The relation between a black hole and a general black body system*

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Taking a black hole as a black body system, using general black body radiation theory, a Schwarzschild black hole and a Kerr–Newman black hole are investigated respectively. It is concluded that a black hole can be regarded as an ideal general black body system exactly for the changing process only. However, a stationary global black hole cannot be smoothly regarded as a general black body system. A black hole has some special characteristics which different from a general thermodynamics system. This conclusion means that a black hole should be inherently dynamical, at least when it is taken as a black body system.

Keywords: black hole, black body radiation, thermodynamics, Hawking radiation

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1. Introduction

According to the classical theory of general relativity, a collapsing star of mass M will shrink. If the mass M and radius r of the star satisfy the relation $r = 2GM/c^2$, it will become a black hole, whose gravitational field becomes so strong that no further radiation or anything else can escape to infinity.[1−3]

The well-known black hole mechanics laws say:[4]

(1) The surface gravity κ on the event horizon of a stationary black hole is constant.

(2) The relation between the change of the black hole mass M and the changes of its area A, angular momentum J, and electric charge Q satisfies

$$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega\delta J + V\delta Q. \tag{1}$$

(3) Hawking's area theorem tells us that the area A of a black hole horizon cannot decrease.

(4) The surface gravity κ cannot be reduced to zero by any finite steps of operations.

Drawing an analogy between the surface gravity κ on the event horizon and the temperature T, which satisfies $T = \kappa/(2\pi)$, an analogy between the area A of the horizon and the entropy S, which satisfies $S = A/4$, it has been found that the black hole mechanics laws are exactly black hole thermodynamics laws, which are quite analogous to the usual thermodynamics laws.[5,6]

In the 1970s, Hawking's astonishing discovery, namely that a black hole radiates thermally, made the analogy between black holes and thermodynamics more meaningful.[7−12] The fact that a black hole absorbs and emits particles makes it more similar to a black body than anything else.[13−17]

In this paper, we shall investigate a black hole using the black body radiation theory.[18,19] In Section 2, we shall take a Schwarzschild black hole as an example. Although a strict definition of volume cannot be given for a black hole, it can be calculated via the analogous black body system. After the temperature of a black body system is given according to a black hole Hawking radiation, we will give two kinds of 'volume': one is related to the black body's energy, the other is due to the entropy. We write them as V_U and V_S respectively. After simple calculation, we find that the two 'volumes' are not consistent with each other when V_U is defined as the quotient of energy U divided by energy density u, and V_S as entropy S divided by s. However, if we define the volume in a dynamic way, i.e. letting $\widetilde{V}_U$ be the quotient of the change of energy dU divided by the change of energy density du, and $\widetilde{V}_S$ is given in a similar way, $\widetilde{V}_U$ and $\widetilde{V}_S$ will be consistent with each other. In Section 3, we shall generalize the work to a more general stationary axis-symmetric Kerr–Newman black hole and draw the same conclusion. For simplicity, natural units with $c = G = k_{\rm B} = \hbar = 1$ are used throughout

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the paper.

2. A Schwarzschild black hole as a black body

The line element of a Schwarzschild black hole can be written as

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2), \tag{2}$$

where M is the mass of the black hole.

The event horizon is given by $r_{\mathrm{h}} = 2M$, so the area A of the event horizon is

$$A = \int \sqrt{g}\,\mathrm{d}\theta\mathrm{d}\varphi = 4\pi(r_{\mathrm{h}})^2 = 16\pi M^2. \tag{3}$$

Therefore using the analogy between the area A of the event horizon and the entropy S of the black hole, we obtain the entropy S

$$S_{\mathrm{BH}} = \frac{A}{4} = 4\pi M^2. \tag{4}$$

The Hawking temperature T_{BH} is given by [1]

$$T_{\mathrm{BH}} = \frac{1}{8\pi M}. \tag{5}$$

Obviously, the differential equation, i.e. energy conservation equation

$$\mathrm{d}M = T_{\mathrm{BH}}\mathrm{d}S_{\mathrm{BH}} \tag{6}$$

is satisfied.

Now, we take the Schwarzschild black hole as a black body system with temperature $T = T_{\mathrm{BH}} = 1/8\pi M$. Its total energy U is exactly the black hole's mass M.

The energy density u of the black body with temperature T is

$$u = aT^4, \tag{7}$$

where a is the Stefan–Boltzmann constant. If we define V_U as the black hole 'volume' relating to energy, which satisfies $V_U \equiv U/u$, we will have

$$V_U \equiv \frac{U}{u} = \frac{M}{aT^4}. \tag{8}$$

In the meantime, the entropy density s of a black body is given by

$$s = \frac{4}{3}aT^3. \tag{9}$$

If we define V_S as the black hole 'volume' relating to entropy, which satisfies $V_S \equiv S/s$, we can have

$$V_S \equiv \frac{S}{s} = \frac{4\pi M^2}{\frac{4}{3}aT^3}. \tag{10}$$

After simple calculation, we can find that the two definitions of black hole 'volume' correlative with energy and entropy respectively are not consistent with each other. They obey the following relationship

$$V_U = \frac{8}{3}V_S. \tag{11}$$

Now we consider a dynamic process with a change δM for the black hole's mass. From the point of a black body, we can write the dynamic definitions of black hole 'volume' corresponding to energy and entropy respectively as

$$\widetilde{V}_U = \frac{\mathrm{d}U}{\mathrm{d}u} = \frac{\delta M}{4aT^3\delta T}, \tag{12}$$

and

$$\widetilde{V}_S = \frac{\mathrm{d}S}{\mathrm{d}s} = \frac{8\pi M\delta M}{4aT^2\delta T}. \tag{13}$$

Therefore, the two 'volumes' are consistent with each other when the temperature satisfies $T = T_{\mathrm{BH}}$.

3. A Kerr–Newman black hole as a black body

The line element of a Kerr–Newman black hole can be written as[1]

$$\begin{aligned}\mathrm{d}s^2 &= -\left(1-\frac{2Mr-Q^2}{\Sigma^2}\right)\mathrm{d}t^2 + \frac{\Sigma^2}{\Delta}\mathrm{d}r^2 + \Sigma^2\mathrm{d}\theta^2 \\ &\quad + \left[\left(r^2+a^2\right) + \frac{\left(2Mr-Q^2\right)a^2\sin^2\theta}{\Sigma^2}\right]\sin^2\theta\,\mathrm{d}\varphi^2 \\ &\quad - \frac{2\left(2Mr-Q^2\right)a\sin^2\theta}{\Sigma^2}\mathrm{d}t\mathrm{d}\varphi,\end{aligned} \tag{14}$$

where $\Delta \equiv r^2 - 2Mr + a^2 + Q^2$, $\Sigma^2 \equiv r^2 + a^2\cos^2\theta$, M, Q, and $a = J/M$ are the total mass, total charge, and angular momentum per unit mass of the black hole respectively.

The event horizon $r_\pm$ is given by

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2},$$

in which the $+/-$ sign corresponds to the outer/inner horizon respectively. The Hawking temperature $T_\pm$ is given by

$$T_\pm = \frac{r_+ - r_-}{4\pi(r_\pm{}^2 + a^2)}. \tag{15}$$

The entropy S of the black hole satisfies

$$S = \frac{1}{4}A = \pi(r_+^2 + a^2), \tag{16}$$

where A is the area of the outer event horizon.

Regarding a Kerr–Newman black hole as a black body system with temperature $T = T_+ = (r_+ - r_-)/4\pi(r_+^2 + a^2)$, we can obtain the 'volumes' as follows:

$$V_U \equiv \frac{U}{u} = \frac{M}{aT_+^4}, \tag{17}$$

and

$$V_S \equiv \frac{S}{s} = \frac{\pi(r_+^2 + a^2)}{4aT_+^3/3}. \tag{18}$$

Obviously, the two 'volumes' are not consistent with each other in the stationary circumstance.

Now we consider a dynamical process in which the black hole's energy changes by δM, while the charge Q and angular momentum J remain as constants. Similarly, we obtain

$$\widetilde{V}_U = \frac{\mathrm{d}U}{\mathrm{d}u} = \frac{\delta M}{4aT_+^3\delta T_+}, \tag{19}$$

and

$$\widetilde{V}_S = \frac{\mathrm{d}S}{\mathrm{d}s} = \frac{\pi\delta(r_+^2 + a^2)}{4aT_+^2\delta T_+}. \tag{20}$$

After the following calculation, we can draw the conclusion that the two 'volumes' are consistent with each other, i.e. $\widetilde{V}_U = \widetilde{V}_S$.

$$\begin{aligned} &\frac{\delta[\pi(r_+^2 + a^2)]}{\delta M} \\ &= \frac{\delta\left[\pi(2M^2 - Q^2 + 2M\sqrt{M^2 - J^2/M^2 - Q^2})\right]}{\delta M} \\ &= \frac{4\pi(r_+^2 + a^2)}{r_+ - r_-} = \frac{1}{T_+}, \end{aligned} \tag{21}$$

in which we have considered the fact that J and Q remain as constants.

4. Discussions and conclusions

Regarding a black hole as an ideal black body system, the thermal character of a black hole has been investigated deeply. We have found that the analogy between a black hole and a black body system is meaningful. We have investigated a Schwarzschild black hole and a Kerr–Newman black hole respectively. From two kinds of definition of black hole 'volume', V_U and V_S, we have obtained the following interesting conclusions.

When the 'volumes' are given in a static way, i.e. $V_U \equiv U/u$ and $V_S \equiv S/s$, they are not consistent with each other. However, if they are defined in a dynamic way, i.e. $\widetilde{V}_U = \frac{\mathrm{d}U}{\mathrm{d}u}$ and $\widetilde{V}_S = \frac{\mathrm{d}S}{\mathrm{d}s}$, they are smoothly consistent. A black hole is a special black body system, whose volume can be defined only in a dynamic way. In other words, it is more suitable that a 'dynamic' black hole is regarded as a black body than a 'static' one. A black hole should be inherently dynamical at least when it is taken as a black body system.

Moreover, there are several points we would like to emphasize, as follows.

(1) The so-called black hole volume in this paper is only a hypothetical quantity calculated via a general black body model. It is not the integral of the volume element on a black hole manifold.[20]

(2) In Schwarzschild circumstances, the black hole is described by only one parameter M, which also corresponds to the black body's only parameter — temperature. When defining the volume in a dynamic way, we only need to take the change of mass δM into account, which is also taken as the change of energy. However, on a Kerr–Newman black hole, which has three parameters — M, Q, and a, the total energy of the black hole is M too. Considering a dynamic process, we only let the mass change while the charge and the angular momentum remain as constants. While the change of mass is related to the thermal property of the black hole, the evolution exactly corresponds to the black body.

(3) As a black body system, a black hole has one peculiar difference from ordinary black bodies, that is, its heat capacity. The heat capacity of a Schwarzschild black hole is defined as

$$C_{\mathrm{BH}} \equiv \frac{\mathrm{d}E}{\mathrm{d}T_{\mathrm{BH}}} = \frac{\mathrm{d}M}{\mathrm{d}T_{\mathrm{BH}}} = -\frac{1}{8\pi T_{\mathrm{BH}}^2} = -8\pi M^2 < 0. \tag{22}$$

Therefore, a black hole is a system with negative heat capacity, which has some peculiar property. The most interesting thing is that when a black hole absorbs energy, its temperature will decrease reversely. It is evident that the equilibrium state is so unstable that even a tiny perturbation will break it down. So our analogy that a black hole is regarded as a black body in equilibrium is suitable only for a very ideal circumstance. However, the crude model gives us a more explicit physics picture to study black holes. To understand more about black hole thermodynamics, further study is needed on, for example, the establishment of a nonequilibrium thermodynamics model.

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