Stochastic resonance in the FitzHugh–Nagumo system driven by bounded noise*

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We investigate stochastic resonance (SR) in the FitzHugh–Nagumo system under combined bounded noise and weak harmonic excitation. Taking a spectral amplification factor as a signal-to-noise ratio, we show numerically that bounded noise can induce SR by adjusting either the intensity of bounded noise or its colour. Moreover, the increase of noise colour can enhance the SR and make the peak of the SR shift toward lower noise intensities, which is more feasible in practice. Since bounded noise is flexible to model random excitation, these findings may have some potential applications in engineering, neuroscience and biology.

Keywords: bounded noise, FitzHugh–Nagumo system, stochastic resonance
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1. Introduction

Stochastic resonance, which was first discovered and proposed by Benzi *et al.* to explain the periodic recurrence of ice ages,[1] has been investigated extensively both theoretically and experimentally due to its potential applications in many fields. As reviewed by Gammaitoni *et al.*[2] a classical SR in a bistable system requires three basic ingredients: an energetic barrier, a weak periodic signal which alone is insufficient to cause the system to surmount the barrier, and a source of noise to surmount the barrier. Given appropriate conditions, the response of the system undergoes a resonant-like behaviour as a function of noise amplitude, while moderate noise causes the transitions across the energetic threshold to become synchronized to follow, on average, the periodicity of the weak periodic input, thereby greatly enhancing the weak signal. Since then, the notion of SR has been extended, by viewing different definitions of the signal-to-noise ratio, to mono-stable systems,[3] chaotic systems,[4] excitable systems,[5] as well as systems without any external deterministic excitation[6] or driving noise.[7] Recently, SR has been investigated in complex networks, such as small-world and scale-free topology.[8]

The majority of previous scientific studies are devoted to the study of SR driven by white noise. White noise is a random process with a constant power spectral density in a range of infinity bandwidth, so that its total power is infinite and in practice impossible to generate. Since coloured noise with bounded power is more practicable than white noise in real physical systems, many contributions have been shifting toward the study of SR driven by coloured noise.[2,5,9–14] It has been found that the colour of noise usually has a drastic influence on SR phenomena. The effect of exponentially correlated Gaussian noise (Ornstein–Uhlenbeck noise) shows that with the increase of noise, colour, SR is suppressed and the peak of the SR is shifted toward larger noise intensities.[15,16] In contrast, driven by harmonic noise, as well as by asymmetric dichotomic noise, SR can be enhanced for effective noise colour which is increasing.[15] Under certain circumstances, the colour of noise may produce SR-type effects and play a significant role in detecting weak subthreshold signals.[17]

Very recently, a versatile model of coloured noise, called bounded noise, has been proposed as a stochastic excitation in dynamical systems by many researchers.[18–24] The bounded noise is expressed as a harmonic function with a constant average frequency and a random phase varying as a unit Wiener process. Actually, it is difficult for an experimental setup to keep the excitation frequency of the shake at a fixed one to experience a long-time excitation. The excitation frequency may slightly deviate from the fixed average frequency at random and the bounded noise is a suitable model for stochastic excitation. In addition,

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the spectrum of bounded noise can be either broad band or narrow band by adjusting the parameters. It has been used, for example, to model a slug flow of a two-phase fluid in a straight pipe,[25] a spatially disordered travelling parametric excitation[26] and a turbulent wind in log-span bridges.[27]

However, to the best of our knowledge, no attention has been paid to stochastic resonance in nonlinear systems under bounded noise excitation. In the present work, we investigate how the colour of bounded noise would influence stochastic resonance in the FitzHugh–Nagumo (FN) system with a weak periodic signal. The rest of the present paper is organized as follows. In Section 2, the dynamics of the unforced FN system and features of bounded noise are presented. Section 3 deals with the FN system under combined bounded noise and weak harmonic excitations and SR induced by bounded noise as well. Finally, some conclusions are drawn in Section 4.

2. The FN system driven by bounded noise

We consider the FN system[28] under combined bounded noise and weak harmonic excitations as follows:

$$\begin{cases} \varepsilon \dot{x} = x - \dfrac{x^3}{3} - y, \\ \dot{y} = x + a + A\xi(t) + f(t), \end{cases} \tag{1}$$

where $x(t)$ is a fast variable representing the membrane voltage of the neuron, $y(t)$ is a slow (recovery) variable, and $\xi(t)$ and $f(t)$ describe bounded noise and a weak periodic signal, respectively. If there is no external excitation and the small time constant is fixed as $\varepsilon = 0.01$, the behaviour of the FN system would be determined by the parameter a. The critical value $a_c = 1$ is a bifurcation point of supercritical Hopf type. When $|a| < a_c$, the system is in the oscillatory regime and a limit cycle appears, while for $|a| > a_c$, it is in the excitable one and only a stable fixed point exists. Here we choose the parameter as $a = 1.1$ and assume that the weak signal takes the form of $f(t) = \mu\cos(\omega_0 t)$ where the small amplitude μ is fixed as 0.05. Hence the signal alone cannot evoke a spike, and without noise the system is under the firing threshold. As a driving noise, the bounded noise can be described as[18−24]

$$\xi(t) = \sin[\Omega t + \sigma W(t) + \varphi], \tag{2}$$

where Ω is an average frequency, $W(t)$ is a unit Wiener process with amplitude σ, and φ is a random phase uniformly distributed in $[0,\ 2\pi]$. The bounded noise $\xi(t)$ is a non-Gaussian stationary process and its probability density is

$$\rho(\xi) = \frac{1}{\pi\sqrt{1-\xi^2}}, \quad \xi \in (-1,\ 1). \tag{3}$$

So its autocorrelation function can be expressed as

$$\begin{aligned} c(\tau) &= \langle \xi(t)\xi(t+\tau)\rangle \\ &= \frac{1}{2}\langle \cos[\Omega\tau + \sigma W(t+\tau) - \sigma W(t)]\rangle \\ &= \frac{1}{2}\exp\left(-\frac{\sigma^2|\tau|}{2}\right)\cos(\Omega\tau) \end{aligned} \tag{4}$$

and its spectral density is

$$S_\xi(\omega) = \frac{\sigma^2}{2\pi}\left(\frac{1}{4(\omega-\Omega)^2+\sigma^4} + \frac{1}{4(\omega+\Omega)^2+\sigma^4}\right). \tag{5}$$

The intensity of bounded noise $\xi(t)$ is fixed at $c(0) = 1/2$, regardless of the parameters, while its spectral bandwidth and shape sensitively depend on Ω and σ. On one hand, when $\Omega/\sigma^2 \gg 1$, $\xi(t)$ becomes a narrow-band random process. Moreover, if $\sigma \to 0$, $S_\xi(\omega)$ vanishes in the entire frequency range except at the singular frequency $\omega = \pm\Omega$ where $S_\xi(\pm\Omega) \to +\infty$. On the other hand, when $\Omega/\sigma^2 \ll 1$, $\xi(t)$ becomes a broad-band random process and as $\sigma \to +\infty$, it leads to a white noise with constant spectral density. Figure 1 shows the spectral densities for the fixed frequency $\Omega = 6$ and several values of σ. It can be

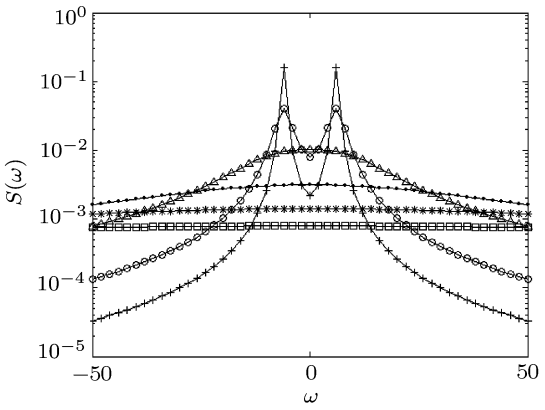


Fig. 1. Curves for spectral density $S(\omega)$ from bounded noise $\xi(t)$ versus frequency ω for different values of the amplitude σ of the unit Wiener process: $\sigma = 1$ (plus sign '+'); $\sigma = 2$ (circle '◯'); $\sigma = 5$ (upward pointing triangle '△'); $\sigma = 10$ (point '•'); $\sigma = 15$ (asterisk '∗'); $\sigma = 20$ (square '□').

seen that for $\sigma = 1,\ 2$, the process appears to be of narrow-band and possesses two peaks near $\Omega = 6$ and for $\sigma = 5,\ 10,\ 15,\ 20$, the two peaks merge at zero and the process tends to be of a broad-band noise.

3. Stochastic resonance in the FN system

The stochastic differential equation (1) is numerically integrated by using the Heun method[29] with a fixed step length of $\Delta t = T/5000$, where $T = 2\pi/\omega_0$ and ω_0 is the frequency of the weak signal. To eliminate direct effects of the bounded noise on the signal, we choose a set of incommensurate frequencies, $\Omega = 6.0$ and $\omega_0 = \pi/4$. For $\sigma = 1$, the driving noise is a narrow-band one. The evolutions of signal and corresponding membrane voltage are shown in Fig. 2 for three different noise amplitudes. It can be seen that for both very small and very large noise amplitudes, the noise induced oscillations appear to be rather

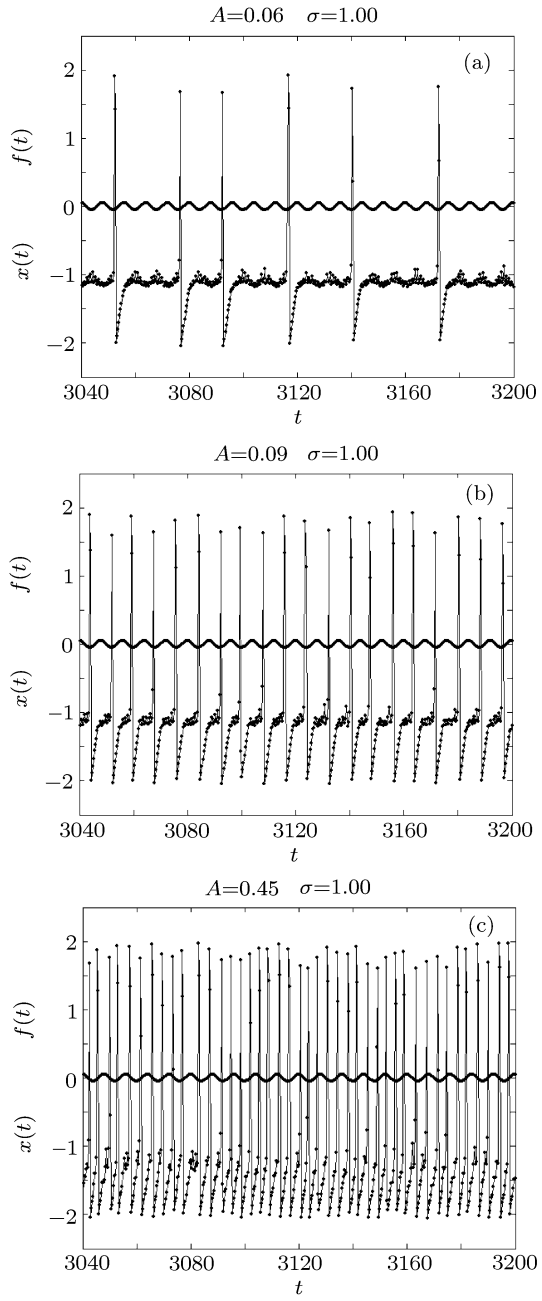


Fig. 2. Time evolution of weak signal $f(t)$ and membrane voltage $x(t)$ for the fixed parameter $\sigma = 1$ and different values of the amplitude A of bounded noise: (a) $A = 0.06$; (b) $A = 0.09$; (c) $A = 0.45$.

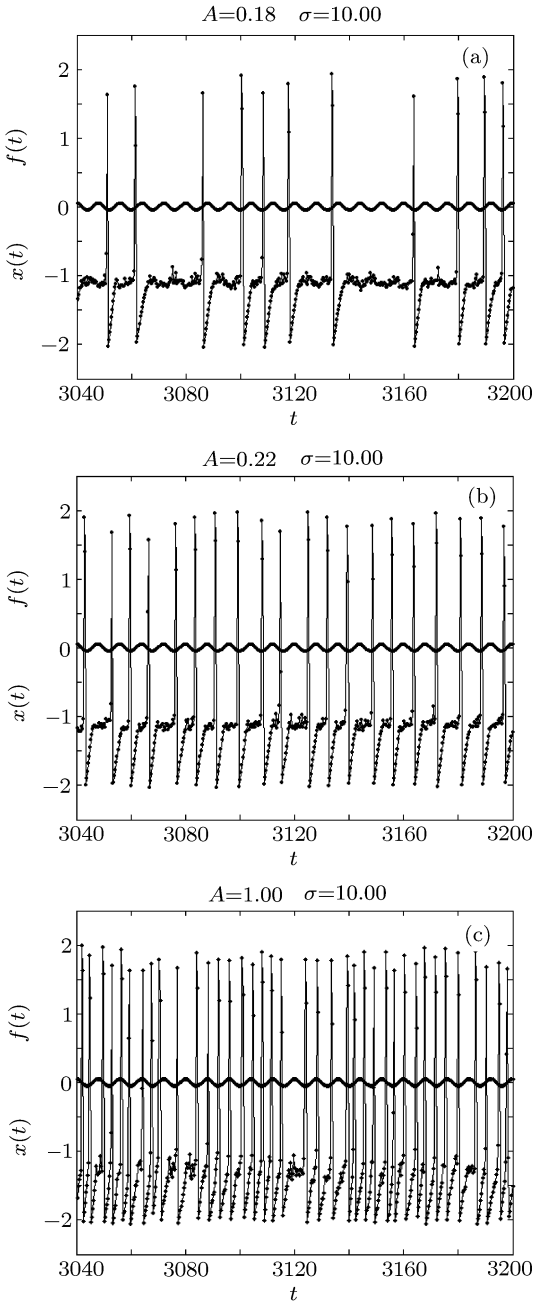


Fig. 3. Time evolution of weak signal $f(t)$ and membrane voltage $x(t)$ for the fixed parameter $\sigma = 10$ and different values of the amplitude A of bounded noise: (a)$A = 0.18$; (b) $A = 0.22$; (c) $A = 1.00$.

irregular, while small noise produces very few spikes and large noise takes place too frequently. In contrast, for moderate noise, the oscillations are relatively regular and to a great extent are synchronized to the weak periodic signal (Fig. 2(b) shows that the period of the spikes is very close to that of the weak signal). This demonstrates that bounded noise can lead to the occurrence of SR and amplify the weak signal. For the case of broad-band noise, we choose $\sigma = 10$ and obtain similar results (see Fig. 3). By comparing these two cases, we can observe that when synchronized to the signal the former seems more regular than the latter and at the same time the noise amplitude of the former is lower (see Figs. 2(b) and 3(b)).

To examine the above phenomena in a broader sense, we quantitatively characterize this resonant effect by the spectral amplification factor[30]

$$Q = 4\mu^{-2} \left| \langle x(t) \exp(\mathrm{i}\,\omega_0 t) \rangle \right|^2 , \tag{6}$$

where $\langle \ldots \rangle$ denotes a time average. To eliminate random fluctuations, all numerical results are obtained by averaging over 50 different initial conditions and realizations. At each realization, the transient 2×10^6 time steps are discarded and the subsequent 2×10^6 time steps are used to compute the spectral amplification factor Q. In Fig. 4, the values of Q are plotted as a function of noise amplitude A for different values of σ. It is shown that for any amplitude of the Wiener process, Q increases with A increasing, reaches a maximum Q_{m} at optimal amplitude A_{m}, and then decreases again. This indicates that SR would occur in the FN system regardless of whether the bounded noise is of a broad-band nature. Besides, we can observe that when σ increases from 1 to 20, the peak for SR Q_{m} is reduced and shifted toward larger values of A_{m}, which accords well with the result in Ref. [15].

To globally investigate the dependence of Q on A and σ, the contour plot is shown in Fig. 5. It is found that the smaller the value of σ is, the larger the value of Q_{m} will be and at the same time the lower the value of A_{m} will be. As σ decreases, the band of bounded noise narrows and its colour increases accordingly. Therefore, the colour of bounded noise can enhance SR and reduce A_{m}. When σ is fixed arbitrarily, we can observe that SR occurs where Q is taken as a function of A. On the other hand, when A is fixed as some value, SR would also occur where Q is taken as a function of σ, which is shown in Fig. 6 as an example. This shows that the colour of noise can induce SR with varying σ although the intensity of bounded noise is invariable.

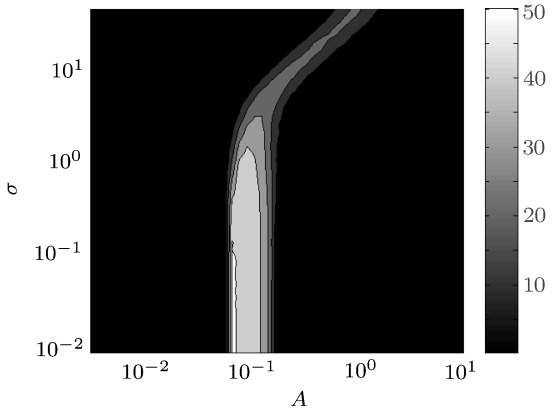


Fig. 5. Contour plot of the spectral amplification factor Q of system (1) in the $A - \sigma$ plane.

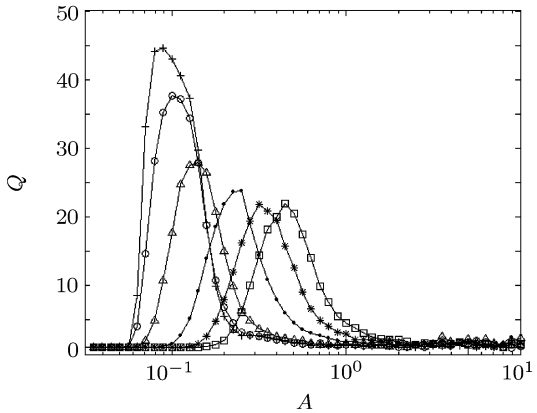


Fig. 4. Curves for spectral amplification factor Q of the stochastic FitzHugh–Nagumo system, Eq. (1), as a function of amplitude A of bounded noise for different values of amplitude σ in a unit Wiener process: $\sigma = 1$ (plus sign '+'); $\sigma = 2$ (circle '◯'); $\sigma = 5$ (upward pointing triangle '△'); $\sigma = 10$ (point '•'); $\sigma = 15$ (asterisk '∗'); $\sigma = 20$ (square '□').

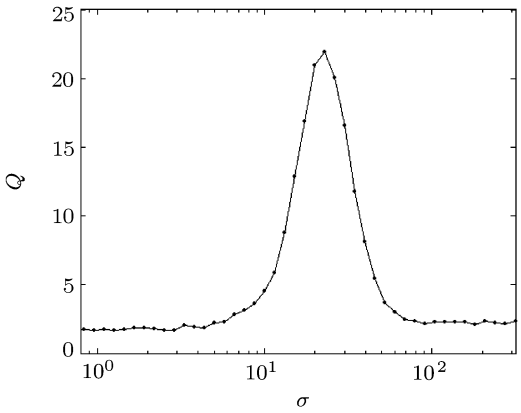


Fig. 6. Spectral amplification factor Q of system (1) as a function of amplitude σ in a unit Wiener process, where $A = 0.5$.

4. Conclusions

In this paper, SR in the FN system driven by bounded noise is investigated. It is found that SR would occur in the FN system regardless of whether the bounded noise is broad-band or narrow-band. From broad-band to narrow-band, with noise colour increasing, bounded noise works better because the SR peak height increases and shifts toward smaller noise intensities (see Figs. 4 and 5). Besides, the resonant phenomena can be obtained by adjusting either the intensity of bounded noise or its colour (see Figs. 4 or 6). Such interesting features, to our knowledge, have not been reported for bounded noise before. Here the excitable FN system is considered as a driven system. Taking into account other systems such as bistable systems, we also find that bounded noise can induce SR and amplify a weak signal. However, theoretical analyses on bounded noise induced SR are to be further studied. Since bounded noise can be used to model random excitations of various types, it may conduce to detecting weak signals in practice, which might have potential applications in engineering, neuroscience and biology.

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