Estimating the bound for the generalized Lorenz system*

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A chaotic system is bounded, and its trajectory is confined to a certain region which is called the chaotic attractor. No matter how unstable the interior of the system is, the trajectory never exceeds the chaotic attractor. In the present paper, the sphere bound of the generalized Lorenz system is given, based on the Lyapunov function and the Lagrange multiplier method. Furthermore, we show the actual parameters and perform numerical simulations.

Keywords: chaos, generalized Lorenz system, Lyapunov function, Lagrange multiplier method
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1. Introduction

Chaos, as a special complex dynamical phenomenon, has been studied for about four decades. In 1963, Lorenz found the first chaotic system[1] which was a third-order autonomous system with only two multiplication-type quadratic terms but which displayed very complex dynamical behaviours. After that, many new chaotic systems were revealed continuously. In 1999, Chen and Ueta[2] found the Chen system and in 2002, Lü and Chen[3] found a Lü system. According to the definition given by Vaněček and Člikovskỳ,[4] these chaotic systems can be classified into different types. After separating the generalized Lorenz system[5] into linear and quadratic parts, it has three eigenvalues λ_1, λ_2, λ_3, which satisfy $-\lambda_2 > \lambda_1 > -\lambda_3 > 0$ in the linear part of the system.

A chaotic system is bounded, and the estimate of its bound is important for chaos control,[6−8] chaos synchronization[9,10] and their applications.[11,12] However, it is difficult to do so. In 1987, Leonov *et al.* estimated the bound of the Lorenz system.[13] In 2003, Zhou *et al.* investigated the bound of the Chen system.[14] In 2005, Li *et al.* derived the bounds for the Lorenz family of chaotic systems.[15,16] In the present paper, based on the Lyapunov function and the Lagrange multiplier method, the bound of the generalized Lorenz system is presented.

2. Estimate of the bound for the generalized Lorenz system

Definition 1[5] If a non-linear ordinary differential equation in R^3 can be described as

$$\dot{\boldsymbol{X}} = \begin{bmatrix} A & 0 \\ 0 & \lambda_3 \end{bmatrix} \boldsymbol{X} + x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{X}, \tag{1}$$

where $\boldsymbol{X} = (x, y, z)^{\mathrm{T}} \in R^3$, $\lambda_3 \in R$, and $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 real matrix with two eigenvalues λ_1 and λ_2, which satisfy $-\lambda_2 > \lambda_1 > -\lambda_3 > 0$, we refer to Eq. (1) as the generalized Lorenz system.

From Eq. (1), we describe the generalized Lorenz system as follows:

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y, \\ \dot{y} = a_{21}x + a_{22}y - xz, \\ \dot{z} = xy + \lambda_3 z. \end{cases} \tag{2}$$

Theorem 1 When $a_{11} < 0$, and $a_{22} < 0$, the system (2) is contained in the sphere $\Omega = \{(x, y, z) | x^2 + y^2 + (z - m)^2 = R^2\}$, with respect to initial conditions located in the Ω, where

$$R = \max\left\{r_1 + \frac{|m|}{2}, r_2\right\}, \quad m = a_{12} + a_{21},$$

$$r_1 = \sqrt{\frac{\lambda_3 m^2}{4\max\{a_{11},\ a_{22},\ \lambda_3\}}}, \quad \text{and} \quad r_2^2 = \begin{cases} \dfrac{\lambda_3^2 m^2}{4a_{11}(\lambda_3 - a_{11})}, & a_{11} \geq a_{22}, a_{11} \geq \dfrac{\lambda_3}{2}, \\ \dfrac{\lambda_3^2 m^2}{4a_{22}(\lambda_3 - a_{22})}, & a_{22} > a_{11}, a_{22} \geq \dfrac{\lambda_3}{2}, \\ m^2, & a_{11} < \dfrac{\lambda_3}{2}, a_{22} < \dfrac{\lambda_3}{2}. \end{cases}$$

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Proof Let $F(x,y,z)=(x-s)^2+(y-l)^2+(z-m)^2$, then its time derivative along the orbits of the system (2) will be

$$\begin{aligned}\frac{1}{2}\dot{F}&=(x-s)\dot{x}+(y-l)\dot{y}+(z-m)\dot{z}\\&=(x-s)(a_{11}x+a_{12}y)+(y-l)(a_{21}x+a_{22}y-xz)+(z-m)(xy+\lambda_3 z)\\&=a_{11}x^2-(sa_{11}+la_{21})x+(a_{12}+a_{21}-m)xy+lxz+a_{22}y^2-(sa_{12}+la_{22})y+\lambda_3 z^2-\lambda_3 mz.\end{aligned}$$

Let $s=l=0$, $m=a_{12}+a_{21}$, then we will have

$$F(x,y,z)=x^2+y^2+(z-m)^2 \quad \text{and} \quad \frac{1}{2}\dot{F}=a_{11}x^2+a_{22}y^2+\lambda_3(z-\frac{m}{2})^2-\lambda_3\frac{m^2}{4}.$$

1) If $\dot{F}/2>0$, that is, $a_{11}x^2+a_{22}y^2+\lambda_3(z-m/2)^2>\lambda_3(m^2/4)$, then

$$x^2+y^2+\left(z-\frac{m}{2}\right)^2<\frac{\lambda_3 m^2}{4\max\{a_{11},\ a_{22},\ \lambda_3\}}.$$

Let

$$r_1=\sqrt{\frac{\lambda_3 m^2}{4\max\{a_{11},\ a_{22},\ \lambda_3\}}},$$

then we will have

$$x^2+y^2+\left(z-\frac{m}{2}\right)^2<r_1^2. \tag{3}$$

2) If $\dot{F}/2\le 0$, where F is a non-increasing function on any trajectory, the maximum point for F on any trajectory satisfies

$$a_{11}x^2+a_{22}y^2+\lambda_3\left(z-\frac{m}{2}\right)^2-\lambda_3\frac{m^2}{4}=0,$$

that is,

$$\frac{x^2}{\frac{\lambda_3 m^2}{4a_{11}}}+\frac{y^2}{\frac{\lambda_3 m^2}{4a_{22}}}+\frac{\left(z-\frac{m}{2}\right)^2}{\frac{m^2}{4}}=1. \tag{4}$$

Denote

$$H^2=\frac{\lambda_3 m^2}{4a_{11}},\quad B^2=\frac{\lambda_3 m^2}{4a_{22}},\ \text{ and }\ C=\frac{m}{2},$$

then $F(x,y,z)=x^2+y^2+(z-2C)^2$ and expression (4) will turn into

$$\frac{x^2}{H^2}+\frac{y^2}{B^2}+\frac{(z-C)^2}{C^2}=1. \tag{5}$$

Next we use the Lagrange multiplier method to obtain the maximum of F under the condition described by expression (5).

Define

$$G=x^2+y^2+(z-2C)^2+\lambda\left(\frac{x^2}{H^2}+\frac{y^2}{B^2}+\frac{(z-C)^2}{C^2}-1\right)$$

and let

$$\frac{1}{2}G'_x=x\left(1+\frac{\lambda}{H^2}\right)=0, \tag{6}$$

$$\frac{1}{2}G'_y=y\left(1+\frac{\lambda}{B^2}\right)=0, \tag{7}$$

$$\frac{1}{2}G'_z=z-2c+\frac{\lambda(z-C)}{C^2}=0. \tag{8}$$

(i) When $\lambda\neq -H^2$, and $\lambda\neq -B^2$, we have $(x,\ y,\ z)=(0,\ 0,\ 0)$ or $(0,\ 0,\ 2C)$ and

$$F_{\max}=0 \quad \text{or} \quad F_{\max}=4C^2=m^2.$$

(ii) When $\lambda=-H^2$, $(H^2\neq B^2)$ and $H^2\ge 2C^2$, that is, $a_{11}\ge\lambda_3/2$, Eqs. (6)–(8) have the following solutions:

$$x=\pm\frac{H^2}{H^2-C^2}\sqrt{H^2-2C^2},$$
$$y=0,$$
$$z=\frac{C(H^2-2C^2)}{H^2-C^2},$$

and

$$F_{\max}=\frac{H^4}{H^2-C^2}=\frac{\lambda_3^2 m^2}{4a_{11}(\lambda_3-a_{11})}.$$

(iii) When $\lambda=-B^2$, $(H^2\neq B^2)$ and $B^2\ge 2C^2$, that is, $a_{22}\ge\lambda_3/2$, Eqs. (6)–(8) have the following solutions:

$$x=0,$$
$$y=\pm\frac{B^2}{B^2-C^2}\sqrt{B^2-2C^2},$$
$$z=\frac{C(B^2-2C^2)}{B^2-C^2},$$

and

$$F_{\max}=\frac{B^4}{B^2-C^2}=\frac{\lambda_3^2 m^2}{4a_{22}(\lambda_3-a_{22})}.$$

(iv) When $\lambda=-H^2$, $(H^2=B^2)$, we have $z=\dfrac{(H^2-2C^2)C}{H^2-C^2}$ and

$$F_{\max}=\frac{H^4}{H^2-C^2}=\frac{\lambda_3^2 m^2}{4a_{11}(\lambda_3-a_{11})}.$$

Summarizing items 1)–2) above, let

$$R = \max\left\{ \frac{|m|}{2} + r_1,\ \sqrt{F_{\max}} \right\},$$

the proof of theorem 1 is completed.

3. Estimate of two-dimensional bound for the generalized Lorenz system

Theorem 2 When the initial conditions is negative and $\dfrac{2a_{11} - \lambda_3}{2a_{12}} < 0$, for the generalized Lorenz system (2), the following inequality holds: $z \geq \dfrac{1}{2a_{12}} x^2$.

Proof Consider the following function:

$$V = \frac{1}{2a_{12}} x^2 - z,$$

then, its time derivative along the orbits of generalized system (2) will be

$$\dot{V} = \frac{1}{a_{12}} x\dot{x} - \dot{z} = \frac{a_{11}}{a_{12}} x^2 - \lambda_3 z.$$

Hence,

$$\begin{aligned} \dot{V} - \lambda_3 V &= \frac{a_{11}}{a_{12}} x^2 - \lambda_3 z - \frac{\lambda_3}{2a_{12}} x^2 + \lambda_3 z \\ &= \left(\frac{2a_{11} - \lambda_3}{2a_{12}} \right) x^2. \end{aligned}$$

Since $(2a_{11} - \lambda_3)/2a_{12} < 0$, therefore $\dot{V} - \lambda_3 V \leq 0$.

According to the comparison theorem, we can obtain $V \leq 0$, that is, $z \geq (1/2a_{12})x^2$.

The proof is completed here.

4. Numerical simulation

Taking $a_{11} = -16$, $a_{12} = 16$, $a_{21} = 30$, $a_{22} = -1$, and $\lambda_3 = -2$, system (2) turns into

$$\begin{cases} \dot{x} = -16x + 16y, \\ \dot{y} = 30x - y - xz, \ , \\ \dot{z} = xy - 2z. \end{cases} \tag{9}$$

The Lyapunov exponents of Eq. (9) are 0.7972, −0.0053, −19.9326, respectively. It implies that the system (9) is chaotic, see Fig. 1.

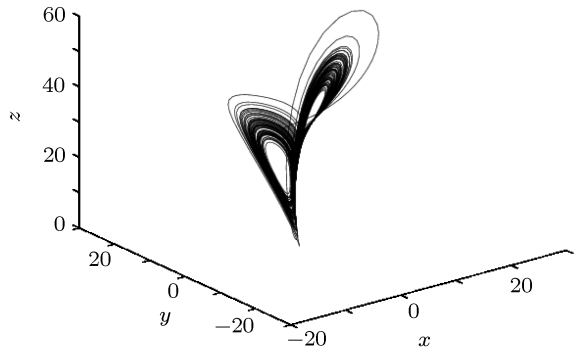


Fig. 1. Three-dimensional phase diagram of system (9).

Because $a_{11} < 0$, and $a_{22} < 0$, system (9) satisfies the conditions of Theorem 1. Its orbit should be contained in the sphere Γ, see Fig. 2, where

$$\Gamma = \{(x, y, z) | x^2 + y^2 + (z - 46)^2 \leq 2116\}.$$

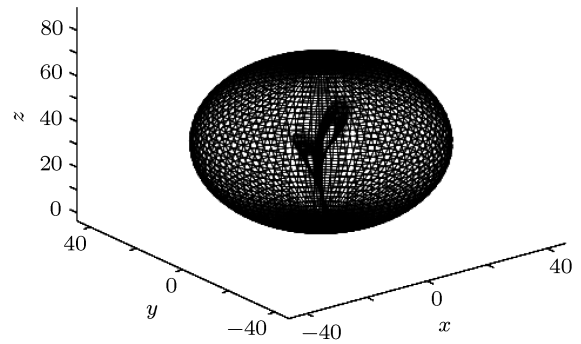


Fig. 2. Three-dimensional phase diagram of system (9) and sphere Γ.

By theorem 2, system (9) satisfies the inequality $z \geq (1/32)x^2$ (see Fig. 3).

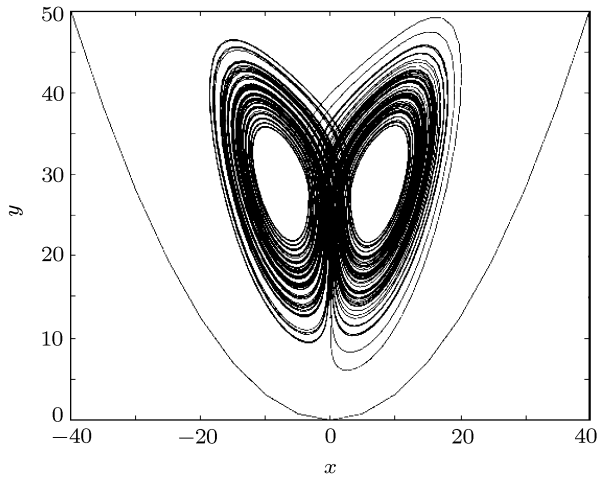


Fig. 3. Orbit of x, z in system (9) and $z = (1/32)x^2$.

5. Conclusions

It is hard work to estimate the bound of a chaotic system. Based on the Lyapunov function and the Lagrange multiplier method, we obtain the bound of the generalized Lorenz system in the condition of $a_{11} < 0$, and $a_{22} < 0$. Then we estimate the two-dimensional bound of system (2) and draw the figures of the bound with the software Matlab.

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