A lattice Boltzmann model with an amending function for simulating nonlinear partial differential equations*

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This paper proposes a lattice Boltzmann model with an amending function for one-dimensional nonlinear partial differential equations (NPDEs) in the form $u_t+\alpha uu_x+\beta u^n u_x+\gamma u_{xx}+\delta u_{xxx}+\zeta u_{xxxx}=0$. This model is different from existing models because it lets the time step be equivalent to the square of the space step and derives higher accuracy and nonlinear terms in NPDEs. With the Chapman–Enskog expansion, the governing evolution equation is recovered correctly from the continuous Boltzmann equation. The numerical results agree well with the analytical solutions.

Keywords: nonlinear partial differential equation, lattice Boltzmann method, Chapman–Enskog expansion, Taylor expansion

PACC: 0540, 0340

1. Introduction

As a new mesoscopic numerical approach, the lattice Boltzmann method (LBM) has achieved much success in studying nonlinear equations and simulating complex fluid systems.[1−7] The LBM is a new technique based on the continuous Boltzmann equation and the mesoscopic kinetic equation; through density distribution function and the Chapman–Enskog expansion, the macroscopic variables are derived and the macroscopic equations are recovered correctly. So it is very different from traditional numerical schemes, such as the finite difference (FD) method,[8,9] the finite element (FE) method,[10] the Chebyshev spectral method[11] and the variational iteration method.[12] From a computational viewpoint, the notable advantages are intrinsic parallelism, geometric flexibility, simplicity of programming, numerical efficiency and ease in incorporating complex boundary conditions.

Recently, the LBM shows potential to simulate nonlinear systems,[13−17] such as the Kuramoto–Sivashinsky equation, the convection–diffusion equation, the reaction–diffusion equation, the Burgers equation, the MKdV equation and the KdV–MKdV equation. As we know, nonlinear partial differential equations (NPDEs) play an important role in different fields of physics and mathematics and there are many interesting and useful features of physical systems hidden in their nonlinear behaviour.[18] In order to study the mechanism of the complicated physical phenomena and dynamical processes modelled by NPDEs, searching and constructing exact solutions for NPDEs is becoming an important aspect.[19,20] In the past few years, the LBM has been extended successfully to simulate the nonlinear systems.[21,22] In the computational fluid mechanics, the time step of the lattice Boltzmann equation (LBE) is equivalent to the space step. This is determined by the qualities of hyperbolic functions, while for the NPDEs, if we also adopt both the time step and the space step in the same order, then the simulation may not be so accurate and effective. Therefore, there is a troublesome problem for solving NPDEs in many existing models, i.e., how to derive higher accuracy and nonlinear terms in NPDEs.

In this paper, by adding an amending function and letting the time step be equivalent to the square of the space step, we construct a more flexible equilibrium distribution function. In this way, we found that the problem is solved completely and readily. In other words, we construct a more flexible five-velocity lattice Boltzmann model with an amending function for the NPDE. With the Chapman–Enskog expansion, the governing evolution equation is recovered correctly. Furthermore, comparisons of numerical re-

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sults and analytical solutions are performed. From the simulations, we found that the results inosculate with the analytical solutions quite well.

2. Lattice Boltzmann model

The one-dimensional NPDEs considered in this paper can be written as

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\beta u^n\frac{\partial u}{\partial x}+\gamma\frac{\partial^2 u}{\partial x^2}+\delta\frac{\partial^3 u}{\partial x^3}+\zeta\frac{\partial^4 u}{\partial x^4}=0, \quad (1)$$

where $\alpha,\beta,\gamma,\delta,\zeta$ are real constants. For the purpose of simplicity but without losing generality, we consider a five velocity lattice BGK model on the one-dimensional case. That is, we discretize one-dimensional space into linear well-proportioned grids, and discretize the velocity into five directions, here the directions of the discrete velocity are defined as $e_i(i=0,1,2,3,4)$,

$$[e_0,e_1,e_2,e_3,e_4]=[0,1,-1,2,-2].$$

The LBE with an amending function is as follows

$$\begin{aligned} &f_i(x+c\varepsilon e_i,t+T\varepsilon^2)-f_i(x,t)\\ &=-\frac{1}{\tau}\left(f_i(x,t)-f_i^{(0)}(x,t)\right)+\varepsilon^2 h_i(x,t), \end{aligned} \quad (2)$$

where $f_i(x,t)$ and $f_i^{(0)}(x,t)$ are defined as density distribution function and equilibrium distribution function, respectively, τ is the dimensionless relaxation time, c is a constant, T is the characteristic time and $h_i(x,t)$ is an amending function. The stability of the equation requires that $\tau>0.5$. We apply the small parameter ε to express the space step. In the classic kinetic theory, ε is called the Knudsen number and it is defined as $\varepsilon=\frac{l}{L}$, l is the mean free path, and L is the characteristic length.

Similar to the general LBM, the macroscopic variable u meets the following conservation law:

$$u(x,t)=\sum_i f_i(x,t)=\sum_i f_i^{(0)}(x,t). \quad (3)$$

$f_i^{(k)}$ $(k=1,2,\ldots)$ are nonequilibrium distribution functions, which satisfy the solvability conditions

$$\sum_i f_i^{(k)}=0, \quad (k=1,2,\ldots). \quad (4)$$

By choosing appropriate local equilibrium distribution and using the Chapman–Enskog expansion, we can retrieve the corresponding macroscopic equation correctly. The role of $h_i(x,t)$ is to derive some nonlinear and derivative terms in NPDEs.

Applying Taylor expansion to the left hand side of Eq. (2) and retaining terms up to $O(\triangle t^5)$, we obtain

$$\begin{aligned} &\varepsilon^2 T\frac{\partial f_i}{\partial t}+\frac{\partial f_i}{\partial x}ce_i\varepsilon+\frac{1}{2}\frac{\partial^2 f_i}{\partial x^2}(ce_i\varepsilon)^2\\ &+\frac{\partial^2 f_i}{\partial x\partial t}(ce_iT\varepsilon^3)+\frac{1}{2}\frac{\partial^2 f_i}{\partial t^2}(\varepsilon^2T)^2+\frac{1}{6}\frac{\partial^3 f_i}{\partial x^3}(ce_i\varepsilon)^3\\ &+\frac{1}{2}\frac{\partial^3 f_i}{\partial x^2\partial t}(ce_i\varepsilon)^2T\varepsilon^2+\frac{1}{24}\frac{\partial^4 f_i}{\partial x^4}(ce_i\varepsilon)^4+O(\varepsilon^5)\\ &=-\frac{1}{\tau}(f_i(x,t)-f_i^{(0)}(x,t))+\varepsilon^2 h_i(x,t). \end{aligned} \quad (5)$$

To derive the macroscopic equation Eq.(1), we apply the Chapman–Enskog expansion in time and space and get

$$\begin{aligned} f_i&=\sum_{n=0}^{\infty}\varepsilon^n f_i^{(n)}\\ &=f_i^{(0)}+\varepsilon f_i^{(1)}+\varepsilon^2 f_i^{(2)}+\varepsilon^3 f_i^{(3)}\\ &\quad+\varepsilon^4 f_i^{(4)}+O(\varepsilon^5), \end{aligned} \quad (6)$$

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t_1}+\varepsilon\frac{\partial}{\partial t_2}+\varepsilon^2\frac{\partial}{\partial t_3}+O(\varepsilon^3). \quad (7)$$

Substituting Eqs.(6) and (7) into Eq.(5), we have

$$\begin{aligned} &\varepsilon^2 T\left(\frac{\partial}{\partial t_1}+\varepsilon\frac{\partial}{\partial t_2}+\varepsilon^2\frac{\partial}{\partial t_3}\right)\left(f_i^{(0)}+\varepsilon f_i^{(1)}+\varepsilon^2 f_i^{(2)}\right)+ce_i\varepsilon\frac{\partial}{\partial x}\left(f_i^{(0)}+\varepsilon f_i^{(1)}+\varepsilon^2 f_i^{(2)}+\varepsilon^3 f_i^{(3)}\right)\\ &+\frac{1}{2}(ce_i\varepsilon)^2\frac{\partial^2}{\partial x^2}\left(f_i^{(0)}+\varepsilon f_i^{(1)}+\varepsilon^2 f_i^{(2)}\right)+ce_iT\varepsilon^3\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t_1}+\varepsilon\frac{\partial}{\partial t_2}\right)\left(f_i^{(0)}+\varepsilon f_i^{(1)}\right)\\ &+\frac{1}{2}T^2\varepsilon^4\left(\frac{\partial}{\partial t_1}\right)^2 f_i^{(0)}+\frac{1}{6}(ce_i\varepsilon)^3\frac{\partial^3}{\partial x^3}\left(f_i^{(0)}+\varepsilon f_i^{(1)}\right)+\frac{1}{2}(ce_i\varepsilon)^2T\varepsilon^2\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t_1}f_i^{(0)}\\ &+\frac{1}{24}(ce_i\varepsilon)^4\frac{\partial^4}{\partial x^4}f_i^{(0)}+O\left(\varepsilon^5\right)\\ &=-\frac{1}{\tau}\left(f_i^{(0)}+\varepsilon f_i^{(1)}+\varepsilon^2 f_i^{(2)}+\varepsilon^3 f_i^{(3)}+\varepsilon^4 f_i^{(4)}-f_i^{(0)}\right)+\varepsilon^2 h_i. \end{aligned} \quad (8)$$

Comparing the two sides of Eq.(8) and treating the term in each order of ε, we have for $O(\varepsilon)$:

$$ce_i\frac{\partial}{\partial x}f_i^{(0)} = -\frac{1}{\tau}f_i^{(1)}, \tag{9}$$

that is

$$f_i^{(1)} = -\tau c\frac{\partial}{\partial x}\left(e_i f_i^{(0)}\right), \tag{10}$$

and for $O(\varepsilon^2)$:

$$T\frac{\partial}{\partial t_1}f_i^{(0)} + c\frac{\partial}{\partial x}\left(e_i f_i^{(1)}\right) + \frac{1}{2}c^2\frac{\partial^2}{\partial x^2}\left(e_i^2 f_i^{(0)}\right) = -\frac{1}{\tau}f_i^{(2)} + h_i. \tag{11}$$

Substituting Eq. (10) into Eq. (11), we obtain

$$f_i^{(2)} = -\tau\left[T\frac{\partial}{\partial t_1}f_i^{(0)} + c^2\left(\frac{1}{2}-\tau\right)\frac{\partial^2}{\partial x^2}\left(e_i^2 f_i^{(0)}\right) - h_i\right]. \tag{12}$$

For $O(\varepsilon^3)$ we have

$$\begin{aligned} &T\frac{\partial}{\partial t_2}f_i^{(0)} + T\frac{\partial}{\partial t_1}f_i^{(1)} + c\frac{\partial}{\partial x}\left(e_i f_i^{(2)}\right) + \frac{1}{2}c^2\frac{\partial^2}{\partial x^2}\left(e_i^2 f_i^{(1)}\right) \\ &+cT\frac{\partial^2}{\partial x\partial t_1}\left(e_i f_i^{(0)}\right) + \frac{1}{6}c^3\frac{\partial^3}{\partial x^3}\left(e_i^3 f_i^{(0)}\right) = -\frac{1}{\tau}f_i^{(3)}. \end{aligned} \tag{13}$$

Substituting Eqs. (10) and (12) into Eq. (13), we obtain

$$f_i^{(3)} = T\tau c(2\tau-1)\frac{\partial^2}{\partial x\partial t_1}\left(e_i f_i^{(0)}\right) - T\tau\frac{\partial}{\partial t_2}f_i^{(0)} - \tau^2 c\frac{\partial}{\partial x}(e_i h_i) - \tau c^3\left(\tau^2-\tau+\frac{1}{6}\right)\frac{\partial^3}{\partial x^3}\left(e_i^3 f_i^{(0)}\right). \tag{14}$$

For $O(\varepsilon^4)$ we have

$$\begin{aligned} &T\frac{\partial}{\partial t_1}f_i^{(2)} + T\frac{\partial}{\partial t_2}f_i^{(1)} + T\frac{\partial}{\partial t_3}f_i^{(0)} + c\frac{\partial}{\partial x}\left(e_i f_i^{(3)}\right) + \frac{1}{2}c^2\frac{\partial^2}{\partial x^2}\left(e_i^2 f_i^{(2)}\right) \\ &+Tc\frac{\partial^2}{\partial x\partial t_1}\left(e_i f_i^{(1)}\right) + Tc\frac{\partial^2}{\partial x\partial t_2}\left(e_i f_i^{(0)}\right) + \frac{1}{2}T^2\frac{\partial^2}{\partial t_1^2}f_i^{(0)} + \frac{1}{6}c^3\frac{\partial^3}{\partial x^3}\left(e_i^3 f_i^{(1)}\right) \\ &+\frac{1}{2}Tc^2\frac{\partial^3}{\partial x^2\partial t_1}\left(e_i^2 f_i^{(0)}\right) + \frac{1}{24}c^4\frac{\partial^4}{\partial x^4}\left(e_i^4 f_i^{(0)}\right) = -\frac{1}{\tau}f_i^{(4)}. \end{aligned} \tag{15}$$

Substituting Eqs. (10), (12) and (14) into Eq.(15), we obtain

$$\begin{aligned} -\frac{1}{\tau}f_i^{(4)} \;=\; &T\frac{\partial}{\partial t_3}f_i^{(0)} + T^2\left(\frac{1}{2}-\tau\right)\frac{\partial^2}{\partial t_1^2}f_i^{(0)} + T\tau\frac{\partial}{\partial t_1}h_i + \tau c^2\left(\frac{1}{2}-\tau\right)\frac{\partial^2}{\partial x^2}(e_i^2 h_i) \\ &+Tc(1-2\tau)\frac{\partial^2}{\partial x\partial t_2}(e_i f_i^{(0)}) + Tc^2\left(3\tau^2-3\tau+\frac{1}{2}\right)\frac{\partial^3}{\partial x^2\partial t_1}\left(e_i^2 f_i^{(0)}\right) \\ &+c^4\left(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\right)\frac{\partial^4}{\partial x^4}\left(e_i^4 f_i^{(0)}\right). \end{aligned} \tag{16}$$

Summing up the two sides of Eqs. (9), (12), (13) and (16) with respect to i, we obtain

$$0 = -\frac{1}{\tau}\sum_i f_i^{(1)} = c\frac{\partial}{\partial x}\left(\sum_i e_i f_i^{(0)}\right), \tag{17}$$

$$0 = -\frac{1}{\tau}\sum_i f_i^{(2)} = T\frac{\partial}{\partial t_1}\left(\sum_i f_i^{(0)}\right) - c^2\left(\tau-\frac{1}{2}\right)\frac{\partial^2}{\partial x^2}\left(\sum_i e_i^2 f_i^{(0)}\right) - \sum_i h_i, \tag{18}$$

$$\begin{aligned} 0 = -\frac{1}{\tau}\sum_i f_i^{(3)} = &T\frac{\partial}{\partial t_2}\left(\sum_i f_i^{(0)}\right) + \tau c\frac{\partial}{\partial x}\left(\sum_i e_i h_i\right) + c^3\left(\tau^2-\tau+\frac{1}{6}\right)\frac{\partial^3}{\partial x^3}\left(\sum_i e_i^3 f_i^{(0)}\right) \\ &-Tc(2\tau-1)\frac{\partial^2}{\partial x\partial t_1}\left(\sum_i e_i f_i^{(0)}\right), \end{aligned} \tag{19}$$

$$
\begin{aligned}
0 = -\frac{1}{\tau}\sum_i f_i^{(4)} &= T\frac{\partial}{\partial t_3}\left(\sum_i f_i^{(0)}\right) + T^2\left(\frac{1}{2}-\tau\right)\frac{\partial^2}{\partial t_1^2}\left(\sum_i f_i^{(0)}\right) \\
&+T\tau\frac{\partial}{\partial t_1}\sum_i h_i + \tau c^2\left(\frac{1}{2}-\tau\right)\frac{\partial^2}{\partial x^2}\left(\sum_i e_i^2 h_i\right) + Tc(1-2\tau)\frac{\partial^2}{\partial x\partial t_2}\left(\sum_i e_i f_i^{(0)}\right) \\
&+Tc^2\left(3\tau^2-3\tau+\frac{1}{2}\right)\frac{\partial^3}{\partial x^2\partial t_1}\left(\sum_i e_i^2 f_i^{(0)}\right) + c^4\left(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\right)\frac{\partial^4}{\partial x^4}\left(\sum_i e_i^4 f_i^{(0)}\right). \quad (20)
\end{aligned}
$$

Using Eq. (12), we obtain

$$
-\frac{1}{\tau}\frac{\partial}{\partial t_1}f_i^{(2)} = T\frac{\partial^2}{\partial t_1^2}f_i^{(0)} + c^2\left(\frac{1}{2}-\tau\right)\frac{\partial^3}{\partial x^2\partial t_1}\left(e_i^2 f_i^{(0)}\right) - \frac{\partial}{\partial t_1}h_i. \quad (21)
$$

Summing up the two sides of the above equation with respect to i and multiply a nonzero term $T\left(\frac{1}{2}-\tau\right)$, we have

$$
T^2\left(\frac{1}{2}-\tau\right)\frac{\partial^2}{\partial t_1^2}\left(\sum_i f_i^{(0)}\right) = -T\left(\frac{1}{2}-\tau\right)^2 c^2\frac{\partial^3}{\partial x^2\partial t_1}\left(\sum_i e_i^2 f_i^{(0)}\right) + T\left(\frac{1}{2}-\tau\right)\frac{\partial}{\partial t_1}\sum_i h_i. \quad (22)
$$

Substituting Eq. (22) into Eq. (20), we obtain

$$
\begin{aligned}
0 &= T\frac{\partial}{\partial t_3}\left(\sum_i f_i^{(0)}\right) - T\left(\frac{1}{2}-\tau\right)^2 c^2\frac{\partial^3}{\partial x^2\partial t_1}\left(\sum_i e_i^2 f_i^{(0)}\right) + T\left(\frac{1}{2}-\tau\right)\frac{\partial}{\partial t_1}\sum_i h_i \\
&+T\tau\frac{\partial}{\partial t_1}\sum_i h_i + \tau c^2\left(\frac{1}{2}-\tau\right)\frac{\partial^2}{\partial x^2}\left(\sum_i e_i^2 h_i\right) + Tc(1-2\tau)\frac{\partial^2}{\partial x\partial t_2}\left(\sum_i e_i f_i^{(0)}\right) \\
&+Tc^2\left(3\tau^2-3\tau+\frac{1}{2}\right)\frac{\partial^3}{\partial x^2\partial t_1}\left(\sum_i e_i^2 f_i^{(0)}\right) + c^4\left(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\right)\frac{\partial^4}{\partial x^4}\left(\sum_i e_i^4 f_i^{(0)}\right). \quad (23)
\end{aligned}
$$

To recover Eq. (1), we select the local equilibrium distribution functions $f_i^{(0)}$ $(i=0,1,2,3,4)$ to satisfy

$$
\sum_i f_i^{(0)} = u,\ \ \sum_i e_i f_i^{(0)} = 0,\ \ \sum_i e_i^2 f_i^{(0)} = 0,\ \ \sum_i e_i^3 f_i^{(0)} = \eta_1 u,\ \ \sum_i e_i^4 f_i^{(0)} = \eta_2 u, \quad (24)
$$

where η_1 and η_2 are some parameters to be determined.

At the same time, we choose the amending functions $h_i(x,t)$ $(i=0,1,2,3,4)$ to satisfy

$$
\sum_i h_i = 0,\ \ \sum_i e_i h_i = \lambda_1 u^2 + \lambda_2 u^{n+1},\ \ \sum_i e_i^2 h_i = \lambda_3 u, \quad (25)
$$

where λ_1, λ_2 and λ_3 are some constants to be determined.

Using Eqs. (24) and (25) to Eqs. (18), (19), (23) we obtain

$$
T\frac{\partial u}{\partial t_1} = 0, \quad (26)
$$

$$
T\frac{\partial u}{\partial t_2} + \tau c\frac{\partial}{\partial x}(\lambda_1 u^2 + \lambda_2 u^{n+1}) + c^3\left(\tau^2-\tau+\frac{1}{6}\right)\frac{\partial^3}{\partial x^3}(\eta_1 u) = 0, \quad (27)
$$

$$
T\frac{\partial u}{\partial t_3} + \tau c^2\left(\frac{1}{2}-\tau\right)\frac{\partial^2}{\partial x^2}(\lambda_3 u) + c^4\left(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\right)\frac{\partial^4}{\partial x^4}(\eta_2 u) = 0. \quad (28)
$$

Finally, from (26) + (27) $\times\,\varepsilon$ + (28) $\times\,\varepsilon^2$ we have

$$
\begin{aligned}
&T\frac{\partial u}{\partial t} + 2\lambda_1\varepsilon\tau c u\frac{\partial u}{\partial x} + (n+1)\varepsilon\tau c\lambda_2 u^n\frac{\partial u}{\partial x} + \lambda_3\varepsilon^2 c^2\tau\left(\frac{1}{2}-\tau\right)\frac{\partial^2 u}{\partial x^2} + \eta\varepsilon c^3\left(\tau^2-\tau+\frac{1}{6}\right)\frac{\partial^3 u}{\partial x^3} \\
&+\eta_2\varepsilon^2 c^4\left(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\right)\frac{\partial^4 u}{\partial x^4} = 0. \quad (29)
\end{aligned}
$$

Furthermore, as the characteristic time T is a positive constant, so the above equation can be rewritten as

$$\frac{\partial u}{\partial t}+2\frac{1}{T}\lambda_1\varepsilon\tau cu\frac{\partial u}{\partial x}+\frac{1}{T}(n+1)\varepsilon\tau c\lambda_2u^n\frac{\partial u}{\partial x}+\frac{1}{T}\lambda_3\varepsilon^2c^2\tau\left(\frac{1}{2}-\tau\right)\frac{\partial^2u}{\partial x^2}+\frac{1}{T}\eta_1\varepsilon c^3\left(\tau^2-\tau+\frac{1}{6}\right)\frac{\partial^3u}{\partial x^3}$$
$$+\frac{1}{T}\eta_2\varepsilon^2c^4\left(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\right)\frac{\partial^4u}{\partial x^4}=0. \quad (30)$$

To recover Eq. (1), just let

$$\Delta x=c\varepsilon,\quad \Delta t=T\varepsilon^2.$$

Moreover, we let

$$\alpha=2\lambda_1\tau\frac{\Delta x}{\Delta t}\varepsilon^2,$$
$$\beta=\lambda_2(n+1)\tau\frac{\Delta x}{\Delta t}\varepsilon^2,$$
$$\gamma=\lambda_3\tau\left(\frac{1}{2}-\tau\right)\frac{\Delta x^2}{\Delta t}\varepsilon^2,$$
$$\delta=\eta_1\left(\tau^2-\tau+\frac{1}{6}\right)\frac{\Delta x^3}{\Delta t},$$
$$\zeta=\eta_2\left(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\right)\frac{\Delta x^4}{\Delta t}, \quad (31)$$

then,

$$\tau=\frac{1}{2}+\sqrt{\frac{1}{12}+\frac{\delta\Delta t}{\Delta x^3\eta_1}},\qquad \left(\tau>\frac{1}{2}\right),$$
$$\lambda_1\varepsilon^2=\frac{\alpha\Delta t}{2\tau\Delta x},\qquad \lambda_2\varepsilon^2=\frac{\beta\Delta t}{(n+1)\tau\Delta x},$$
$$\lambda_3\varepsilon^2=\frac{\gamma\Delta t}{\tau\left(\frac{1}{2}-\tau\right)\Delta x^2},$$
$$\eta_2=\frac{\zeta\Delta t}{\left(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\right)\Delta x^4},$$

where η_1 is free parameter to be determined. It is used for adjusting the equilibrium distribution function and amending function. If we choose a proper value of η_1, then the numerical accuracy and stability may be improved.

Coupling Eqs. (3) and (24), we can derive the equilibrium distribution function,

$$f_0^{(0)}=\left(1+\frac{1}{4}\eta_2\right)u,$$
$$f_1^{(0)}=-\frac{1}{6}(\eta_1+\eta_2)u,$$
$$f_2^{(0)}=\frac{1}{6}(\eta_1-\eta_2)u,$$
$$f_3^{(0)}=\frac{1}{24}(2\eta_1+\eta_2)u,$$
$$f_4^{(0)}=\frac{1}{24}(-2\eta_1+\eta_2)u. \quad (32)$$

From Eq. (25), the amending functions can be defined. For simplicity, only one case for h_i is given,

$$h_0=\lambda_1u^2+\lambda_2u^{n+1}-\lambda_3u,$$
$$h_1=-\frac{1}{6}\lambda_1u^2-\frac{1}{6}\lambda_2u^{n+1}+\frac{1}{2}\lambda_3u,$$
$$h_2=-\frac{7}{6}\lambda_1u^2-\frac{7}{6}\lambda_2u^{n+1}+\frac{1}{2}\lambda_3u,$$
$$h_3=h_4=\frac{1}{6}\lambda_1u^2+\frac{1}{6}\lambda_2u^{n+1}. \quad (33)$$

3. Numerical experiments

In this section, to test the lattice BGK model proposed in the above section, numerical simulations of some important NPDEs are performed. In comparison with the analytical solutions and results derived by existing literature, the efficiency of the proposed model is validated. The macroscopic variable $u(x,t)$ in Eq. (1) is initialized by the initial condition. The distribution function $f_i(x,t)$ is initialized by setting to equal $f_i^{(0)}(x,t)$ for all nodes at $t=0$. To the boundary conditions we use the non-equilibrium extrapolation scheme proposed by Guo *et al.*[23] In addition, the global relative error (GRE) is presented for testing the model's precision, and defined as

$$\mathrm{GRE}=\frac{\sum_i|u(x_i,t)-u^*(x_i,t)|}{\sum_i|u^*(x_i,t)|},$$

where $u(x_i,t)$, $u^*(x_i,t)$ are the numerical solution and analytical solution, respectively. The summation is taken over all grid points. It is found that the simulating results are in excellent agreement with the analytical solutions after a relatively long period of time of evolution.

Example 1 We show an accuracy test for the generalized Kuramoto–Sivashinsky (GKS) equation in Refs. [24] and [25]

$$u_t+uu_x+u_{xx}+4u_{xxx}+u_{xxxx}=0$$

with the exact solitary wave solution

$$u(x,t)$$

$$= 9 + 2c + 15\tanh\left(-\frac{1}{2}x + ct\right)$$
$$-15\tanh^2\left(-\frac{1}{2}x + ct\right) - 15\tanh^3\left(-\frac{1}{2}x + ct\right),$$

the boundary conditions correspond to the data from the solitary wave solution.

In the proceeding, we take $c = 0.001,\ \alpha = \gamma = \zeta = 1,\ \beta = 0,\ \delta = 4,\ \Delta x = 0.1,\ \Delta t = 0.0001$ and $\eta_1 = 0.78$.

The computational domain is fixed on $I = [-30, 30]$, so we choose 600 grids; we present the comparison at $t = 1,\ t = 2,\ t = 3,\ t = 4,\ t = 5$ and $t = 5.5$. Some simulated results are given in Fig. 1. The GRE for the solutions of GKS equation at different times can be found in Table 1.

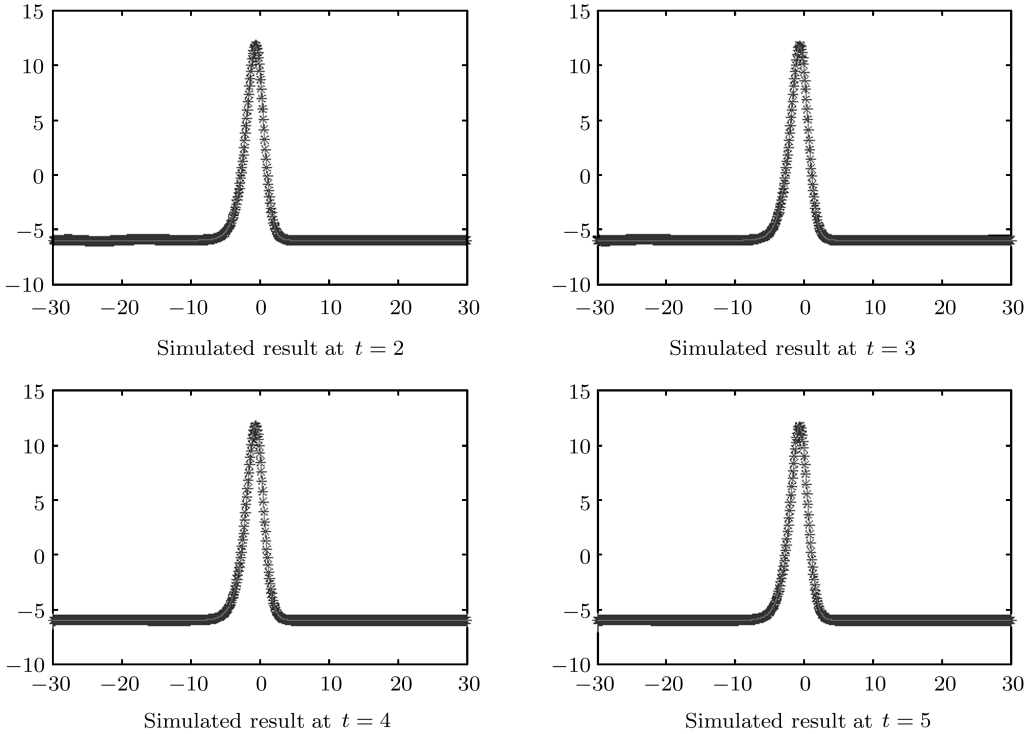


Fig.1. Simulated results of the GKS equation at different times. The solid lines represent the analytical solutions.

Table 1. The GRE for solutions of the GKS equation at different times.

t	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 5.5$
GRE	7.0×10^{-3}	6.8×10^{-3}	8.1×10^{-3}	5.6×10^{-3}	3.4×10^{-3}	3.2×10^{-3}

Example 2 We show an accuracy test for the GKS equation in Ref. [24]

$$u_t + uu_x + u_{xx} + \frac{12}{\sqrt{47}}u_{xxx} + u_{xxxx} = 0,$$

with the exact solitary wave solution

$$u(x,t) = \frac{45 - 4418c}{47\sqrt{47}} + \frac{15}{47\sqrt{47}}[3\tanh\xi - 3\tanh^2\xi + \tanh^3\xi],$$

where $\xi = (1/2\sqrt{47})x + ct$. The boundary conditions correspond to the data from the solitary wave solution.

In the proceeding, we take $c = 0.001,\ \alpha = \gamma = \zeta = 1,\ \beta = 0,\ \delta = 12/\sqrt{47},\ \Delta x = 0.1,\ \Delta t = 0.0001$ and $\eta_1 = 1.0 \times 10^{-8}$.

The computational domain is fixed on $I = [-30, 30]$, so we choose 600 grids; we present the two-dimensional visual comparison at $t = 0.005,\ t = 0.01,\ t = 0.015,\ t = 0.02,\ t = 0.05$ and $t = 0.1$. The global relative error

for the solutions of GKS equation at different times can be found in Table 2. In Fig. 2 we give some simulated results at different times.

Table 2. The GRE for solutions of the equation in Example 2 at different times.

t	$t = 0.005$	$t = 0.01$	$t = 0.015$	$t = 0.05$	$t = 0.1$
GRE	1.7991×10^{-6}	4.2864×10^{-6}	7.6767×10^{-6}	3.2871×10^{-5}	6.5827×10^{-5}

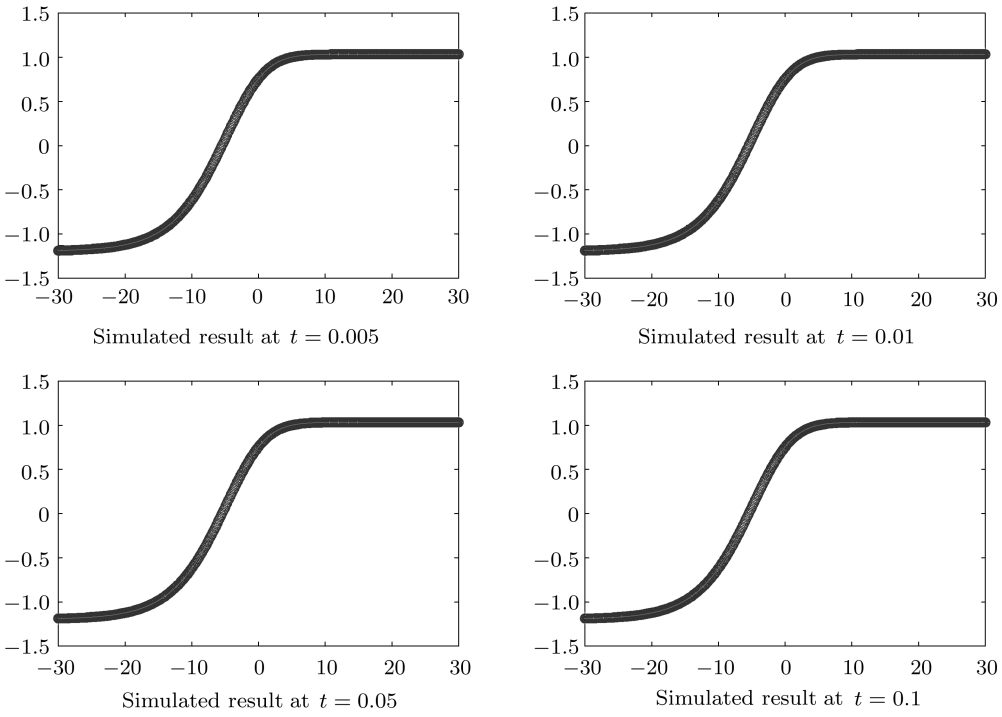


Fig.2. Some simulated results of Example 2. The solid lines represent the analytical solutions.

Example 3 We show an accuracy test for the combined KdV–MKdV equation in Ref. [26]

$$u_t + \alpha u u_x + \beta u^2 u_x + \delta u_{xxx} = 0$$

with the exact solution

$$u(x,t) = -\frac{\alpha}{2\beta} + \sqrt{\frac{6c_2}{\beta}} \tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right), \quad \beta < 0,\ c_2 < 0,$$

where c_2 is constant, $\xi = x + (\alpha^2 + 4\beta c_2)t/4\beta$.

In the present numerical example, we take $n = 2,\ \alpha = 5,\ \beta = -400,\ \gamma = 0,\ \delta = 1.0,\ \zeta = 0,\ c_2 = -0.0001,\ \Delta x = 0.1,\ \Delta t = 0.01$ and $\eta_1 = 0.015$.

The GRE for the solutions of the KdV–MKdV equation at different times can be found in Table 3. In Fig. 3 we give some simulated results at different times.

Table 3. The GRE for solutions of the KdV–MKdV equation in Example 3 at different times.

t	$t = 5$	$t = 10$	$t = 15$	$t = 20$
GRE	5.1277×10^{-6}	6.1981×10^{-6}	8.9897×10^{-6}	1.4327×10^{-5}

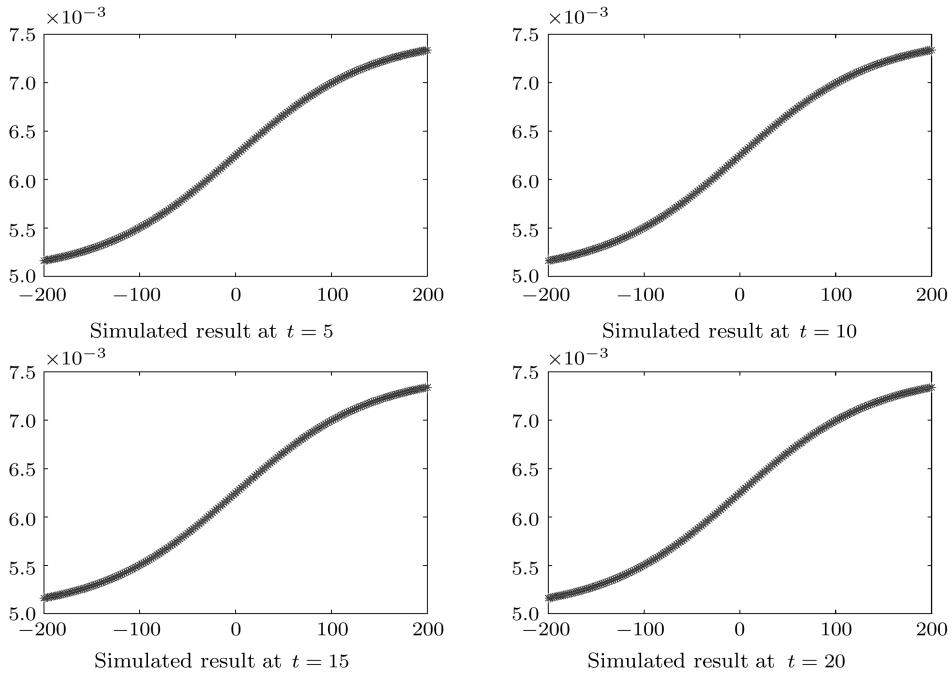


Fig.3. Some simulated results of the KdV–MKdV equation in Example 3. The solid lines represent the analytical solutions.

4. Conclusion

In this paper, we propose a lattice Boltzmann model with an amending function for the NPDEs. By Taylor expansion and the Chapman–Enskog expansion, some NPDEs are precisely recovered from the model. Unlike other models, we let the time step be equivalent to the square of the space step. The higher accuracy and nonlinear terms are derived by adding an appropriate amending function. Numerical simulations for some test problems validate the efficiency of the present model. Based on our experiences, the choice of the parameter η_1 makes our model more flexible. The numerical results agree well with the analytic solutions. In the near future, we may extend our model to two-dimensional problems or more, and a stability analysis is also needed in further study.

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