Figs. 5(a) and 5(b). Meanwhile, the coherent phase trajectories and time history diagrams of DMR and SRM for $a = 0.1$, 0.15 and 0.18 are shown in Figs. 3(c), 3(d), Figs. 4(c), 4(d) and Figs. 5(c), 5(d) respectively. From Figs. 3, 4 and 5, it is obvious that DMR and SRM are very close to each other while SSR coincides with EMR very well, so these coherences illustrate the validity of the Chebyshev polynomial approximation method. It is also certain that the stochastic period-doubling bifurcation must happen between $a = 0.1$ and $a = 0.18$.

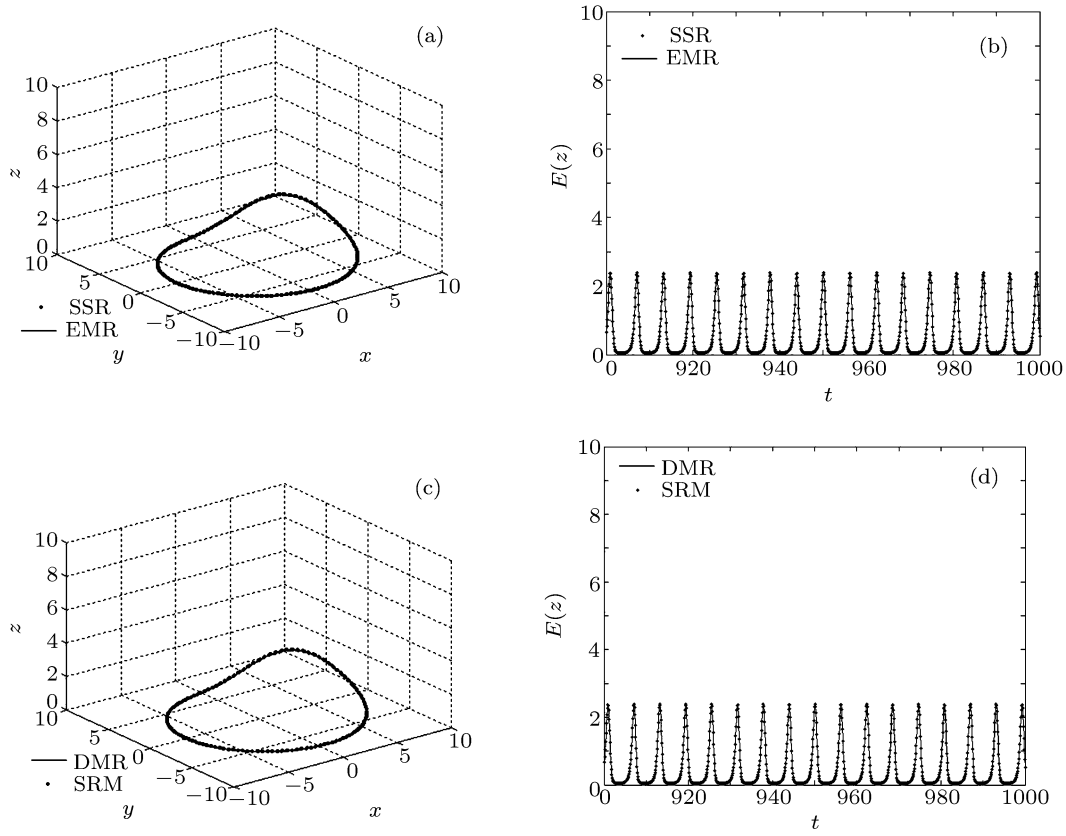


Fig. 3. Phase trajectories and time history diagrams for EMR of period-doubling bifurcation for $a = 0.1$, $b = 0.6$, $r = 6.0$ and $\delta = 0.01$. (a) Phase trajectories of SSR and EMR; (b) time history diagrams of SSR and EMR; (c) phase trajectories of DMR and SMR; (d) time history diagrams of DMR and SMR.

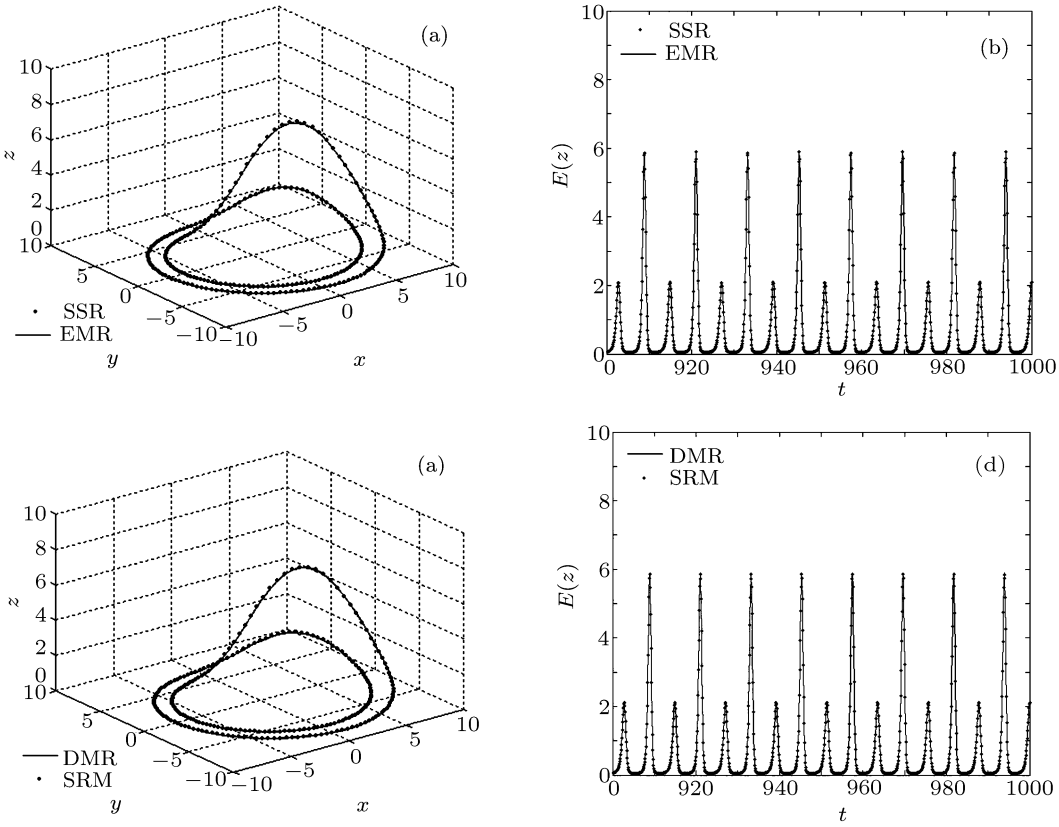


Fig. 4. Phase trajectories and time history diagrams of period-doubling bifurcation for $a = 0.15$, $b = 0.6$, $r = 6.0$ and $\delta = 0.01$. (a) Phase trajectories of SSR and EMR; (b) time history diagrams of SSR and EMR; (c) phase trajectories of DMR and SMR; (d) time history diagrams of DMR and SMR.

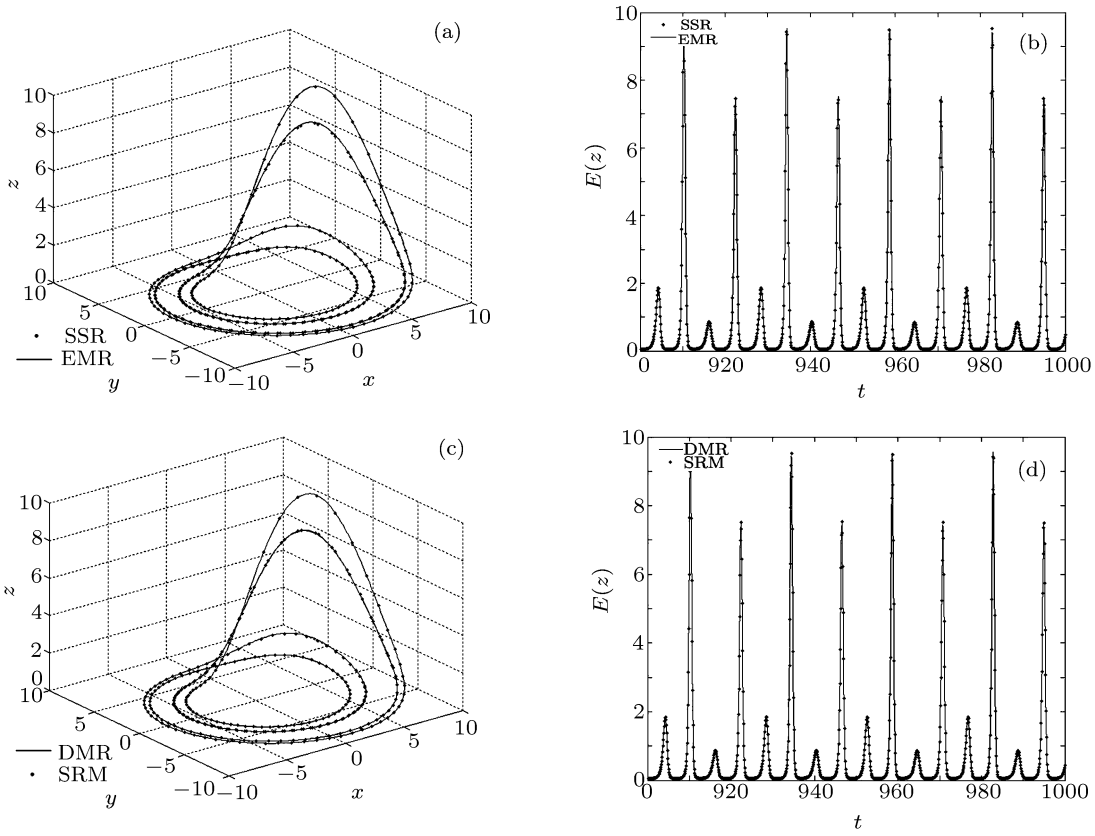


Fig. 5. Phase trajectories and time history diagrams of period-doubling bifurcation for $a = 0.18$, $b = 0.6$, $r = 6.0$ and $\delta = 0.01$. (a) Phase trajectories of SSR and EMR; (b) time history diagrams of SSR and EMR; (c) phase trajectories of DMR and SMR; (d) time history diagrams of DMR and SMR.

4.2. Influence of δ on stochastic period-doubling bifurcation phenomenon

From Subsection 4.1, we know that stochastic period-doubling bifurcation phenomena may take place in the equivalent deterministic Rössler system (15) at a fixed intensity $\delta = 0.01$ when a takes different values. In addition, we find that the stochastic bifurcation can also be affected by the intensity of the random parameter. For example, if we take Eq. (19) as the IC and choose the system parameters to be $b = 0.6$ and $r = 6$, numerical simulation results show that a stochastic period-doubling bifurcation 1T→2T takes place in the equivalent deterministic Rössler system (15) with the increase of a. For different values of intensity δ, the bifurcation diagrams of EMR against a are shown in Figs. 6(a), 6(b), 6(c) and 6(d) respectively.

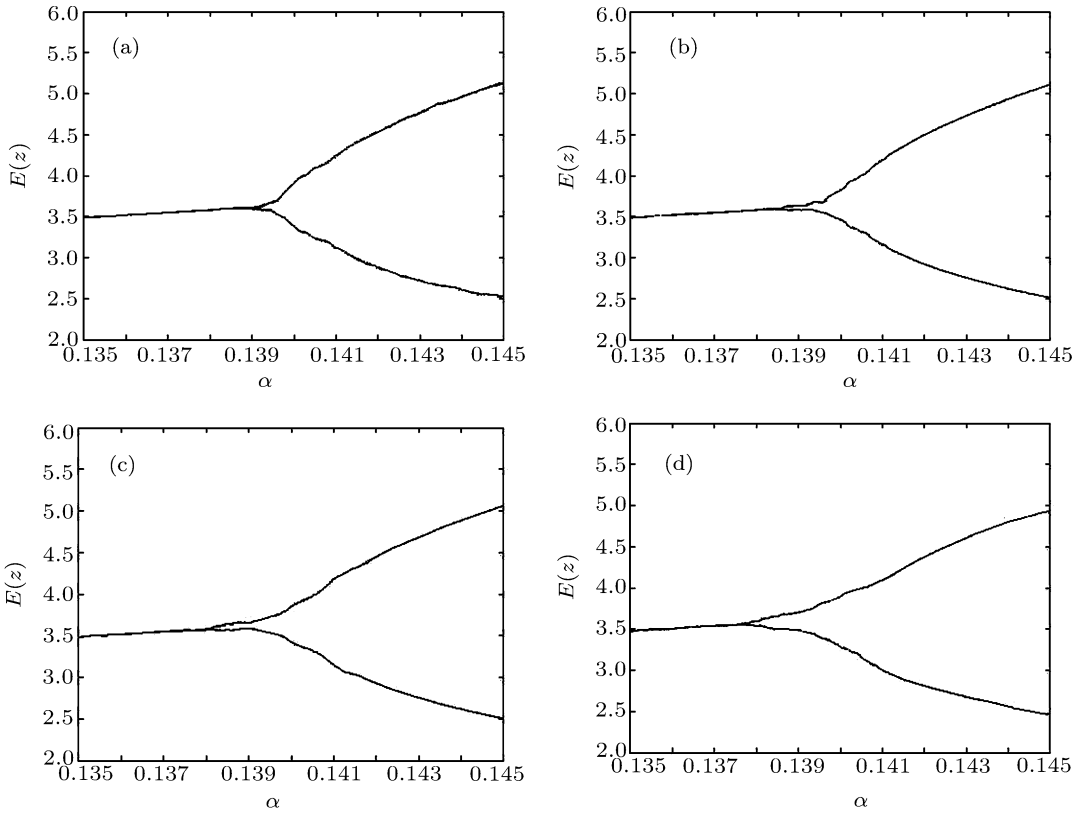


Fig. 6. Bifurcation diagrams of EMR versus a for different values of δ, (a) $\delta = 0.01$; (b) $\delta = 0.05$; (c) $\delta = 0.08$; (d) $\delta = 0.12$.

Figure 6 shows that the critical value at which the stochastic period-doubling bifurcation appears will be brought forward with the increase of intensity δ.

5. Conclusion

The bifurcation phenomena of the stochastic Rössler system have been studied by means of the orthogonal polynomial approximation method in this paper. Analysis shows that the Chebyshev polynomial approximation method is effective in reducing a stochastic Rössler system with a bounded random parameter of an arch-like PDF into its equivalent deterministic system. Numerical simulation results show that stochastic period-doubling bifurcation can also occur in a stochastic Rössler system with a bounded random parameter. Moreover, the stochastic Rössler system has its own feature, i.e., the intensity of the random parameter can greatly influence the bifurcation behaviour. The stochastic bifurcation analysis is helpful for not only gaining insight into the stochastic bifurcation phenomena, but also further investigating stochastic chaos and its control.

References

[1] Spanos P D and Ghanem R G 1989 *J. Eng. Mech. Div. ASCE* **115** 1035
[2] Jense H and Iwan W D 1992 *J. Eng. Mech. Div. ASCE* **118** 1012
[3] Li J 1996 *Acta Mech. Sin.* **28** 63 (in Chinese)
[4] Fang T, Leng X L and Song C Q 2003 *J. Sound. Vib.* **226** 198
[5] Fang T, Leng X L, Ma X P and Meng G 2004 *Acta Mech. Sin.* **20** 292 (in Chinese)
[6] Leng X L, Wu C L, Ma X P, Meng G and Fang T 2005 *Nonlinear Dynamics* **42** 185
[7] Ma S J, Xu W, Li W and Fang T 2006 *Chin. Phys.* **15** 1231
[8] Sun X J, Xu W and Ma S J 2006 *Acta Phys. Sin.* **55** 610 (in Chinese)
[9] Zhang Y, Xu W, Sun X J and Fang T 2007 *Acta Phys. Sin.* **56** 5665 (in Chinese)
[10] Zhang Y, Xu W and Fang T 2007 *Chin. Phys.* **16** 1923
[11] Li G J, Xu W, Wang L and Feng J Q 2008 *Acta Phys. Sin.* **57** 2107 (in Chinese)
[12] Xu W, Ma S J and Xie W X 2008 *Chin. Phys.* B **17** 857
[13] Ma S J and Xu W 2008 *Commun. Nonlinear Sci. Numer. Simulat.* **13** 2256
[14] Feng J Q, Wu W and Wang R 2006 *Acta Phys. Sin.* **55** 5733 (in Chinese)
[15] Ma S J, Xu W and Fang T 2008 *Nonlinear Dynamics* **52** 289
[16] Rössler O E 1976 *Phys. Lett.* A **57** 397
[17] Genesio R and Innocenti G 2008 *Phys. Lett.* A **372** 1799
[18] Vadivasova T E, Balanov A G, Sosnovtseva O V, Postnov D E and Mosekilde E 1999 *Phys. Lett.* A **253** 66
[19] Castro V, Monti M, Pardo W B and Walkenstein J A 2007 *Int. J. Bifur. Chaos* **17** 965
[20] Svetoslav N and Valko P 2004 *Int. J. Bifur. Chaos* **14** 293
[21] Jaume L and Claudia V 2007 *Int. J. Bifur. Chaos* **17** 3289