One-loop renormalizability of noncommutative U(1) gauge theory with scalar fields*

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(Received 24 February 2009; revised manuscript received 20 March 2009)

This paper uses the background field method to calculate one-loop divergent corrections to the gauge field propagators in noncommutative U(1) gauge theory with scalar fields. It shows that for a massless scalar field, the gauge field propagators are renormalizable to θ^2-order, but for a massive scalar field they are renormalizable only to θ-order.

Keywords: noncommutative geometry, quantum field theory, renormalization

PACC: 0370, 1110G, 1117

1. Introduction

The discovery of noncommutative (NC) structures in string theory put noncommutative field theories into the focus of vigorous investigation in the last few years.[1−4] Theoretical and experimental aspects of NC theories have been studied extensively.[5−10] To construct a field theory on NC space, we can follow Weyl's prescription of quantum phase space.[5] The most common NC spacetime is defined by the following canonical structure:

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\,\theta^{\mu\nu}, \tag{1}$$

where $\hat{x}^\mu, \hat{x}^\nu$ are NC coordinates and $\theta^{\mu\nu}$ are constant real numbers relating to the NC scale. In geometry, the Gelfand–Naimark theorem tells us that the algebra of a manifold can be discussed through the algebra of functions on it with a certain algebraic multiplication. So Eq. (1) can be replaced by algebra of functions on ordinary spacetime with the Moyal–Weyl star product which is defined as

$$f \star g(x) = \exp\left(\frac{\mathrm{i}}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right) f(x)g(y)|_{y\to x}. \tag{2}$$

Then, we can get

$$[\hat{x}^\mu, \hat{x}^\nu] = [x^\mu, \star x^\nu] = \mathrm{i}\,\theta^{\mu\nu}. \tag{3}$$

The action of a field theory on NC spacetime can be obtained by replacing the ordinary multiplication with the Moyal–Weyl star product. However, to construct a gauge theory on NC spacetime is not so straightforward. In a general NC gauge theory, the gauge algebra cannot be closed in Lie algebra because of the noncommutativity of the Moyal–Weyl star product, and the NC gauge theory has to be extended to an enveloping algebra-valued one.[11−13] Based on the studies of gauge theories in the context of string theory, Seiberg and Witten[4] argued that there is a map (S–W map) between the NC gauge fields and commutative ones. The NC gauge fields can be expressed by the corresponding commutative gauge fields and their derivatives. This map can be determined by the equivalence between NC gauge transformation and commutative gauge transformation.[14,15]

Renormalizability is an important property of a field theory. Whether NC field theories are acceptable physical theories, renormalizability properties of these theories should be considered. In perturbation theory, non-planar diagrams yield UV/IR mixing.[16−18] This well-known phenomenon leads NC field theories to appear non-renormalizable. However, many authors have shown that NC gauge theories are renormalizable by using the θ-expanded approach which is based on S–W maps.[19−24] For pure U(1) gauge theory, the field propagator is renormalizable to all orders in θ at the one-loop level,[19] where the non-uniqueness of S–W maps provides additional freedom to absorb the divergent terms. For NC quantum electrodynamics (QED), it was proved that the theory is one-loop renormalizable to first order in θ except a four fermionic vertex in Ref. [25], and it was demonstrated that the propagators are renormalizable (θ^2-order) only when the fermions are massless in Ref. [21]. The renormaliz-

*Project supported by the National Natural Science Foundation of China (Grant No. 90303003).

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ability of the U(1) gauge field coupled with a scalar field was discussed in Ref. [26], where the gauge sector is renormalizable only to first order in θ. So, it is natural to consider the renormalizability of the U(1) gauge field coupled with a scalar field to θ^2 order. In Ref. [27], the authors used the free parameters of the S–W map to achieve one-loop renormalizability of the gauge sector of the NC standard model. It is interesting to check whether this procedure is applicable to gauge theory with matter fields.

In this paper, we consider an NC U(1) field coupled with a complex scalar field in the θ-expanded approach, discuss the renormalizability of a gauge field propagator to θ^2 order and check whether the free parameters of the S–W map can render the divergences renormalizable. In Section 2, we review the general S–W map and expand the NC action to first order in θ. In Section 3, the divergent one-loop corrections are computed and the renormalizability is discussed. Section 4 is for the conclusions.

2. θ-expanded action

In order to get the general θ-expanded action, we first review the S–W map. Let A_μ, Φ be the NC fields and Λ be the NC gauge transformation parameter. The commutative counterparts are a_μ, ϕ, λ. The NC infinitesimal gauge transformation is

$$\begin{aligned}\delta_\Lambda A_\mu &= \partial_\mu \Lambda + \mathrm{i}[\Lambda, \star A_\mu],\\ \delta_\Lambda \Phi &= \mathrm{i}\,\Lambda \star \Phi.\end{aligned} \tag{4}$$

The S–W maps, $A_\mu = A_\mu(a_\mu)$, $\Phi = \Phi(\phi, a_\mu)$, $\Lambda = \Lambda(\lambda, a_\mu)$, are defined by equivalence between commutative and NC gauge transformations in the following sense:

$$\begin{aligned}\delta_\Lambda A_\mu &= A_\mu(a_\mu + \delta_\Lambda a_\mu) - A_\mu(a_\mu),\\ \delta_\Lambda \Phi &= \Phi(a_\mu + \delta_\Lambda a_\mu, \phi + \delta_\Lambda \phi) - \Phi(a_\mu, \phi).\end{aligned} \tag{5}$$

Considering two infinitesimal gauge transformations, one can get an equation for gauge transformation parameters

$$\delta_{\lambda_1}\Lambda_2 - \delta_{\lambda_2}\Lambda_1 - \mathrm{i}[\Lambda_1, \star\Lambda_2] + \mathrm{i}\,\Lambda_{[\lambda_1,\lambda_2]} = 0. \tag{6}$$

Using the above equations, one can get the solution of the S–W map up to the desired order in θ.[15,28,29] For NC U(1) gauge theory, the S–W maps of first order in θ are given by

$$\Lambda = \lambda + \frac{1}{2}\theta^{\mu\nu}\partial_\mu\lambda\, a_\nu, \tag{7}$$

$$\Phi = \phi - \frac{1}{2}\theta^{\mu\nu}a_\mu\partial_\nu\phi + \frac{1}{2}c\,\theta^{\mu\nu}f_{\mu\nu}\phi, \tag{8}$$

$$\begin{aligned}A_\mu = a_\mu &- \frac{1}{2}\theta^{\alpha\beta}a_\alpha(f_{\beta\mu} + \partial_\beta a_\mu)\\ &+ z\theta^{\alpha\beta}\partial_\mu\partial_\alpha a_\beta + w\theta_{\mu\alpha}\partial_\beta f^{\alpha\beta},\end{aligned} \tag{9}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the commutative gauge field strength. We introduce a complex number $c = a + \mathrm{i}\,b$ and real numbers z, w to represent the degree of freedom of the S–W map at the first order in θ.

The classical action of scalar QED on NC space is

$$\begin{aligned}S = \int \mathrm{d}^4x\Big(&-\frac{1}{4}F^{\mu\nu}\star F_{\mu\nu}\\ &+ (D_\mu\Phi)^*\star D^\mu\Phi - m^2\Phi^*\star\Phi\Big),\end{aligned} \tag{10}$$

where the NC covariant derivative $D_\mu\Phi$ and the NC field strength $F_{\mu\nu}$ are defined as

$$D_\mu\Phi = \partial_\mu\Phi - \mathrm{i}\,A_\mu\star\Phi, \tag{11}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, \star A_\nu]. \tag{12}$$

Using S–W maps Eqs. (8) and (9), one can write the NC covariant derivative of Φ as

$$\begin{aligned}D_\mu\Phi = \partial_\mu\phi - \mathrm{i}\,a_\mu\phi &- \frac{1}{2}\partial_\mu(\theta^{\alpha\beta}a_\alpha\partial_\beta\phi) + \frac{1}{2}c\theta^{\alpha\beta}\partial_\mu(f_{\alpha\beta}\phi) + \frac{1}{2}\theta^{\alpha\beta}\partial_\alpha a_\mu\partial_\beta\phi + \frac{\mathrm{i}}{2}\theta^{\alpha\beta}a_\mu a_\alpha\partial_\beta\phi - \frac{\mathrm{i}}{2}c\theta^{\alpha\beta}a_\mu f_{\alpha\beta}\phi\\ &+ \frac{\mathrm{i}}{2}\theta^{\alpha\beta}a_\alpha(f_{\beta\mu} + \partial_\beta a_\mu)\phi - \mathrm{i}\,z\theta^{\alpha\beta}\partial_\mu\partial_\alpha a_\beta\phi - \mathrm{i}\,w\theta^{\mu\alpha}\partial^\beta f_{\alpha\beta}\phi.\end{aligned} \tag{13}$$

Then, one can get the θ-expanded classical action to first order in θ,

$$\begin{aligned}S = \int \mathrm{d}^4x\Big[&-\frac{1}{4}f^{\mu\nu}f_{\mu\nu} + (D_\mu\phi)^*D^\mu\phi - m^2\phi^*\phi - \frac{1}{2}\theta^{\mu\nu}f_{\alpha\mu}f_{\beta\nu}f^{\alpha\beta} + \frac{1}{8}\theta^{\alpha\beta}f_{\alpha\beta}f_{\mu\nu}f^{\mu\nu}\\ &+ \left(\frac{1}{2}\theta^{\alpha\beta}f_{\alpha\mu}(D^\mu\phi)^*D_\beta\phi + \mathrm{i}\,w\theta^{\mu\alpha}\partial^\beta f_{\alpha\beta}(D_\mu\phi)\phi^* + \mathrm{C.C.}\right) + \left(\left(c - \frac{1}{8}\right)\theta^{\alpha\beta}f_{\alpha\beta}(D^\mu\phi)^*D_\mu\phi\right.\\ &+ \left.\left(c - \mathrm{i}\,\frac{z}{2}\right)\theta^{\alpha\beta}\partial_\mu f_{\alpha\beta}(D^\mu\phi)^*\phi + \mathrm{C.C.}\right) + \left(\frac{1}{4} - a\right)m^2\theta^{\alpha\beta}f_{\alpha\beta}\phi^*\phi\Big],\end{aligned} \tag{14}$$

where $D_\mu\phi = \partial_\mu\phi - \mathrm{i}\,a_\mu\phi$ is the commutative covariant derivative of scalar field ϕ, and C.C. represents complex conjugation.

3. One-loop divergent terms

Using the background field method and dimensional regularization,[30,31] we calculate the divergent parts of the 2-point function of the U(1) gauge field at the one-loop level. The contributions to the divergent terms come from both the gauge sector and the scalar field sector. The gauge sector contributions are renormalizable.[19,21] So we only need to consider the contributions from the scalar field. For a quantum theory, we should add the gauge fixing term

$$S_{gf} = -\frac{1}{2}\int \mathrm{d}^4x(\partial_\mu a^\mu)^2$$

and ghost field term $\int \mathrm{d}^4x\bar{C}\partial^2 C$ in the classical action. But these two terms are not related to our calculation.

To carry out the calculation, we express the complex scalar field as two real scalar fields $\phi = \phi_1 + i\phi_2$ and then work out terms quadratic in fields ϕ_1, ϕ_2 from the action. These terms can be written as

$$L_{\phi^2} = \phi_i(B_{ij}^{(0,0)} + B_{ij}^{(1,0)} + B_{ij}^{(2,0)} + B_{ij}^{(1,1)} + B_{ij}^{(2,1)} + B_{ij}^{(3,1)})\phi_j, \tag{15}$$

where $i, j = 1, 2$ and $B_{ij}^{(m,n)}$ denote terms containing m factors of a_μ and n factors of θ. Writing these terms explicitly according to Eq. (14), we have

$$B_{ij}^{(0,0)} = \delta_{ij}(\Box - m^2), \tag{16}$$

$$B_{ij}^{(1,0)} = \epsilon_{ij}(\theta^{\alpha\beta}\overleftarrow{\partial_\alpha}a^\alpha - a^\alpha\partial_\alpha), \tag{17}$$

$$B_{ij}^{(2,0)} = \delta_{ij}(a^\alpha a_\alpha), \tag{18}$$

$$\begin{aligned} B_{ij}^{(1,1)} = {} & \delta_{ij}(\theta^{\alpha\beta}\overleftarrow{\partial_\mu}\partial_\alpha a^\mu\partial_\beta - \theta^{\alpha\beta}\overleftarrow{\partial_\mu}\partial^\mu a_\alpha\partial_\beta + \left(4a - \frac{1}{2}\right)\theta^{\alpha\beta}\overleftarrow{\partial_\mu}\partial_\alpha a_\beta\partial^\mu \\ & + 4a\theta^{\alpha\beta}\partial_\mu\partial_\alpha a_\beta\partial^\mu + \left(\frac{1}{2} - 2a\right)m^2\theta^{\alpha\beta}\partial_\alpha a_\beta) \\ & + \epsilon_{ij}((4b - 2z)\theta^{\alpha\beta}\partial_\mu\partial_\alpha a_\beta\partial^\mu - 2w\theta^{\mu\alpha}\partial_\alpha\partial^\beta a_\beta\partial_\mu + 2w\theta^{\mu\alpha}\partial^2 a_\alpha\partial_\mu), \end{aligned} \tag{19}$$

$$\begin{aligned} B_{ij}^{(2,1)} = {} & \delta_{ij}[(2z - 4b)\theta^{\alpha\beta}\partial_\mu\partial_\alpha a_\beta a^\mu + 2w\theta^{\alpha\beta}\partial_\beta\partial^\mu a_\mu a_\alpha - 2w\theta^{\alpha\beta}\partial^2 a_\beta a_\alpha] \\ & + \epsilon_{ij}\left[\theta^{\alpha\beta}\overleftarrow{\partial^\mu}f_{\alpha\mu}a_\beta - \theta^{\alpha\beta}f_{\alpha\mu}a^\mu\partial_\beta + \left(4a - \frac{1}{2}\right)\theta^{\alpha\beta}\overleftarrow{\partial^\mu}\partial_\alpha a_\beta a_\mu \right. \\ & \left. - \left(4a - \frac{1}{2}\right)\theta^{\alpha\beta}\partial_\alpha a_\beta a^\mu\partial_\mu\right], \end{aligned} \tag{20}$$

$$B_{11}^{(3,1)} = \theta^{\alpha\beta}f_{\alpha\mu}a^\mu a_\beta + \left(4a - \frac{1}{2}\right)\theta^{\alpha\beta}\partial_\alpha a_\beta a_\mu a^\mu, \tag{21}$$

where $\Box = -\partial^2$ and $\epsilon_{12} = 1$.

The one-loop scalar field contributions for the gauge field propagators can be evaluated by functional integral as

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\,\Gamma_\phi^1[a_\mu]} &= \int \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp\left(\mathrm{i}\int \mathrm{d}^4x L_{\phi^2}\right) \\ &= [\det(B_{ij}^{(0,0)} + B_{ij}^{(1,0)} + B_{ij}^{(2,0)} + B_{ij}^{(1,1)} + B_{ij}^{(2,1)} + B_{ij}^{(3,1)})]^{-1/2}. \end{aligned} \tag{22}$$

Then

$$\begin{aligned} \Gamma_\phi^1[a_\mu] &= \frac{\mathrm{i}}{2}\ln\det(B_{ij}^{(0,0)} + B_{ij}^{(1,0)} + B_{ij}^{(2,0)} + B_{ij}^{(1,1)} + B_{ij}^{(2,1)} + B_{ij}^{(3,1)}) \\ &= \frac{\mathrm{i}}{2}\mathrm{Tr}\ln(\Box - m^2 + B_{ij}^{(1,0)} + B_{ij}^{(2,0)} + B_{ij}^{(1,1)} + B_{ij}^{(2,1)} + B_{ij}^{(3,1)}) \\ &= \frac{\mathrm{i}}{2}\mathrm{Tr}\ln(\mathcal{I} + \Box^{-1}(-m^2 + B_{ij}^{(1,0)} + B_{ij}^{(2,0)} + B_{ij}^{(1,1)} + B_{ij}^{(2,1)} + B_{ij}^{(3,1)})) \\ &\quad + \frac{\mathrm{i}}{2}\mathrm{Tr}\ln\Box. \end{aligned} \tag{23}$$

In the above equation, the last term does not involve gauge fields and yields a constant to the one-loop effective action, so we can ignore it and the first functional determinant can be expanded by using a power

series

$$\begin{aligned}\Gamma^1_\phi =&\frac{\mathrm{i}}{2}\mathrm{Tr}\ln(\mathcal{I}+\Box^{-1}(-m^2+B^{(1,0)}_{ij}+B^{(2,0)}_{ij}+B^{(1,1)}_{ij}+B^{(2,1)}_{ij}+B^{(3,1)}_{ij}))\\ =&\frac{\mathrm{i}}{2}\mathrm{Tr}\sum_{i=1}^{\infty}\frac{(-1)^{n+1}}{n}(\Box^{-1}(-m^2+B^{(1,0)}_{ij}+B^{(2,0)}_{ij}+B^{(1,1)}_{ij}+B^{(2,1)}_{ij}+B^{(3,1)}_{ij}))^n. \end{aligned}\tag{24}$$

We will discuss the divergent parts of the gauge field 2-point functions, so we only need to consider terms which are quadratic in a_μ in the above expansions. Since each $\Box^{-1}$ gives a contribution of $1/p^2$ in momentum space, terms are convergent when n is large enough.

The divergent parts, to θ^1-order, will be considered first. For $n=1$, there are two terms which may give divergence: $B^{(2,0)}, B^{(2,1)}$. In fact, there are no divergences from these two terms. For $n=2$, the nonvanishing divergent parts from the trace are

$$\mathrm{Tr}(\Box^{-1}B^{(1,0)}_{ij}\Box^{-1}B^{(1,0)}_{ij})=\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{2}{3}a_\alpha(k^\alpha k^\beta-g^{\alpha\beta}k^2)a_\beta,\tag{25}$$

$$\mathrm{Tr}(-\Box^{-1}m^2\Box^{-1}B^{(2,0)}_{ij}-\Box^{-1}B^{(2,0)}_{ij}\Box^{-1}m^2)=\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}(-4m^2)a_\mu a^\mu,\tag{26}$$

$$\mathrm{Tr}(-\Box^{-1}m^2\Box^{-1}B^{(2,1)}_{ij}-\Box^{-1}B^{(2,1)}_{ij}\Box^{-1}m^2)=\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}4m^2(2z-4b-2w)\theta^{\alpha\beta}a_\mu k^\mu k_\alpha a_\beta.\tag{27}$$

For $n=3$, the divergent corrections are

$$\begin{aligned}&\mathrm{Tr}(-\Box^{-1}m^2(\Box^{-1}B^{(1,0)}_{ij})^2-\Box^{-1}B^{(1,0)}_{ij}\Box^{-1}m^2\Box^{-1}B^{(1,0)}_{ij}-(\Box^{-1}B^{(1,0)}_{ij})^2\Box^{-1}m^2)\\ =&\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}(-6m^2)a_\mu a^\mu,\end{aligned}\tag{28}$$

$$\begin{aligned}&\mathrm{Tr}(-\Box^{-1}m^2\Box^{-1}B^{(1,0)}_{ij}\Box^{-1}B^{(1,1)}_{ij}-\Box^{-1}B^{(1,0)}_{ij}\Box^{-1}m^2\Box^{-1}B^{(1,1)}_{ij}\\ &-\Box^{-1}B^{(1,0)}_{ij}\Box^{-1}B^{(1,1)}_{ij}\Box^{-1}m^2-\Box^{-1}m^2\Box^{-1}B^{(1,1)}_{ij}\Box^{-1}B^{(1,0)}_{ij}\\ &-\Box^{-1}B^{(1,1)}_{ij}\Box^{-1}m^2\Box^{-1}B^{(1,0)}_{ij}-\Box^{-1}B^{(1,1)}_{ij}\Box^{-1}B^{(1,0)}_{ij}m^2\Box^{-1})\\ =&\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}12m^2(z-w-2b)\theta^{\alpha\beta}a_\mu k^\mu k_\alpha a_\beta.\end{aligned}\tag{29}$$

For $n=4,5,\ldots$, no divergent terms appear. Summing the above divergent corrections, one can check that the divergent parts are θ^0-order and there are no θ^1-order corrections. The θ^0-order divergences can be absorbed by gauge field strength renormalization. The gauge field propagator is renormalizable up to θ^1-order no matter whether the scalar field is massive or massless. This is compatible with the conclusion in Ref. [26].

Then, we consider the divergent corrections at θ^2-order. One can see that these divergent corrections come from the terms: $(-\Box^{-1}m^2)^k(\Box^{-1}B^{(1,1)}_{ij})^2$ $(k=0,1,\ldots)$. If the scalar field is massless, only one term $(\Box^{-1}B^{(1,1)}_{ij})^2$ needs to be considered. From Eq. (19) with $m=0$, one can get

$$\mathrm{Tr}((\Box^{-1}B^{(1,1)}_{ij})^2)=2\mathrm{Tr}(\Box^{-1}B^{(1,1)}_{11}\Box^{-1}B^{(1,1)}_{11}-\Box^{-1}B^{(1,1)}_{12}\Box^{-1}B^{(1,1)}_{12}),\tag{30}$$

and

$$\begin{aligned}\mathrm{Tr}(\Box^{-1}B^{(1,1)}_{11}\Box^{-1}B^{(1,1)}_{11})=&\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\bigg[\frac{1}{48}(\theta^2)^{\alpha\beta}a_\alpha k^6a_\beta\\ &+\frac{1}{240}(\theta^2)^{\alpha\beta}a_\mu k^4k_\alpha k_\beta a^\mu\frac{1}{60}(\theta^2)^{\alpha\beta}a_\mu k^\mu k^2k_\alpha k_\beta k^\nu a_\nu\\ &-\frac{1}{24}(\theta^2)^{\alpha\beta}a_\alpha k_\beta k^4k^\mu a_\mu+\frac{17}{240}\theta^{\alpha\beta}\theta^{\mu\nu}a_\beta k_\alpha k^4k_\mu a_\nu\bigg],\end{aligned}\tag{31}$$

$$\begin{aligned}\mathrm{Tr}(\Box^{-1}B^{(1,1)}_{12}\Box^{-1}B^{(1,1)}_{12})=&\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\bigg[\frac{w^2}{3}((\theta^2)^{\alpha\beta}a_\alpha k^6a_\beta\\ &+(\theta^2)^{\alpha\beta}a_\mu k^\mu k^2k_\alpha k_\beta k^\nu a_\nu-2(\theta^2)^{\alpha\beta}a_\alpha k_\beta k^4k^\mu a_\mu)\\ &\times\bigg(\frac{2w^2}{3}+(2b-z)(2b-z+2w)\bigg)\theta^{\alpha\beta}\theta^{\mu\nu}a_\beta k_\alpha k^4k_\mu a_\nu\bigg],\end{aligned}\tag{32}$$

where $(\theta^2)^{\mu\nu} = \theta^{\alpha\mu}\theta^{\beta\nu}g_{\alpha\beta}$.

It is known that the S–W map is not unique. At each order in θ, it allows freedom of field redefinitions.[19] Let $A_\mu^{(n)}$ be a solution of the S–W map at θ^n-order. Then, a gauge covariant term $\mathcal{A}_\mu^{(n)}$ of exactly θ^n-order can be added to $A_\mu^{(n)}$ to form another solution. These field redefinitions add an additional terms to the original action, which can be used to absorb divergences,

$$\Delta S = \int \mathrm{d}^4x \mathrm{tr}(D_\mu F^{\mu\nu}\mathcal{A}_\nu^{(n)}). \tag{33}$$

At θ^2-order, there are four independent terms in the U(1) gauge field redefinitions, which lead to four additional terms in the original action. In momentum space, these additional terms can be expressed as

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\theta^{\alpha\beta}\theta_{\alpha\beta}a_\mu(k^2g^{\mu\nu}-k^\mu k^\nu)k^4a_\nu, \tag{34}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\theta^{\alpha\beta}\theta^{\mu\nu}a_\mu k_\nu k^4k_\beta a_\alpha, \tag{35}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}(\theta^2)^{\alpha\beta}(a_\mu k^4k_\alpha k_\beta a^\mu - a_\mu k^\mu k^2k_\alpha k_\beta k^\nu a_\nu), \tag{36}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}[(\theta^2)^{\alpha\beta}a_\alpha k^6a_\beta + (\theta^2)^{\alpha\beta}a_\mu k^\mu k^2k_\alpha k_\beta k^\nu a_\nu - 2(\theta^2)^{\alpha\beta}a_\alpha k_\beta k^4k^\mu a_\mu]. \tag{37}$$

To see whether the θ^2-order corrections are renormalizable, one should compare the divergent corrections from $\mathrm{Tr}(\Box^{-1}B_{11}^{(1,1)}\Box^{-1}B_{11}^{(1,1)}) - \mathrm{Tr}(\Box^{-1}B_{12}^{(1,1)}\Box^{-1}B_{12}^{(1,1)})$ with Eqs. (34)–(37). It is easy to check that these divergent corrections are all of the same forms as Eqs.(34)–(37). So, the gauge field 2-point functions are one loop renormalizable at θ^2 order when they are coupled with a massless scalar field.

Then we consider the case with massive scalar field. For $n = 2$, we just need to consider the terms containing mass m in $\mathrm{Tr}((\Box^{-1}B_{ij}^{(1,1)})^2)$. The divergent corrections from these terms are

$$\begin{aligned}\mathrm{Tr}((\Box^{-1}B_{ij}^{(1,1)})^2)_{\mathrm{mass}} = \frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\Bigg[&\frac{1}{3}\left(2a-\frac{1}{2}\right)m^2(\theta^{\alpha\beta}\theta^{\mu\nu}a_\mu k_\nu k^4k_\beta a_\alpha)\\ &+2\left(2a-\frac{1}{2}\right)^2m^4(\theta^{\alpha\beta}\theta^{\mu\nu}a_\mu k_\nu k^4k_\beta a_\alpha)\Bigg].\end{aligned} \tag{38}$$

For $n = 3$, the divergent corrections come from the terms $-m^2\Box^{-1}(\Box^{-1}B_{ij}^{(1,1)})^2$,

$$\begin{aligned}&\mathrm{Tr}(-m^2\Box^{-1}(\Box^{-1}B_{ij}^{(1,1)})^2 - (\Box^{-1}B_{ij}^{(1,1)})^2m^2\Box^{-1} - \Box^{-1}B_{ij}^{(1,1)}\Box^{-1}m^2\Box^{-1}B_{ij}^{(1,1)})\\ &= \frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}m^2\Bigg[\frac{1}{8}(\theta^2)^{\alpha\beta}a_\mu k^2k_\alpha k_\beta a^\mu + \frac{1}{4}(\theta^2)^{\alpha\beta}a_\mu k^\mu k_\alpha k_\beta k^\nu a_\nu\\ &\quad+\frac{1}{4}(\theta^2)^{\alpha\beta}a_\alpha k^4a_\beta - \frac{3}{4}(\theta^2)^{\alpha\beta}a_\alpha k_\beta k^2k^\mu a_\mu + \left(2a-\frac{3}{8}\right)\theta^{\alpha\beta}\theta^{\mu\nu}a_\mu k_\nu k^2k_\beta a_\alpha\\ &\quad+48a\left(2a-\frac{1}{2}\right)m^2\theta^{\alpha\beta}\theta^{\mu\nu}a_\mu k_\nu k_\beta a_\alpha\Bigg] + \frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}6m^2[(-w^2)((\theta^2)^{\alpha\beta}a_\alpha k^4a_\beta\\ &\quad+(\theta^2)^{\alpha\beta}a_\mu k^\mu k_\alpha k_\beta k^\nu a_\nu - 2(\theta^2)^{\alpha\beta}a_\alpha k_\beta k^2k^\mu a_\mu)\\ &\quad+(2w(2b-z)-(2b+z)^2)\theta^{\alpha\beta}\theta^{\mu\nu}a_\mu k_\nu k^2k_\beta a_\alpha].\end{aligned} \tag{39}$$

For $n = 4$, the divergent corrections are

$$\begin{aligned}\mathrm{Tr}[(m^2\Box^{-1})(\Box^{-1}B_{ij}^{(1,1)})^2 + \mathrm{P.T.}] = \frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}m^4\Bigg[&\frac{1}{2}(\theta^2)^{\alpha\beta}a_\alpha k^2a_\beta + \frac{1}{2}(\theta^2)^{\alpha\beta}a_\mu k_\alpha k_\beta a^\mu\\ &+192a^2\theta^{\alpha\beta}\theta^{\mu\nu}a_\mu k_\nu k_\beta a_\alpha - (\theta^2)^{\alpha\beta}a_\alpha k_\beta k^\mu a_\mu\Bigg].\end{aligned} \tag{40}$$

The above are the all divergent corrections to the gauge field propagator with mass m. We rewrite the sum

of them in coordinates space,

$$\begin{aligned}\Gamma^1_m = \frac{-1}{(4\pi)^2}\frac{2}{\epsilon}\int \mathrm{d}^4x\Bigg[&\left(\frac{1}{8}-2w^2\right)m^2(\theta^2)^{\alpha\beta}\partial^\mu f_{\mu\alpha}\partial^\nu f_{\nu\beta} - \frac{1}{8}m^4(\theta^2)^{\alpha\beta}f_{\mu\alpha}f^\mu{}_\beta \\ &+\frac{1}{4}\left((2b-z)(4w-4b+2z)+\frac{a}{3}-\frac{1}{24}\right)m^2\theta^{\alpha\beta}\theta^{\mu\nu}f_{\alpha\beta}\Box f_{\mu\nu} \\ &-\frac{1}{4}\left(2a+\frac{1}{2}\right)\left(10a+\frac{1}{2}\right)m^4\theta^{\alpha\beta}\theta^{\mu\nu}f_{\alpha\beta}f_{\mu\nu} - \frac{1}{24}m^2(\theta^2)^{\alpha\beta}f^{\mu\nu}\partial_\alpha\partial_\beta f_{\mu\nu}\Bigg]. \end{aligned} \tag{41}$$

It is easy to check that if one takes $a = -1/4$, $w = \pm 1/4$, $2b - z = \pm 1/4$, the corresponding divergent terms vanish. Term $(m^4/12)(\theta^2)^{\alpha\beta}f^{\mu\nu}\partial_\alpha\partial_\beta f_{\mu\nu}/(4\pi^2\epsilon)$ can be renormalized by gauge field redefinitions. But the one-loop renormalizability of the gauge field propagator is broken by the only term $(m^4/4)(\theta^2)^{\alpha\beta}f_{\mu\alpha}f^\mu{}_\beta/(4\pi^2\epsilon)$. This is very similar to the case where the U(1) gauge field couples with fermionic fields,[21] where a term $-(m^4/4)\widetilde{F}_{\mu\nu}\widetilde{F}^{\nu\mu}/(4\pi^2\epsilon)$ spoils the renormalizability of the gauge field propagator.

4. Conclusion

In this paper, we use the θ-expanded approach and the background field method to study the one-loop renormalizability of an NC U(1) gauge field propagator when it is coupled with a massive scalar field. Because the gauge sector contributions are renormalizable, we just need to consider the scalar field contributions. In the discussion, the non-uniqueness of S–W maps is important.

To θ-linear order, the divergent corrections to gauge field propagators can be renormalized no matter whether the scalar field is massive or massless. This result is compatible with that in Ref. [26]. To θ^2-order, the divergent corrections can be renormalized by field redefinitions when the scalar field is massless. But when the scalar field is massive, a term $(m^4/4)(\theta^2)^{\alpha\beta}f_{\mu\alpha}f^\mu{}_\beta/(4\pi^2\epsilon)$ makes one-loop renormalizability unachievable. This non-renormalizability is very similar to the NC QED case.[21] So the renormalizability of an NC gauge field coupled with matter fields needs further study. Maybe for θ^2-order renormalizability, we should expand the classical NC action to θ^2 order.

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