Optical scaled Fresnel–Fourier transform obtained via intermediate coordinate–momentum representation

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(Received 25 April 2009; revised manuscript received 19 June 2009)

Using the intermediate coordinate–momentum representation $|x\rangle_{s,r}$, we introduce a new Hadamard transform. It is found that the operator U corresponding to this transform can be considered as a combination of the Fresnel operator $F(r,s)$ and the Fourier transform operator $\mathcal{F}$ by decomposing U. We also find that the matrix element ${}_{s,r}\langle x|U|f\rangle$ just corresponds to an optical scaled Fresnel–Fourier transform.

Keywords: Fresnel transform, Fresnel operator, Fourier transform, intermediate coordinate–momentum representation

PACC: 0365, 4250

1. Introduction

Linear integral transforms, by which lots of optical processes can be demonstrated, play an important role in wave optics. Among them, the Fresnel transform, a well-known linear transform, has been used to describe the light propagation in an optical system. In Ref. [1], James and Agarwal proposed a generalized Fresnel transform (GFT) in the 1-dimensional (1D) case,

$$g(x') = \frac{1}{\sqrt{2\pi \mathrm{i}B}} \int_{-\infty}^{\infty} \times \exp\left[\frac{\mathrm{i}}{2B}\left(Ax^2 - 2x'x + Dx'^2\right)\right] f(x)\,\mathrm{d}x, \tag{1}$$

where $f(x)$ is an input field, $g(x')$ is the output field pattern, the exponent in Eq. (1) is the integral transform kernel with the real parameters A, B, C and D restricted by the relation $AD - BC = 1$. The GFT stands for a class of optical transforms which are of great importance for their applications to describe various optical systems. From the point of view of quantum optics theory,[2−4] Fan and Lu[5] pointed out that Eq. (1) corresponds to the single-mode 3-parameter unitary squeezing operator $F(r,s)$ (or the Fresnel operator (FO)),

$$F(r,s) = \frac{1}{\sqrt{s^*}} \exp\left(-\frac{r}{2s^*}a^{\dagger 2}\right) \times \exp\left[\left(a^\dagger a + \frac{1}{2}\right)\ln\frac{1}{s^*}\right] \exp\left(\frac{r^*}{2s^*}a^2\right), \tag{2}$$

where $a^\dagger$ and a are Bose creation and annihilation operators obeying $\left[a, a^\dagger\right] = 1$, respectively, and the following relations are introduced,[6]

$$s = \frac{1}{2}\left[A + D - \mathrm{i}\left(B - C\right)\right],$$
$$r = -\frac{1}{2}\left[A - D + \mathrm{i}\left(B + C\right)\right], \tag{3}$$

so the condition $AD - BC = 1$ becomes $|s|^2 - |r|^2 = 1$. That is to say, the matrix element of $F(r,s)$ between the coordinate eigenstate $\langle x'|$ and $|x\rangle$ just leads to the integral transform kernel in Eq. (1), i.e.,

$$\langle x'|F(r,s)|x\rangle = \frac{1}{\sqrt{2\pi \mathrm{i}B}} \times \exp\left[\frac{\mathrm{i}}{2B}\left(Ax^2 - 2x'x + Dx'^2\right)\right]. \tag{4}$$

Here if we denote $g(x')$ as $\langle x'|\,g\rangle$ and $f(x)$ as $\langle x|\,f\rangle$, the wave functions of the quantum states $|g\rangle$ and $|f\rangle$ in coordinate representation, then we see that Eq. (1) corresponds to

$$g(x') = \int_{-\infty}^{\infty} \mathrm{d}x\,\langle x'|F(r,s)|x\rangle f(x) = \langle x'|F(r,s)|f\rangle, \tag{5}$$

which is the form of the optical Fresnel transform in the context of quantum mechanics, i.e., the quantum version of the optical Fresnel transform.

In recent years, the continuous Hadamard transform has attracted lots of interest due to the applications in quantum computation. This transform, used

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to go from the position basis $|x\rangle$ to the momentum basis, is defined as[7]

$$\mathcal{F}|x\rangle = \frac{1}{\sqrt{\pi}\sigma}\int_{-\infty}^{\infty} \mathrm{e}^{2\mathrm{i}xy/\sigma^2}|y\rangle\,\mathrm{d}y, \tag{6}$$

where σ is the scale length (it also makes the expression in the exponential dimensionless), $|x\rangle$ and $|y\rangle$ are the eigenvectors of the coordinate operator Q. Fan and Guo[8] have derived the explicit form of $\mathcal{F}$ by using the technique of integration within an ordered product (IWOP) of operators,[9,10] and found that it can be decomposed into a single-mode squeezing operator and a position–momentum mutual transform operator. Then they also introduced the two-mode Hadamard transform with continuous variables. From Eq. (6), we see that the Hadamard transform is in fact just the Fourier transform, thus here for simplicity we name the operator $\mathcal{F}$ and the transform in Eq. (6) the Fourier transform operator (FTO) and the generalized Fourier transform, respectively.

On the other hand, there are many quantum mechanical representations constructed by Fan *et al.* through the IWOP technique, for example, the intermediate coordinate–momentum state $|x\rangle_{s,r}$ (see Eq. (9) in Section 2), which makes up a complete quantum mechanical representation and has a close relation to the Fresnel transform (see Eq. (17)).[11] Then what can be obtained by replacing Eq. (6) with the following transform,

$$U|x\rangle_{s,r} = \frac{1}{\sqrt{\pi}\sigma}\int_{-\infty}^{\infty} \mathrm{e}^{2\mathrm{i}xy/\sigma^2}|y\rangle_{s,r}\,\mathrm{d}y? \tag{7}$$

Accordingly, two questions naturally arise: what is the explicit form of operator U, and what will the transform ${}_{s,r}\langle x|U^{\dagger}|f\rangle$ correspond to? The motivation to consider this transform lies in the following two facts: 1) this operator U can be considered as a combination of the Fresnel operator $F(r,s)$ and the Fourier transform operator $\mathcal{F}$; 2) as one can see later, this transform ${}_{s,r}\langle x|U^{\dagger}|f\rangle$ just corresponds to the optical scaled Fresnel–Fourier transform (OSFFT).

In this work, the OSFFT ${}_{s,r}\langle x|U^{\dagger}|f\rangle$ and the optical scaled Fresnel–Fourier transform operator (OSFFO) U can be obtained conveniently by virtue of the intermediate coordinate–momentum state $|x\rangle_{s,r}$. Our work is arranged as follows. In Section 2 we review the intermediate coordinate–momentum state $|x\rangle_{s,r}$ and $|p\rangle_{s,r}$. In Section 3, based on $|x\rangle_{s,r}$ and the IWOP technique we shall derive the explicit expression of U and then analyse its properties, i.e., the decomposition of the OSFFT operator in Section 4. Section 5 is devoting to presenting OSFFT by using the properties of $U^{\dagger}$ and $|x\rangle_{s,r}$. Conclusions are presented in the last section.

2. Intermediate coordinate–momentum state

In this section, we firstly review the intermediate coordinate–momentum state. It has been shown that one may derive some new quantum mechanical states possessing a completeness relation by constructing miscellaneous normally ordered Gaussian integration unity operators and then by considering their decomposition of unity. Recall that in Ref. [11] Fan and Hu have constructed two new quantum mechanical representations,

$$|p\rangle_{s,r} = \frac{\pi^{-1/4}}{\sqrt{s^*-r^*}}\exp\left\{-\frac{s^*+r^*}{s^*-r^*}\frac{p^2}{2}+\frac{\mathrm{i}\sqrt{2}p}{s^*-r^*}a^{\dagger}+\frac{s-r}{s^*-r^*}\frac{a^{\dagger 2}}{2}\right\}|0\rangle, \tag{8}$$

and

$$|x\rangle_{s,r} = \frac{\pi^{-1/4}}{\sqrt{s^*+r^*}}\exp\left\{-\frac{s^*-r^*}{s^*+r^*}\frac{x^2}{2}+\frac{\sqrt{2}x}{s^*+r^*}a^{\dagger}-\frac{s+r}{s^*+r^*}\frac{a^{\dagger 2}}{2}\right\}|0\rangle, \tag{9}$$

where $|x\rangle_{s,r}$ is conjugate to $|p\rangle_{s,r}$. In fact, operating a on Eq. (12) yields the eigen-equation for $|p\rangle_{s,r}$,

$$\left[(s^*-r^*)\,a-(s-r)\,a^{\dagger}\right]|p\rangle_{s,r} = \sqrt{2}\mathrm{i}p\,|p\rangle_{s,r}. \tag{10}$$

By using Eq. (3) the eigen-equation changes into

$$(AP-CQ)\,|p\rangle_{s,r} = p\,|p\rangle_{s,r}, \tag{11}$$

where $Q=\left(a+a^{\dagger}\right)/\sqrt{2}, P=\left(a-a^{\dagger}\right)/\left(\sqrt{2}\mathrm{i}\right)$ are the coordinate and the momentum operators, respectively. From Eq. (14) one can see that when $A=1$ and $C=0$ ($A=0$ and $C=1$), $|p\rangle_{s,r}$ is just the eigenstate of the momentum operator P (coordinate operator Q), thus we call $|p\rangle_{s,r}$ the intermediate coordinate–momentum state. In addition, $|p\rangle_{s,r}$ is complete and orthogonal,

$$\int \mathrm{d}p\,|p\rangle_{s,r\ s,r}\langle p| = 1,\ {}_{s,r}\langle p'\,|p\rangle_{s,r} = \delta\left(p-p'\right). \quad (12)$$

In a similar way, for $|x\rangle_{s,r}$ one can see that

$$\left(DQ - BP\right)|x\rangle_{s,r} = x\,|x\rangle_{s,r}, \quad (13)$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\,|x\rangle_{s,r\ s,r}\langle x| = 1, \quad (14)$$

$${}_{s,r}\langle x'\,|x\rangle_{s,r} = \delta\left(x-x'\right). \quad (15)$$

Furthermore, using the IWOP technique, Fan and Hu have found the following relations:[11]

$$\int_{-\infty}^{\infty} \mathrm{d}p\,|p\rangle_{s,r}\langle p| = \int_{-\infty}^{\infty} \mathrm{d}x\,|x\rangle_{s,r}\langle x| = F\left(r,s\right). \quad (16)$$

Thus the two intermediate coordinate–momentum states can be generated by operating the FO on coordinate (momentum) eigenstate, i.e.,

$$F\left(r,s\right)|x\rangle = |x\rangle_{s,r}, \quad F\left(r,s\right)|p\rangle = |p\rangle_{s,r}. \quad (17)$$

This property shall often be used in our derivation.

3. Optical scaled Fresnel–Fourier transform operator via $|x\rangle_{s,r}$

Motivated by the Hardamard transform (which can also be called the (quantum) Fourier transform), in this section, we introduce a new optical scaled Fresnel–Fourier transform (OSFFT) on the basis of the intermediate coordinate–momentum state $|x\rangle_{s,r}$, i.e., we introduce the transform in Eq. (7). Using the completeness relation of $|x\rangle_{s,r}$ in Eq. (12), we can put the OSFFO U into the following integration expression,

$$U = \frac{1}{\sqrt{\pi}\sigma}\int_{-\infty}^{\infty} \mathrm{e}^{2\mathrm{i}xy/\sigma^2}\,|y\rangle_{s,r\ s,r}\langle x|\,\mathrm{d}x\mathrm{d}y. \quad (18)$$

Further employing the Fock representation of $|x\rangle_{s,r}$ in Eq. (9), and the vacuum projector's normally ordered form $|0\rangle\langle 0| =: \mathrm{e}^{-a^\dagger a}:$ (where : : denotes normal ordering) as well as the IWOP technique, one can directly perform the integration involved in Eq. (18) and obtain

$$\begin{aligned} U &= \frac{1}{\pi\sigma\,|s+r|}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}y{:}\exp\left\{-\frac{s^*-r^*}{2\left(s^*+r^*\right)}y^2 + \left(\frac{\sqrt{2}a^\dagger}{s^*+r^*} + \frac{2\mathrm{i}x}{\sigma^2}\right)y\right. \\ &\quad \left. -\frac{s-r}{2\left(s+r\right)}x^2 + \frac{\sqrt{2}x}{s+r}a - \frac{s+r}{s^*+r^*}\frac{a^{\dagger 2}}{2} - \frac{s^*+r^*}{s+r}\frac{a^2}{2} - a^\dagger a\right\}: \\ &= \frac{2\sigma}{\sqrt{M}}{:}\exp\left\{\frac{N}{2}a^{\dagger 2} + \left(\frac{4\mathrm{i}\sigma^2}{M}-1\right)a^\dagger a + \frac{N^*}{2}a^2\right\}:, \end{aligned} \quad (19)$$

where we have set

$$M = 4\,|s+r|^2 + \sigma^4\,|s-r|^2, \quad N = \frac{2\sigma^4}{M}\frac{s-r}{s^*+r^*} - \frac{s+r}{s^*+r^*}, \quad (20)$$

and have used the integral formula

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\alpha x^2+\beta x}\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}\mathrm{e}^{\frac{\beta^2}{4\alpha}}, \quad \mathrm{Re}\,\alpha > 0. \quad (21)$$

Further using the operator identity, $\mathrm{e}^{\lambda a^\dagger a} =: \exp\left[\left(\mathrm{e}^\lambda - 1\right)a^\dagger a\right]:$, we can obtain the explicit expression of U,

$$U = \frac{2\sigma}{\sqrt{M}}\exp\left\{\frac{N}{2}a^{\dagger 2}\right\}\exp\left\{a^\dagger a\ln\frac{4\mathrm{i}\sigma^2}{M}\right\}\exp\left\{\frac{N^*}{2}a^2\right\}, \quad (22)$$

which is the normal-ordered form of the OSFFT operator U. In particular, when $r=0$, $s=1$, Eq. (22) just reduces to

$$\begin{aligned} U &= \frac{2\sigma}{\sqrt{\sigma^4+4}}{:}\exp\left\{\frac{\sigma^4-4}{\sigma^4+4}\frac{a^2}{2} + \left(\frac{4\mathrm{i}\sigma^2}{\sigma^4+4}-1\right)a^\dagger a + \frac{\sigma^4-4}{\sigma^4+4}\frac{a^{\dagger 2}}{2}\right\}: \\ &= \frac{2\sigma}{\sqrt{\sigma^4+4}}\exp\left\{\frac{\sigma^4-4}{\sigma^4+4}\frac{a^{\dagger 2}}{2}\right\}\exp\left\{a^\dagger a\ln\frac{4\mathrm{i}\sigma^2}{\sigma^4+4}\right\}\exp\left\{\frac{\sigma^4-4}{\sigma^4+4}\frac{a^2}{2}\right\}, \end{aligned} \quad (23)$$

which agrees with Eq. (7) in Ref. [8].

4. Decomposition of the OSFFO U

From Eqs. (12), (14) and (15), it can be easily seen that the OSFFO U is unitary, i.e., $UU^\dagger = U^\dagger U = 1$. It then follows from $Q = \left(a + a^\dagger\right)/\sqrt{2}, P = \left(a - a^\dagger\right)/\left(\sqrt{2}\mathrm{i}\right)$ and Eq. (22) that

$$\begin{aligned} UaU^\dagger &= \frac{M}{4\mathrm{i}\sigma^2}\left(a - Na^\dagger\right) \\ &= \frac{M}{4\mathrm{i}\sigma^2}\left\{\frac{Q}{\sqrt{2}}\left(1-N\right) + \frac{\mathrm{i}P}{\sqrt{2}}\left(1+N\right)\right\}, \end{aligned} \tag{24}$$

$$Ua^\dagger U^\dagger = \frac{M}{-4\mathrm{i}\sigma^2}\left\{\frac{Q}{\sqrt{2}}\left(1-N^*\right) - \frac{\mathrm{i}P}{\sqrt{2}}\left(1+N^*\right)\right\}, \tag{25}$$

which leads to

$$\begin{aligned} UQU^\dagger = \frac{M}{8\mathrm{i}\sigma^2}\big\{&Q\left(N^* - N\right) \\ &+ \mathrm{i}\left(2 + N + N^*\right)P\big\}, \end{aligned} \tag{26}$$

$$\begin{aligned} UPU^\dagger = \frac{M}{8\mathrm{i}\sigma^2}\big\{&P\left(N - N^*\right) \\ &- \mathrm{i}\left(2 - N - N^*\right)Q\big\}, \end{aligned} \tag{27}$$

which indicates that the role of U is to combine the coordinate operator Q and momentum operator P in a certain way (26) and (27). In particular, when $N^* = N$ ($r = 0, s = 1$ as a special case), Eqs. (26) and (27) become

$$\begin{aligned} UQU^\dagger &= \frac{M}{4\sigma^2}\left(N+1\right)P, \\ UPU^\dagger &= \frac{M}{4\sigma^2}\left(N-1\right)Q, \end{aligned} \tag{28}$$

i.e., the mutual exchanging of coordinate–momentum operators.

On the other hand, by noticing Eqs. (17) and (7), we can put the OSFFO U into the following expression,

$$U = FS^{-1}\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}F^\dagger, \tag{29}$$

where F is the Fresnel operator, and S^{-1} is the single-mode squeezing operator,[9]

$$\begin{aligned} S &= \frac{\sqrt{2}}{\sigma}\int_{-\infty}^{\infty}\mathrm{d}x\left|2x/\sigma^2\right\rangle\left\langle x\right| \\ &= \exp\left[\mathrm{i}\left(QP - \frac{\mathrm{i}}{2}\right)\ln\frac{\sigma^2}{2}\right], \end{aligned} \tag{30}$$

as well as $\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}$ being a mutual exchanging operator for coordinate and momentum operators. From Eq. (29) we see that the OSFFO U can be decomposed. Further, noting Eq. (17) again, we see

$$\begin{aligned} U\left|x\right\rangle_{s,r} &= FS^{-1}\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\left|x\right\rangle \\ &= FS^{-1}\left|p_{=x}\right\rangle = \frac{\sqrt{2}}{\sigma}\left|p_{=\frac{2}{\sigma^2}x}\right\rangle_{s,r}, \end{aligned} \tag{31}$$

$$\begin{aligned} U\left|p\right\rangle_{s,r} &= FS^{-1}\left|x_{=-p}\right\rangle \\ &= \frac{\sigma}{\sqrt{2}}F\left|x_{=-\frac{\sigma^2}{2}p}\right\rangle = \frac{\sigma}{\sqrt{2}}\left|x_{=-\frac{\sigma^2}{2}p}\right\rangle_{s,r}. \end{aligned} \tag{32}$$

Thus the OSFFO U can also be expressed as

$$\begin{aligned} U &= \frac{\sqrt{2}}{\sigma}\int_{-\infty}^{\infty}\mathrm{d}x\left|p_{=\frac{2}{\sigma^2}x}\right\rangle_{s,r}\,{}_{s,r}\left\langle x\right| \\ &= \frac{\sigma}{\sqrt{2}}\int_{-\infty}^{\infty}\mathrm{d}p\left|x_{=-\frac{\sigma^2}{2}p}\right\rangle_{s,r}\,{}_{s,r}\left\langle p\right|. \end{aligned} \tag{33}$$

5. Optical scaled Fresnel–Fourier transform obtained from U

Motivated by the quantum version of the Fresnel transform in Eq. (5), we can also consider Eq. (7) as another optical transform, whose quantum version is given by

$$g_{s,r}\left(x'\right) = {}_{s,r}\left\langle x'\right|U^\dagger\left|f\right\rangle. \tag{34}$$

If we denote $\left\langle x\right|\left.f\right\rangle \equiv f\left(x\right)$ and use the complete relation of the coordinate eigenvector, $\int_{-\infty}^{\infty}\mathrm{d}x\left|x\right\rangle\left\langle x\right| = 1$, then Eq. (34) can be rewritten as

$${}_{s,r}\left\langle x'\right|U^\dagger\left|f\right\rangle = \int_{-\infty}^{\infty}\mathrm{d}xK_{s,r}\left(x',x\right)f\left(x\right), \tag{35}$$

where the integral kernel $K_{s,r}\left(x',x\right)$ is defined as $K_{s,r}\left(x',x\right) \equiv {}_{s,r}\left\langle x'\right|U^\dagger\left|x\right\rangle$.

In order to obtain the explicit expression of this transform, we calculate first the matrix element ${}_{s,r}\left\langle x'\right|U^\dagger\left|x\right\rangle$. Noticing Eq. (17) we have

$$\begin{aligned} K_{s,r}\left(x',x\right) &= {}_{s,r}\left\langle x'\right|F\mathrm{e}^{-\mathrm{i}\pi a^\dagger a/2}SF^\dagger\left|x\right\rangle \\ &= \left\langle p_{=x'}\right|SF^\dagger\left|x\right\rangle = \frac{\sqrt{2}}{\sigma}\left\langle p_{=\frac{2}{\sigma^2}x'}\right|F^\dagger\left|x\right\rangle \\ &= \frac{\sqrt{2}}{\sigma}\,{}_{s,r}\left\langle p_{=\frac{2}{\sigma^2}x'}\right|\left.x\right\rangle. \end{aligned} \tag{36}$$

Thus Eq. (35) can also be expressed as

$$\begin{aligned} g_{s,r}\left(x'\right) &= \int_{-\infty}^{\infty}\mathrm{d}x\frac{\sqrt{2}}{\sigma}\,{}_{s,r}\left\langle p_{=\frac{2}{\sigma^2}x'}\right|\left.x\right\rangle f\left(x\right) \\ &= \frac{\sqrt{2}}{\sigma}\,{}_{s,r}\left\langle p_{=\frac{2}{\sigma^2}x'}\right|\left.f\right\rangle. \end{aligned} \tag{37}$$

On the other hand, noticing Eqs. (8), (9) and using the completeness relation of the coherent state,[12,13] $\int\frac{\mathrm{d}^2z}{\pi}\left|z\right\rangle\left\langle z\right| = 1$, Fan and Lu have obtained[5]

$${}_{s,r}\left\langle p\right|\left.x\right\rangle = \frac{1}{\sqrt{2\pi A}}\exp\left[\frac{\mathrm{i}}{2A}\left(Bp^2 - 2px - Cx^2\right)\right]. \tag{38}$$

Then substituting Eq. (38) into Eq. (36) yields

$$K_{s,r}\left(x',x\right)=\frac{1}{\sigma\sqrt{\pi A}}\exp\left[\frac{\mathrm{i}}{2A}\left(\frac{4}{\sigma^4}Bx'^2-\frac{4}{\sigma^2}x'x-Cx^2\right)\right], \tag{39}$$

thus the optical scaled Fourier–Fourier transform in Eq. (42) is given by

$$g_{s,r}\left(x'\right)=\frac{1}{\sigma\sqrt{\pi A}}\int_{-\infty}^{\infty}\mathrm{d}x\exp\left[\frac{\mathrm{i}}{2A}\left(\frac{4B}{\sigma^4}x'^2-\frac{4}{\sigma^2}x'x-Cx^2\right)\right]f\left(x\right), \tag{40}$$

which and Eq. (1) are similar but different in form. It can be found that the above transform is determined not only by the scale parameter σ but also by the Fresnel parameters s and r (or A, B, C, D), similar to the classical optical Fresnel transform[14] but a little different, thus it seems to be new and we call it the optical scaled Fresnel–Fourier transform (OSFT).

Formula (37) can help us to derive the OSFFT of some wave functions, for example, when $f\left(x\right)=\left\langle x\left|n\right.\right\rangle$, i.e., the wave function of a one-mode number state $\left|n\right\rangle=a^{\dagger n}/\sqrt{n!}\left|n\right\rangle$, using the un-normalized coherent state, $\left|z\right\rangle=\exp\left(za^{\dagger}\right)\left|0\right\rangle$, we have

$$\left\langle n\right|=\frac{1}{\sqrt{n!}}\frac{\partial^n}{\partial z^{*n}}\left\langle z\right|\Big|_{z^*=0}, \tag{41}$$

it then follows that

$$\begin{aligned}\left\langle n\right|\left.p\right\rangle_{s,r}&=\frac{1}{\sqrt{n!}}\frac{\partial^n}{\partial z^{*n}}\left\langle z\right|\left.p\right\rangle_{s,r}\\&=\frac{1}{\sqrt{n!}}\frac{\pi^{-1/4}}{\sqrt{s^*-r^*}}\exp\left\{-\frac{s^*+r^*}{s^*-r^*}\frac{p^2}{2}\right\}\frac{\partial^n}{\partial z^{*n}}\exp\left\{\frac{\mathrm{i}\sqrt{2}pz^*}{s^*-r^*}-\frac{s-r}{s^*-r^*}\frac{z^{*2}}{2}\right\}\Big|_{z^*=0}.\end{aligned} \tag{42}$$

Using the following generating formula,[15]

$$H_n\left(\zeta\right)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\exp\left(2t\zeta-t^2\right)\Big|_{t=0}, \tag{43}$$

where $H_n\left(\zeta\right)$ is the Hermite polynomials, Eq. (42) can be rewritten as

$$\left\langle n\right|\left.p\right\rangle_{s,r}=\frac{\pi^{-1/4}}{\sqrt{2^n n!}}\frac{\left(s-r\right)^{n/2}}{\left(s^*-r^*\right)^{(n+1)/2}}\exp\left\{-\frac{s^*+r^*}{s^*-r^*}\frac{p^2}{2}\right\}H_n\left(\frac{\mathrm{i}p}{\left|s-r\right|}\right). \tag{44}$$

Thus we have

$$g_{s,r}^{n}\left(x'\right)=\frac{\sqrt{2}}{\sigma}\frac{\pi^{-1/4}}{\sqrt{2^n n!}}\frac{\left(s^*-r^*\right)^{n/2}}{\left(s-r\right)^{(n+1)/2}}\exp\left\{-\frac{s+r}{s-r}\frac{2x'^2}{\sigma^4}\right\}H_n\left(\frac{-2\mathrm{i}x'}{\left|s-r\right|\sigma^2}\right). \tag{45}$$

6. Conclusions

In summary, on the basis of the intermediate coordinate–momentum representation $\left|x\right\rangle_{s,r}$, which can make up a complete quantum mechanical representation, and the idea of the continuous Hadamard transform, we find that the operator U can be decomposed into two Hermite-conjugated Fresnel operators $F\left(r,s\right)$, $F^{\dagger}\left(r,s\right)$ and a Fourier transform operator $\mathcal{F}=S^{-1}\mathrm{e}^{\mathrm{i}\pi a^{\dagger}a/2}$ by introducing a new Hadamard transform on $\left|x\right\rangle_{s,r}$. It is also found that the matrix element ${}_{s,r}\left\langle x\right|U\left|f\right\rangle$ just corresponds to an optical scaled Fresnel–Fourier transform. The properties of the optical scaled Fresnel–Fourier transform can be seen more clearly from the viewpoint of a representation transform in quantum mechanics. In our whole derivation, the representation $\left|x\right\rangle_{s,r}$ and the IWOP technique are fully used. For more discussions on classical optical transforms in the context of quantum optics, please refer to Refs. [16] and [17].

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