Generalized uncertainty principle and tunneling radiation of the $SAdS_5$ black hole*

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After considering the generalized uncertainty principle, we discuss the quantum tunneling radiation of a five-dimensional Schwarzschild anti de Sitter black hole. The radiation spectrum and the correction value of the Bekenstein–Hawking entropy are derived. In a five-dimensional black hole the one order correction term in the Bekenstein–Hawking entropy correction term is proportional to the third power of the area, and the logarithmic correction term is a two-order small quantity. The correction term is related to the dimension constant introduced in the generalized uncertainty principle. Because the black hole entropy is not divergent, the lowest value of the five-dimensional Schwarzschild anti de Sitter black hole horizon radius is obtained. After considering the generalized uncertainty principle, the radiation spectrum is still consistent with normalization theory.

Keywords: generalized uncertainty principle, correction of Bekenstein–Hawking entropy, tunneling radiation

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1. Introduction

Hawking[1] interpreted the quantum effect of the black hole as emitting a thermal radiant spectrum from an event horizon, which sets a milestone in black hole physics. The discovery of this effect not only solves the problem in black hole thermodynamics but also announces the relation between quantum mechanics, thermodynamics and gravitation. Discovering the thermodynamic properties of various black holes is an important subject of black hole physics. Hawking pointed out that vacuum fluctuations near the surface of the black hole would produce virtual particle pairs. When the virtual particles with negative energy go into the black hole via the tunnel effect, the energy of the black hole will decrease. At the same time, the particle with positive energy may thread out of the gravitation region outside the black hole. Equivalently, the black hole radiates a particle. Gibbons and Hawking[2] proved that the radiation spectrum of the black hole is a strict black body spectrum. This discovery is of positive significance for understanding and investigating star evolution, which greatly promotes the development of black hole thermodynamics. In the past decades, based on the research on black hole radiation and the property of black hole thermodynamics, the four laws of black hole thermodynamics are derived.[3,4] The thermal radiation of various black holes has been studied.[5−13] They all derived that the radiation spectrum of the black hole is an exact black body spectrum. There are two obvious disputes: the first is information loss. The Hawking radiation makes people realize that the black hole is no longer an ultimate state of a star. The black hole can evolve and finally disappear. The Hawking radiation spectrum is an exact black body spectrum and we can only derive a temperature parameter from a black body spectrum. Thus the black hole radiation will not bring any information on the inner matter of the black hole. This means that when the black hole has vaporized all information including unitary property will be lost. The information loss of the black hole means that the pure quantum state will decay to a mixed state. This violates the unitary principle in quantum mechanics. This challenges the foundation theory of quantum mechanics. The second is the re-

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action of the radiation to the spacetime. Because the black hole produces Hawking radiation, state parameters (energy and charge) describing the black hole will fluctuate. However, in the process of proving Hawking radiation, the effect of radiation on the black hole state parameters was not considered in the past. That is, the reaction of the radiation to spacetime is not considered. So only under the condition that the spacetime is invariable does Hawking radiation derive the purely thermal spectrum.

In 2000, Parikh and Wilczek put forward a semiclassical method for calculating the modified spectrum of the black hole Hawking radiation.[14] In this method, the black hole Hawking radiation is understood as a sort of quantum tunneling. The potential barrier is determined by the energy of emission particles. The key of this method is emphasizing energy conservation during the particle emission process and establishing a good coordinate system at the horizon. Using this method Parikh and Wilczek have calculated the radiation spectrum of particles when it traverse the horizons of a Schwarzschild and a Reissner–Nordström black hole. The result departs from the purely thermal spectrum. It satisfies the unitary principle and supports the result of information conservation. Subsequently, the Hawking radiation modified spectra of various black holes have been calculated.[15−42] The outgoing rate of the black hole radiation particle is

$$\Gamma = \exp[\Delta S], \tag{1}$$

where ΔS is the change of Bekenstein–Hawking (B–H) entropy before and after black hole radiation. The results all satisfy unitary theory and support the result of the information conservation in Hawking radiation. Recently, Refs. [43] and [44] started from Damour–Ruffini's method,[5] and also derived that the outgoing rate of the black hole radiation particle is Eq. (1).

The correction to the B–H entropy of a black hole has attracted researchers' attention. Various methods have been used to discuss the correction value to the black hole B–H entropy.[45−56] Many researchers think that the expression of the correction to the B–H entropy of a Schwarzschild black hole is

$$S = \frac{A}{4G} + \chi \ln \frac{A}{4G}, \tag{2}$$

where A is the area of the black hole horizon, χ is a dimensionless constant. However, the exact coefficient of the logarithmic term is not derived. Some references give $\chi = -3/2$, for example Ref. [46], some works lead to $\chi = -1/2$, for example Ref. [52]; the author of Ref. [53] argued that χ should be a positive integer and on the other hand, the author of Ref. [54] argued that χ should be equal to zero.

Because Hawking radiation of a black hole is a quantum effect, for the radiation particle or absorption particle we must consider the uncertainty principle. However, when gravity action exists, the Heisenberg uncertainty principle does not hold, and it can be replaced with the generalized uncertainty principle. Much attention has been paid to the correction to the entropy and radiation spectrum of the black hole due to the generalized uncertainty principle.[47−49,55] In this paper, considering the generalized uncertainty principle and applying it to the tunneling method, we discuss the radiation spectrum of a five-dimensional Schwarzschild anti de Sitter ($SAdS_5$) black hole. The radiation spectrum and the correction value of the Bekenstein–Hawking entropy of the $SAdS_5$ black hole are derived. Throughout the study, the units $G_5 = c = \hbar = 1$.

2. Tunneling property of particles at the black hole horizon

The metric of a $SAdS_5$ black hole is give by[49]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t_{\mathrm{s}}^2 + f^{-1}(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2, \tag{3}$$

where

$$f(r) = 1 - \frac{16\pi M}{3\Omega_3 r^2} + \frac{r^2}{l^2}, \tag{4}$$

Ω_3 is the metric of the unit S^3. The event horizon is located at $r = r_{\mathrm{h}}$; since $f(r_{\mathrm{h}}) = 0$, we have

$$r_{\mathrm{h}}(M) = \frac{l}{2}\sqrt{\frac{-1 + \sqrt{1 + \dfrac{64\pi M}{3l^2\Omega_3}}}{2}}. \tag{5}$$

The B–H entropy of the black hole is

$$S_{\mathrm{BH}} = \frac{1}{4}A_{\mathrm{h}} = \frac{1}{4}\Omega_3 r_{\mathrm{h}}^3. \tag{6}$$

The coordinate singularity at the horizon is removed by introducing Painleve coordinates.[57] Under the transformation

$$\mathrm{d}t_s = \mathrm{d}t - \frac{\sqrt{1 - f(r)}}{f(r)}\mathrm{d}r, \tag{7}$$

the metric (3) takes the form

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + 2\sqrt{1 - f(r)}\mathrm{d}t\mathrm{d}r$$

$$+\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{n+1}. \tag{8}$$

The basic idea of this method is to find the radial null geodesics for the metric (8):

$$\dot{r} \equiv \frac{\mathrm{d}r}{\mathrm{d}t} = \pm 1 - \sqrt{1 - f(r)}, \tag{9}$$

where $+(-)$ sign gives outgoing (ingoing) null radial geodesics. Using this one has to calculate the imaginary part of the action for a shell with energy ω. In the original work,[14] the imaginary part of the action is defined as,

$$\begin{aligned}\mathrm{Im}\, S &= \mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_{\mathrm{r}}\mathrm{d}r \\ &= \mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^{p_{\mathrm{r}}} \mathrm{d}p_r'\mathrm{d}r \\ &= \mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^{H} \frac{\mathrm{d}H'}{\dot{r}}\mathrm{d}r, \end{aligned}\tag{10}$$

where in the last step we multiply and divide the integrand by the two sides of Hamilton's equation $\dot{r} = \frac{\mathrm{d}H}{\mathrm{d}p_r}\Big|_r$.

Near the horizon one can expand $f(r)$ about the horizon r_{h}.

$$f(r) = f'(r_{\mathrm{h}})(r - r_{\mathrm{h}}) + o((r - r_{\mathrm{h}})^2). \tag{11}$$

Substituting these into (9) $\dot{r}$ can be approximately expressed as

$$\dot{r} \approx \frac{1}{2}f'(r_{\mathrm{h}})(r - r_{\mathrm{h}}). \tag{12}$$

The Hawking temperature and the energy of black hole are as[50]

$$T = \frac{2r_{\mathrm{h}}^2 + l^2}{2\pi l^2 r_{\mathrm{h}}} = \frac{f'(r_{\mathrm{h}})}{4\pi}, \tag{13}$$

$$E = \frac{3\Omega_3 r_{\mathrm{h}}^2}{16\pi}\left(1 + \frac{r_{\mathrm{h}}^2}{l^2}\right). \tag{14}$$

From Eqs. (13) and (12), we have

$$\dot{r} \approx 2\pi T(M)(r - r_{\mathrm{h}}). \tag{15}$$

Now taking into account the self-gravitation effects,[58] the above integration can be expressed as

$$\begin{aligned}\mathrm{Im}\, S &= \mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_M^{M-\omega} \frac{\mathrm{d}H'}{\dot{r}}\mathrm{d}r \\ &= -\mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^{\omega} \frac{\mathrm{d}\omega'}{\dot{r}}\mathrm{d}r, \end{aligned}\tag{16}$$

where we used the fact that the Hamiltonian $H = M - \omega$, with M being the original mass of the black hole. Here $\dot{r}$ can be approximated by (15) as follows

$$\dot{r} \approx 2\pi(r - r_{\mathrm{h}})T(M - \omega), \tag{17}$$

where r_{h} is the modified SAdS_5 radius and $T(M-\omega)$ is the modified Hawking radiation temperature. Substituting Eq.(17) into Eq.(16), we obtain

$$\mathrm{Im}\, S = -\mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^{\omega} \frac{\mathrm{d}\omega'}{2\pi(r - r_{\mathrm{h}})T(M - \omega')}\mathrm{d}r. \tag{18}$$

But now the integration over r can be done by deforming the contour. Guaranteeing that the positive energy solutions decay in time we have after r integration[14]

$$\mathrm{Im}\, S = \pi\int_0^{\omega} \frac{\mathrm{d}\omega'}{2\pi T(M - \omega')}. \tag{19}$$

After considering the generalized uncertainty principle, the Hawking radiation temperature of the black hole is[50]

$$T'(M) = \left(1 - \frac{\alpha^2}{4r_{\mathrm{h}}^2}\right)T(M), \tag{20}$$

where α is a dimensionless constant of order one.

Substituting Eq.(20) into Eq.(19), we derive the imaginary part of the action after considering the generalized uncertainty principle

$$\begin{aligned}\mathrm{Im}\, S &= \int_0^{\omega} \frac{\mathrm{d}\omega'}{2T'(M - \omega')} \\ &= \int_0^{\omega} \frac{2r_{\mathrm{h}}^2(M - \omega')\mathrm{d}\omega'}{(4r_{\mathrm{h}}^2(M - \omega') - \alpha^2)T(M - \omega')}. \end{aligned}\tag{21}$$

Based on Eq.(14), we have

$$\mathrm{d}\omega' = -\frac{3\Omega_3 r_{\mathrm{h}}}{8\pi}\left(1 + \frac{2r_{\mathrm{h}}^2}{l^2}\right)\mathrm{d}r_{\mathrm{h}}. \tag{22}$$

Thus Eq.(21) can be rewritten as

$$\begin{aligned}\mathrm{Im}\, S &= -\int_{r_{\mathrm{h}}(M)}^{r_{\mathrm{h}}(M-\omega)} \frac{3\Omega_3 r_{\mathrm{h}}^4\mathrm{d}r_{\mathrm{h}}}{2(4r_{\mathrm{h}}^2 - \alpha^2)} = -\int_{r_{\mathrm{h}}(M)}^{r_{\mathrm{h}}(M-\omega)} \frac{3\Omega_3 r_{\mathrm{h}}^2}{8}\left(1 + \frac{\alpha^2}{4r_{\mathrm{h}}^2 - \alpha^2}\right)\mathrm{d}r_{\mathrm{h}} \\ &= -\left[\frac{3\Omega_3}{8}\left(\frac{r_{\mathrm{h}}^3}{3} + \frac{\alpha^2}{4}r_{\mathrm{h}} + \frac{\alpha^3}{16}\ln\frac{2r_{\mathrm{h}} - \alpha}{2r_{\mathrm{h}} + \alpha}\right)\right]_{r_{\mathrm{h}}=r_{\mathrm{h}}(M)}^{r_{\mathrm{h}}=r_{\mathrm{h}}(M-\omega)}. \end{aligned}\tag{23}$$

Now according to the WKB-approximation method the tunneling probability is given by

$$\Gamma \sim \mathrm{e}^{-2\mathrm{Im}\,S}. \tag{24}$$

From Eq.(23), considering the generalized uncertainty principle the probability of a particle through the potential barrier is

$$\begin{aligned}\Gamma = & \exp\left[-\frac{3\Omega_3}{4}\left(\frac{r_{\mathrm{h}}^3(M)}{3}+\frac{\alpha^2}{4}r_{\mathrm{h}}(M)+\frac{\alpha^3}{16}\ln\frac{2r_{\mathrm{h}}(M)-\alpha}{2r_{\mathrm{h}}(M)+\alpha}\right)\right] \\ & \times\exp\left[\frac{3\Omega_3}{4}\left(\frac{r_{\mathrm{h}}^3(M-\omega)}{3}+\frac{\alpha^2}{4}r_{\mathrm{h}}(M-\omega)+\frac{\alpha^3}{16}\ln\frac{2r_{\mathrm{h}}(M-\omega)-\alpha}{2r_{\mathrm{h}}(M-\omega)+\alpha}\right)\right].\end{aligned} \tag{25}$$

3. Correction to B–H entropy after considering the generalized uncertainty principle

It is known[14] that the change in the B–H entropy due to tunneling through the horizon is related to $\mathrm{Im}\,S$ by the following relation

$$\Delta S_{\mathrm{bh}} = -2\mathrm{Im}\,S. \tag{26}$$

Therefore the corrected change in B–H entropy is

$$\begin{aligned}\Delta S_{\mathrm{bh}} = & \left[-\frac{3\Omega_3}{4}\left(\frac{r_{\mathrm{h}}^3(M)}{3}+\frac{\alpha^2}{4}r_{\mathrm{h}}(M)+\frac{\alpha^3}{16}\ln\frac{2r_{\mathrm{h}}(M)-\alpha}{2r_{\mathrm{h}}(M)+\alpha}\right)\right] \\ & +\left[\frac{3\Omega_3}{4}\left(\frac{r_{\mathrm{h}}^3(M-\omega)}{3}+\frac{\alpha^2}{4}r_{\mathrm{h}}(M-\omega)+\frac{\alpha^3}{16}\ln\frac{2r_{\mathrm{h}}(M-\omega)-\alpha}{2r_{\mathrm{h}}(M-\omega)+\alpha}\right)\right].\end{aligned} \tag{27}$$

Equation (26) is the change of B–H entropy after considering the generalized uncertainty principle.

$$S_{\mathrm{final}}(M-\omega)-S_{\mathrm{initial}}(M)=\Delta S_{\mathrm{bh}}. \tag{28}$$

So the B–H entropy of the black hole with mass M is

$$S_{\mathrm{bh}}=\frac{\Omega_3}{4}\left(r_{\mathrm{h}}^3+\frac{3\alpha^2}{4}r_{\mathrm{h}}+\frac{\alpha^3}{16}\ln\frac{2r_{\mathrm{h}}(M)-\alpha}{2r_{\mathrm{h}}(M)+\alpha}\right). \tag{29}$$

4. Conclusion

Based on $\alpha^3\Omega_3/64\ln[(2r_{\mathrm{h}}-\alpha)/(2r_{\mathrm{h}}+\alpha)]$ in the correction term of entropy, whether the dimensional factor α is positive or negative, the correction terms are all positive. According to the convergence of the logarithmic term, the radius of the SAdS_5 black hole horizon satisfies the inequality $2r_{\mathrm{h}} > |\alpha|$. Whether this radius is the lowest radius of the SAdS_5 black hole will be studied. Because α introduced in the generalized uncertainty principle is a small quantity, after considering the generalized uncertainty principle, the one-order small quantity correction term in the correction to B–H entropy of the SAdS_5 black hole is positive, which is proportional to the third power of the area. The logarithmic correction term is a two-order small quantity. The correction term is related to the dimension constant introduced in the generalized uncertainty principle. After considering the generalized uncertainty principle, the radiation spectrum still corresponds to normalization theory.

Based on Eq.(29), the correction terms to B–H entropy of a SAdS_5 black hole are positive, whether they are order one or order two. However, the correction term given by Ref. [49] is negative. The coefficients of the correction to B–H entropy depend on α introduced in the generalized uncertainty principle. If we can determine α, the coefficients of the correction to B–H entropy will be determined.

To determine the coefficients of the correction to B–H entropy of the SAdS_5 black hole, we need consider many factors, such as thermal fluctuation and the reaction of the radiation. In this paper, we only derive the correction to entropy of the black hole due to the generalized uncertainty principle. We provide a method for discussing the correction to the B–H entropy of a higher-dimensional black hole considering every factor.

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