Operating a geometric quantum gate by external controllable parameters*

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A scheme to perfectly preserve an initial qubit state in geometric quantum computation is proposed for a single-qubit geometric quantum gate in a nuclear magnetic resonance system. At first, by adjusting some magnetic field parameters, one can let the dynamic phase be proportional to the geometric phase. Then, by controlling the azimuthal angle in the initial state, we may realize a geometric quantum gate whose fidelity is equal to one under cyclic evolution. This means that the quantum information is no distortion in the process of geometric quantum computation.

Keywords: geometric phase, geometric quantum gate, fidelity

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1. Introduction

Geometric phases have been attracting increasing interest in understanding and implementing quantum computation in real physical systems.[1,2] Geometric quantum computation is a scheme that is potentially intrinsically fault tolerant and therefore resilient to certain types of computational errors.[3−6] Theoretically, a pure geometric quantum gate can be achieved based only on adiabatic geometric phases. However, it is difficult to experimentally realize quantum computation under adiabatic evolution because a long operating time is required, especially for solid-state systems whose decoherent time is very short. To solve this problem, the Aharonov and Anandan phase (A–A phase) was suggested to realize geometric quantum gates, which allow for faster gate-operation time and have an intrinsically geometric feature.[7] Conventionally, there exist two approaches to obtain the geometric quantum gate: (i) driving the qubits to undergo an appropriately adiabatic or nonadiabatic cyclic evolution and (ii) displacing a harmonic oscillator along a closed path condition on the state of the qubits in the rotating frame.[8−13] However, the complexity of controlling the process will lead to many calculational mistakes. Recently, a different approach was proposed to implement the geometric quantum computation by directly controlling the magnetic field parameters, i.e., the strength and direction of magnetic field, where the dynamic phase is proportional to the geometric phase in a nuclear magnetic resonance system (NMR).[14] It is obvious that this third project makes quantum regulation easier in the laboratory. An essential requirement in quantum computation is to maintain quantum coherence in a computing system since the coherent interference pattern between the multitude of superpositions is necessary for taking advantage of quantum parallelism. Therefore an interesting issue about the magnetic field parameters is what is a better effect of the proportional constants between the dynamic and geometric phases on the fidelity of the quantum gate.

Fidelity is a degree of closeness to the original state of information in the transmission, and of fundamental importance in information science as well as in quantum optics. It has been adopted broadly as an important physical parameter in quantum communication and quantum computation.[15−17] Fidelity can measure the performance of quantum teleportation and describe the similarity between the input state and the teleported state. In order to obtain a better output result, the efforts of increasing the fidelity of quantum teleportation in continuous-variable systems have greatly been made.[18−20] In order to perfectly preserve the initial quantum state in the geometric quantum computation, therefore, it is interesting to study the relations between the fidelity and propor-

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tional constant between the dynamic and geometric phases.

2. Evolving equations

Considering the NMR Hamiltonian for a single-qubit system,[21]

$$H(t) = -\frac{1}{2}\Omega_1 \sin\theta\left(\sigma_x \cos(\omega t) + \sigma_y \sin(\omega t)\right) - \frac{1}{2}\Omega_2 \sigma_z \cos\theta, \tag{1}$$

where $\hat{\sigma}_i$ $(i = x,\ y,\ z)$ are the Pauli operators and $\Omega_i = g(\mu)B_i/\hbar$ are the gyromagnetic, B_i $(i = 1, 2)$ and θ act as external controllable parameters and can be experimentally changed.

Suppose that $|\Psi(t)\rangle$ is a state vector of the NMR system that evolves according to the time-dependent Schringer equation ($\hbar = 1$). Such as

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle. \tag{2}$$

Reference [22] gives the general solution of Eq. (2). Then we will give a simple solution of Eq. (2). Under a canonical transformation, the state vector may be written as[23]

$$|\Psi(t)\rangle = \exp\left(-\frac{\mathrm{i}}{2}\omega t\sigma_z\right)\left|\tilde{\Psi}(t)\right\rangle. \tag{3}$$

Inserting Eq. (3) into Eq. (2), one finds that $\left|\tilde{\Psi}(t)\right\rangle$ satisfies the following equation,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\left|\tilde{\Psi}(t)\right\rangle = \tilde{H}\left|\tilde{\Psi}(t)\right\rangle, \tag{4}$$

where the effective Hamiltonian is given by

$$\tilde{H}(t) = -\frac{1}{2}\Omega\left(\sigma_x \sin\chi + \sigma_z \cos\chi\right), \tag{5}$$

with

$$\Omega = \sqrt{(\Omega_1 \sin\theta)^2 + (\Omega_2 \cos\theta + \omega)^2}, \tag{6}$$

$$\sin\chi = \frac{\Omega_1 \sin\theta}{\Omega}, \quad \cos\chi = \frac{\Omega_2 \cos\theta + \omega}{\Omega}. \tag{7}$$

The eigenequations of the effective Hamiltonian $\tilde{H}$ may be expressed as

$$\tilde{H}|\pm\rangle = \mp\frac{1}{2}\Omega|\pm\rangle, \tag{8}$$

where

$$|+\rangle = \cos\frac{\chi}{2}|0\rangle + \sin\frac{\chi}{2}|1\rangle,$$
$$|-\rangle = \cos\frac{\chi}{2}|1\rangle - \sin\frac{\chi}{2}|0\rangle. \tag{9}$$

From Eqs. (3)–(5), we find

$$|\Psi(t)\rangle = \exp\left(-\mathrm{i}\omega t\sigma_z/2\right)\exp\left(-\mathrm{i}t\tilde{H}\right)|\Psi(0)\rangle = U(t)|\Psi(0)\rangle, \tag{10}$$

where $|\Psi(0)\rangle$ is an initial state of the system and the unitary transformation matrix may be given by

$$\boldsymbol{U}(t) = \begin{pmatrix} \left(\cos\dfrac{\Omega t}{2} + \mathrm{i}\sin\dfrac{\Omega t}{2}\cos\chi\right)\mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}} & \mathrm{i}\sin\dfrac{\Omega t}{2}\sin\chi\mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}} \\ \mathrm{i}\sin\dfrac{\Omega t}{2}\sin\chi\mathrm{e}^{\mathrm{i}\frac{\omega t}{2}} & \left(\cos\dfrac{\Omega t}{2} - \mathrm{i}\sin\dfrac{\Omega t}{2}\cos\chi\right)\mathrm{e}^{\mathrm{i}\frac{\omega t}{2}} \end{pmatrix}. \tag{11}$$

From Eq. (10), the total phases of the system under the cyclic evolution is written as

$$\alpha_k = k\pi\left(1 - \frac{\Omega}{\omega}\right), \quad k = \pm 1, \tag{12}$$

and the corresponding dynamic phases may be obtained by

$$\beta_k = \int_0^T \langle\Psi(t)|H(t)|\Psi(t)\rangle\,\mathrm{d}t = k\pi\left(\cos\chi - \frac{\Omega}{\omega}\right). \tag{13}$$

Thus, the geometric phases may be expressed as

$$\gamma_k = \alpha_k - \beta_k = k\pi\left(1 - \cos\chi\right). \tag{14}$$

In order to get the geometric quantum gate, one may control the magnetic field and let the dynamic phase be proportional to the geometric phase, i.e.,

$$\beta = x\gamma, \tag{15}$$

where

$$\beta = \cos\chi - \frac{\Omega}{\omega}, \quad \gamma = 1 - \cos\chi.$$

From Eqs. (13) and (14), it is easy to satisfy the relation (15) by adjusting the magnetic field parameters, i.e.,

$$x = -\frac{(\Omega_1 \sin\theta)^2 + (\Omega_2 \cos\theta)^2 + \omega\Omega_2 \cos\theta}{\omega\left(\Omega - \omega - \Omega_2 \cos\theta\right)}. \tag{16}$$

Under the computational basis $\{|0\rangle, |1\rangle\}$ the unitary transformation matrix $\boldsymbol{U}(T)$ under the cyclic evolution, between the input and output states, can be written as

$$\boldsymbol{U}(T) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(1+x)\gamma}\cos^2\frac{\chi}{2} + \mathrm{e}^{\mathrm{i}(1+x)\gamma}\sin^2\frac{\chi}{2} & \frac{1}{2}\sin\chi\left(\mathrm{e}^{-\mathrm{i}(1+x)\gamma} - \mathrm{e}^{\mathrm{i}(1+x)\gamma}\right) \\ \frac{1}{2}\sin\chi\left(\mathrm{e}^{-\mathrm{i}(1+x)\gamma} - \mathrm{e}^{\mathrm{i}(1+x)\gamma}\right) & \mathrm{e}^{-\mathrm{i}(1+x)\gamma}\sin^2\frac{\chi}{2} + \mathrm{e}^{\mathrm{i}(1+x)\gamma}\cos^2\frac{\chi}{2} \end{pmatrix}. \tag{17}$$

Thus, a geometric quantum gate is constructed based entirely on purely geometric operations (holonomies). The geometric quantum computation demands that logical gates in computing are realized by using the geometric phase shifts, so that it may have the built-in fault-tolerant advantage due to the fact that the geometric phases depend only on some global geometric features.

3. Fidelity

It is well-known that fidelity can measure the performance of quantum information and describe the similarity between the input and the output states. In order to obtain a better output result in the geometric quantum computation, we study the relation between the fidelity and the proportional constant x between the dynamic and geometric phases. The fidelity was defined by Schumacher as[24]

$$F(t_1, t_2) = \left\{\mathrm{Tr}\left(\sqrt{\rho(t_1)}\rho(t_2)\sqrt{\rho(t_1)}\right)^{1/2}\right\}^2, \tag{18}$$

where $\rho(t_1) = |\Psi(t_1)\rangle\langle\Psi(t_1)|$ and $\rho(t_2) = |\Psi(t_2)\rangle\langle\Psi(t_2)|$ are the density operators corresponding to the initial and final pure states, respectively. Equation (18) is a more satisfactory definition of the fidelity in general. Its ranges are obviously between 0 and 1. If the initial and the final states are orthogonal each other, then $F(t_1, t_2) = 0$, which indicates that the quantum information is totally lost in the quantum computation. If the initial and final states are coincided, then $F(t_1, t_2) = 1$, which implies that the quantum information is perfectly protected in the transmission. If $0 < F(t_1, t_2) < 1$, certain distortion exists in the transmission process of information.

For the evolution of pure states, based on Eq. (18), we have

$$F(t_1, t_2) = |\langle\Psi(t_1)|\Psi(t_2)\rangle|^2. \tag{19}$$

Consider that the initial state of the system is

$$|\Psi(0)\rangle = \cos\frac{\varphi}{2}|+\rangle + \sin\frac{\varphi}{2}|-\rangle, \tag{20}$$

where φ is an azimuthal angle of the Bloch sphere in the initial state. In terms of Eqs. (10), (11) and (17), at the time $t > 0$, the state vector of the system may be written as

$$|\Psi(t)\rangle = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}m}\cos^2\frac{\chi}{2} + \mathrm{e}^{-\mathrm{i}n}\sin^2\frac{\chi}{2} & \frac{1}{2}\sin\chi\left(\mathrm{e}^{-\mathrm{i}m} - \mathrm{e}^{-\mathrm{i}n}\right) \\ -\frac{1}{2}\sin\chi\left(\mathrm{e}^{\mathrm{i}m} - \mathrm{e}^{\mathrm{i}n}\right) & \mathrm{e}^{\mathrm{i}m}\cos^2\frac{\chi}{2} + \mathrm{e}^{\mathrm{i}n}\sin^2\frac{\chi}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} \end{pmatrix}, \tag{21}$$

where

$$m = \frac{\omega t}{2\pi}(1+x)\gamma, \quad n = \frac{\omega t}{2\pi}\left[2\pi - (1+x)\gamma\right].$$

The state vector $|\Psi(t)\rangle$ is constructed based on the geometric phase shifts and therefore called the geometric qubit state.

Substituting Eqs. (20) and (21) into Eq. (19), we find that the fidelity may be expressed as

$$F(t) = \left(\cos^2\frac{\omega t}{2} + \sin^2\frac{\omega t}{2}\cos^2\varphi\right)\cos^2\left[\frac{\omega t}{2}\left(\cos\chi - 2x\sin^2\frac{\chi}{2}\right)\right] + \left[\cos^2\frac{\omega t}{2}\cos^2(\varphi-\chi) + \sin^2\frac{\omega t}{2}\cos^2\chi\right]\sin^2\left[\frac{\omega t}{2}\left(\cos\chi - 2x\sin^2\frac{\chi}{2}\right)\right] + \frac{1}{2}\left[\cos\chi - \cos\varphi\cos(\varphi-\chi)\right]\sin^2\left[\frac{\omega t}{2}\left(\cos\chi - 2x\sin^2\frac{\chi}{2}\right)\right]\sin(\omega t), \quad (22)$$

which depends obviously on the proportional constant x and the initial angle φ.

When $x = 0$, $\beta_k = 0$ and $\alpha_k = \gamma_k$, in this case, the eigenvalues of Hamiltonian $H(t)$ are zero. The NMR system is in dark states and its dynamic phases disappear. When $x \neq 0$, the dynamic phases are proportional to the geometric phases. Figures 1 and 2 show that the fidelity evolves according to the ωt for the geometric quantum qubit that is initially in singlet state and superposition state in the case of $x = 0$, respectively. We see that the curves of fidelity oscillate with evolving time. Moreover, the oscillated periods and amplitudes depend on the initial conditions. The same situations are shown at Figs. 3 and 4 in case of $x = 2$ in the non-dark states. However, there are differently oscillated periods for the two cases, respectively.

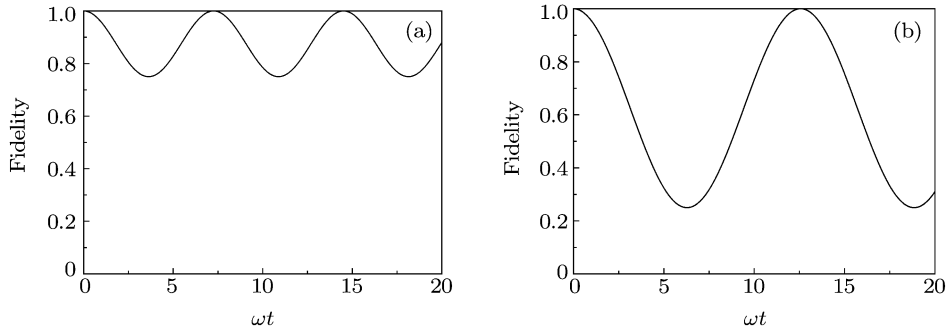


Fig. 1. Fidelity for dark state with an initial singlet state in a single-qubit geometric quantum gate as a function of evolving time with parameters (a) $x = 0$, $\varphi = 0$, $\chi = \pi/6$ and (b) $x = 0$, $\varphi = 0$, $\chi = 2\pi/3$, respectively.

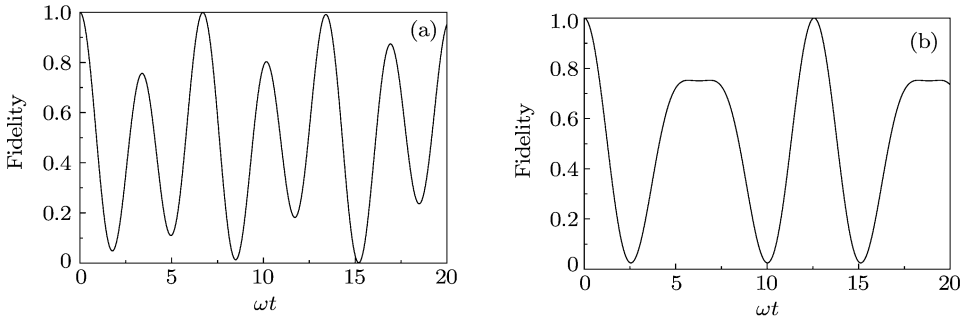


Fig. 2. Fidelity for dark state with an initial superposition state in a single-qubit geometric quantum gate as a function of evolving time with the parameters (a) $x = 0$, $\varphi = \pi/2$, $\chi = \pi/6$ and (b) $x = 0$, $\varphi = \pi/2$, $\chi = 2\pi/3$, respectively.

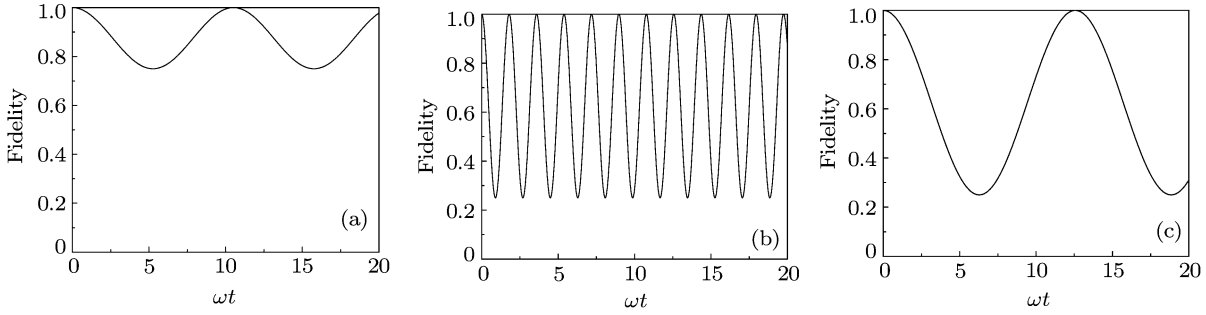


Fig. 3. Fidelity for an initial singlet state in a single-qubit geometric quantum gate as a function of evolving time with parameters (a) $x = 2$, $\varphi = 0$, $\chi = \pi/6$; (b) $x = 2$, $\varphi = 0$, $\chi = 2\pi/3$ and (c) $x = 2$, $\varphi = 0$, $\chi = 5\pi/3$, respectively.

Figures 1–4 indicate that the amplitude and period of fidelity evolving with time are related to not only the field parameters but also the initial state of the system and the proportional constant between the dynamic and geometric phases. The phenomenon of complete distortion will not appear during the whole evolving process.

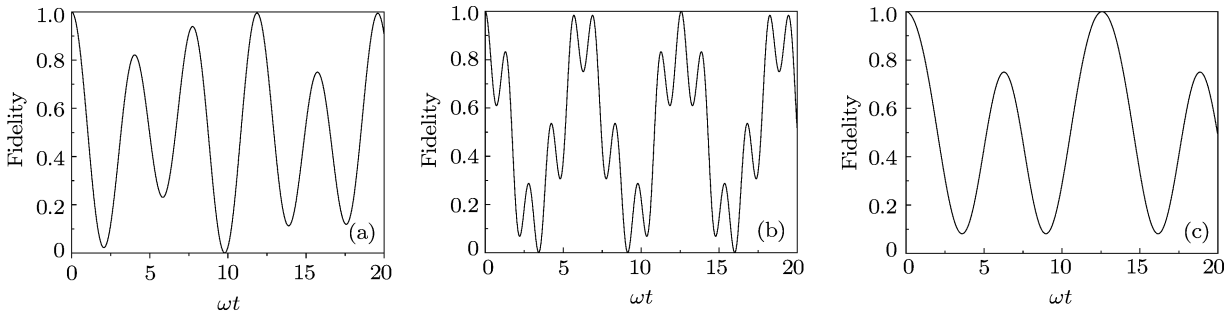


Fig. 4. Fidelity for an initial superposition state in a single-qubit geometric quantum gate as a function of evolving time with the parameters (a) $x = 2$, $\varphi = \pi/2$, $\chi = \pi/6$; (b) $x = 2$, $\varphi = \pi/2$, $\chi = 2\pi/3$ and (c) $x = 2$, $\varphi = \pi/2$, $\chi = 5\pi/3$, respectively.

When $\omega + \Omega_2 \cos\theta = 0$, $\varphi = \pi/2$, we get

$$F(t) = \frac{1}{2}\left(1 + \cos(\omega t)\right), \tag{23}$$

which implies that, for a geometric quantum gate in an initial superposition, when resonance interaction between the geometric quantum qubit and the magnetic field occurs, the fidelity in the evolution of the quantum state is not related to the strength and direction of the magnetic field.

Under a periodic evolution, the fidelity in Eq. (22) of the quantum state is expressed as

$$\begin{aligned} F(T) = {} & \cos^2\left[\pi\cos\chi - \pi x\left(1 - \cos\chi\right)\right] \\ & + \sin^2\left[\pi\cos\chi - \pi x\left(1 - \cos\chi\right)\right] \\ & \times \cos^2\left(\varphi - \chi\right). \end{aligned} \tag{24}$$

From the above equation, we find that, when

$$\varphi - \chi = k\pi, k = 0, 1, 2\ldots, \tag{25}$$

then one has

$$F(T) = 1, \tag{26}$$

which means that one may perfectly preserve an initial qubit state in the geometric quantum computation. It is very important for the process of quantum computation and communication. Thus, we may control the initial angle of qubit in the initial state to satisfy $\varphi - \chi = k\pi$, which will make that the fidelity of the quantum state always come back to one after the cyclic evolution. This also provides a new idea to implement the quantum information transmission and geometric quantum computation.

At first, the magnetic field parameters are adjusted to make the dynamic phase be proportional to the geometric phase in terms of Eqs. (15) and (16). Thus, the proportional constant x is determined by Eq. (16). Then, by solving Eqs. (6), (7), (16) and (25), we can get a relation between the initial angle φ and the proportional constant x. The relation offers a wide choice of between φ and x. Thus, one may prepare the geometric quantum qubit in a certain original state satisfying Eq. (25), which makes the fidelity under the cyclic evolution be equal to one. As an example, we choose that $\Omega_1 = \Omega_2 = \omega$. The relation between φ and x may be written as

$$\varphi = k\pi \pm \arctan\left(\sqrt{\left(\frac{1+x}{x}\right)^2 - 1}\right), \tag{27}$$

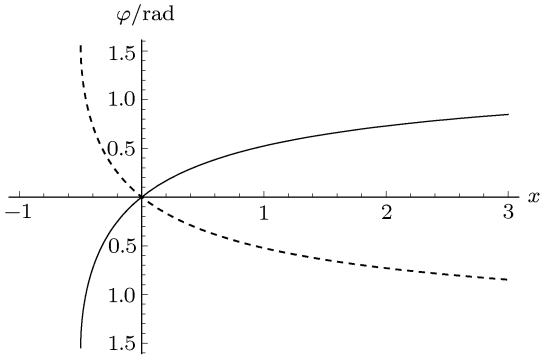


Fig. 5. Initial azimuthal angle of the Bloch sphere for the initial state as a function of the proportional rate x between dynamic and geometric phases in case of $k = 0$ and $\Omega_1 = \Omega_2 = \omega$. It is shown that there indeed exist two solutions of Eq. (27) by the solid and dashed lines, respectively. Therefore, it is easy to control φ to satisfy Eq. (27) so that one may get perfect fidelity in the geometric quantum computation.

which implies that it is easy to satisfy Eq. (27) by operating φ as shown by Fig. 5 so that one obtains perfect fidelity in the geometric quantum computation.

4. Conclusion

It is known in Refs. [14] and [20] that, for given x and frequency ω, one may have a corresponding design to the magnetic field. In contrast, by adjusting the magnetic field parameter plane, we will give x and ω. If the system is in a dark state under cyclic evolution, the result is similar to the conventionally geometric quantum computation, where the dynamic phase was cancelled by rotating operations in single-loop or multi-loop approaches. For $x \neq 0$, the dynamic and total phases are proportional to the conditional geometric phase. This result is similar to unconventionally geometric quantum computation, where the global geometric features are used in the rotating frame at the cavity frequency and therefore the total and geometric phases are not distinguished. For our approach, it is consistent to include the two cases by controlling the parameters in the Hamiltonian and it does not need any operations, which may be a distinct advantage for experimentally implementing the geometric quantum computation. Furthermore, we may get a perfect fidelity for the evolution of the geometric qubit. Our results show that, for any proportional constant between the dynamic and geometric phases in the NMR system, when we may control the initial parameters to satisfy $\varphi - \chi = k\pi$, the fidelity of the quantum state will always come back to one after a period of evolution. It is known that the two-qubit geometric quantum gates may be constructed by our single-qubit geometric quantum gate as well as by a unified way. Therefore, our scheme may be expanded to the two-qubit system and other physical systems. In conclusion, we propose a scheme to perfectly preserve the geometric quantum information by controlling the magnetic field and initial state. It is very helpful in experiments to realize geometric quantum computation.

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