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Invariant operator theory for the single-photon energy in time-varying media*

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After the birth of quantum mechanics, the notion in physics that the frequency of light is the only factor that determines the energy of a single photon has played a fundamental role. However, under the assumption that the theory of Lewis–Riesenfeld invariants is applicable in quantum optics, it is shown in the present work that this widely accepted notion is valid only for light described by a time-independent Hamiltonian, i.e., for light in media satisfying the conditions, $\epsilon(t) = \epsilon(0)$, $\mu(t) = \mu(0)$, and $\sigma(t) = 0$ simultaneously. The use of the Lewis–Riesenfeld invariant operator method in quantum optics leads to a marvelous result: the energy of a single photon propagating through time-varying linear media exhibits nontrivial time dependence without a change of frequency.

Keywords: single-photon energy, invariant operator theory, time-varying media
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1. Introduction

Since the first indication of the quantum aspect of light by Planck in 1900, the study of the quantum features of light has occupied a special position in our understanding of nature. The essence of the quantum theory of light is that each mode of the radiation field behaves as a quantized simple harmonic oscillator. The solution to the ultraviolet catastrophe led to the birth of quantum mechanics which requires that the spectrum of allowed energy for the harmonic oscillator should be discrete with equal spacing. Planck found that monochromatic light consists of photons which all have the same energy, directly proportional to their frequency ω. In other words, the energy of a single photon is $\hbar\omega$ where $\hbar$ is Planck's constant. This means that the single-photon energy cannot be changed so long as the frequency is constant, i.e., the only factor related to determination of its energy is frequency. This concept became a fundamental notion in quantum optics and, consequently, was accepted in physics for a long time after the birth of quantum mechanics.

However, Stowe indicated that the reason for this phenomenon has not yet been exactly understood.[1] It is shown in recent years that single-photon energy in a conducting linear medium undergoes continuous decay over time without change of frequency,[2] when we admit a proposition that the Lewis–Riesenfeld (LR) invariant operator theory,[3,4] which was first introduced by Lewis, is applicable in quantum optics. In the present work, we wish to extend this idea to a more complicated case, the so-called problem of the quantization of light in time-varying media. Time-varying media are defined as those media whose parameters, such as electric permittivity, magnetic permeability, and conductivity, vary with time.

The LR invariant operator is very useful in the formulation of quantum mechanics when the Hamiltonian of the system is explicitly dependent on time. In fact, a pioneer in the research of adiabatic invariants is Ermakov[5] who played an active role in theoretical physics in the late 19th century. His research, however, was inevitably limited within the context of classical mechanics since quantum mechanics was not yet known at that time. Thus the actual originator of the invariant operator theory associated with quantum mechanics is Lewis. Many researchers have used the LR invariant operator theory in order to solve the time-dependent Schrödinger equation for a time-dependent Hamiltonian systems (TDHSs), such as a damped harmonic oscillator,[6,7] a time-dependent forced oscillator,[8,9] a time-dependent singular oscillator,[10,11] and a two-dimensional electronic circuit with/without power source.[12,13] The relation between geometric phase and Hannay's angle for the

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general time-dependent oscillator is studied through consideration of the LR invariant operator theory by Song.[14] Kim and Yee presented a unified treatment of the anharmonic and double-well oscillators by employing a variational perturbation method in a complete Hilbert space formulated on the basis of the LR invariant operator.[15] For a more detailed and complete study relevant to invariant operator theory in quantum mechanics, one can refer to Ref. [16] and references therein.

In particular, we have investigated various quantum properties of light in homogeneous conducting linear media[2,17–19] and light propagating in time-varying linear media[20,21] by means of the LR invariant operator method. For several decades, the classical and quantum optics of light propagating in time-varying media have attracted considerable interest in the literature.[20–26] Wave propagation in time-varying media is applicable to the research of various fields such as magnetoelastic delay lines,[27] wave propagation in ionized plasmas,[28] modulation of microwave power,[29] etc. We are going to show theoretically in this paper that the behaviour of single-photon energy in time-varying linear media exhibits nontrivial time dependence without change of frequency, under the assumption that the invariant operator theory is equally applicable in quantum optics as it is in other systems.

2. The LR invariant operator theory for quantum optics

In time-varying linear media, the relations between fields and current are

$$\boldsymbol{D} = \epsilon(t)\boldsymbol{E}, \qquad \boldsymbol{B} = \mu(t)\boldsymbol{H}, \qquad \boldsymbol{J} = \sigma(t)\boldsymbol{E}, \tag{1}$$

where the time-dependent parameters $\epsilon(t)$, $\mu(t)$, and $\sigma(t)$ are electric permittivity, magnetic permeability, and conductivity of the media, respectively. For the sake of simplicity, we suppose that the media have no net charge distribution. Then, with the choice of Coulomb gauge, the scalar potential vanishes and, consequently, the electric and magnetic fields can be represented by expanding only the vector potential. To decouple the functions of variables $\boldsymbol{r}$ and t for vector potential $\boldsymbol{A}$, we put

$$\boldsymbol{A}(\boldsymbol{r}, t) = \boldsymbol{u}(\boldsymbol{r})q(t). \tag{2}$$

By substituting the above equation into a damped wave equation of the form

$$\nabla^2\boldsymbol{A} - \mu(t)[\sigma(t) + \dot{\epsilon}(t)]\frac{\partial \boldsymbol{A}}{\partial t} - \epsilon(t)\mu(t)\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0, \tag{3}$$

we find two separate differential equations related to $\boldsymbol{u}(\boldsymbol{r})$ and $q(t)$, respectively:[20]

$$\nabla^2\boldsymbol{u}(\boldsymbol{r}) + k^2\boldsymbol{u}(\boldsymbol{r}) = 0, \tag{4}$$

$$\ddot{q}(t) + \frac{\sigma(t) + \dot{\epsilon}(t)}{\epsilon(t)}\dot{q}(t) + \omega^2(t)q(t) = 0, \tag{5}$$

where k is a constant wave number and $\omega(t)$ is the time-dependent natural frequency of light. Note that $\omega(t)$ can be represented in terms of the time-dependent velocity of light $c(t)$ such that

$$\omega(t) = \omega(0)c(t)/c(0), \tag{6}$$

under the well known relation $c(t) = [\epsilon(t)\mu(t)]^{-1/2}$. The normalized mode function $\boldsymbol{u}(\boldsymbol{r})$ is determined by applying boundary conditions of the system considered. On the other hand, without loss of generality, a c-number solution of Eq. (5) may be supposed in the form

$$q(t) = M(t)\mathrm{e}^{\mathrm{i}[\gamma(t)+\delta_0]}, \tag{7}$$

where $M(t)$ and $\gamma(t)$ are some time-variable real functions and δ_0 is a real constant. With the replacement of $q(t)$ by operator $\hat{q}$ and the introduction of its canonical conjugate operator $\hat{p} = -\mathrm{i}\hbar\partial/(\partial q)$, the Hamiltonian related to the system is expressed as[20]

$$\hat{H}(\hat{q}, \hat{p}, t) = \mathrm{e}^{-\Lambda(t)}\frac{\hat{p}^2}{2\epsilon_0} + \frac{1}{2}\mathrm{e}^{\Lambda(t)}\epsilon_0\omega^2(t)\hat{q}^2, \tag{8}$$

where $\epsilon_0 = \epsilon(0)$ and

$$\Lambda(t) = \int_0^t \frac{\sigma(t') + \dot{\epsilon}(t')}{\epsilon(t')}\mathrm{d}t'. \tag{9}$$

From basic Hamiltonian dynamics, we can readily show that Eq. (8) gives the exact classical equation given in Eq. (5). The commutation relations for $\hat{q}$ and $\hat{p}$ obey noncommutative algebra, namely, $[\hat{q}, \hat{p}] = \mathrm{i}\hbar$. Due to the explicit time dependence of Eq. (8), the system of light in time-varying media is a kind of TDHS.

The LR invariant operator method is indeed very useful in treating a TDHSs like this, especially in view of quantum mechanics. The powerfulness of the LR invariant operator method in solving the Schrödinger equation of TDHS has been proved through its extensive use in the literature.[2–4,6–14,16–21,30,31] From

$$\frac{\mathrm{d}\hat{I}}{\mathrm{d}t} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{\mathrm{i}\hbar}[\hat{I}, \hat{H}] = 0, \tag{10}$$

the LR invariant operator of the system can be constructed in terms of an annihilation operator $\hat{a}$ and a creation operator $\hat{a}^\dagger$ as[21]

$$\hat{I}(\hat{q}, \hat{p}, t) = \hbar\Omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{11}$$

where Ω is a time-constant given by

$$\Omega = \exp\left[\Lambda(t)\right]\dot{\gamma}(t)M^2(t)/M^2(0). \tag{12}$$

Here, the definitions of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ take the following form

$$\hat{a}(t) = \frac{1}{\sqrt{2}}(\hat{X} + \mathrm{i}\,\hat{Y}), \quad \hat{a}^\dagger(t) = \frac{1}{\sqrt{2}}(\hat{X} - \mathrm{i}\,\hat{Y}), \tag{13}$$

where

$$\hat{X} = \sqrt{\frac{\epsilon_0\dot{\gamma}(t)}{\hbar}}\mathrm{e}^{\Lambda(t)/2}\hat{q}, \tag{14}$$

$$\hat{Y} = \frac{1}{\sqrt{\hbar\epsilon_0\dot{\gamma}(t)}}\left(\mathrm{e}^{-\Lambda(t)/2}\hat{p} - \epsilon_0\frac{\dot{M}(t)}{M(t)}\mathrm{e}^{\Lambda(t)/2}\hat{q}\right). \tag{15}$$

Note that the time-evolution of these ladder operators yields

$$\hat{a}(t) = \hat{a}(0)\exp\left[-\mathrm{i}\,\gamma(t)\right], \tag{16}$$

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)\exp\left[\mathrm{i}\,\gamma(t)\right]. \tag{17}$$

Moreover, they exactly follow the boson commutation relation, such that $[\hat{a}, \hat{a}^\dagger] = 1$. On the other hand, $\hat{a}$ and $\hat{a}^\dagger$ considered in Refs. [32] and [33] in order to describe the quantum theory of damped light are merely the same as that of the simple harmonic oscillator. In that case, it is well known that the ladder operators do not satisfy the boson commutation relation (for example, see Eq. (9.1.17) of Ref. [32]). Therefore, there is a good possibility that this fact raises some difficulty in describing the quantum features of damped light. However, as you can see, the analysis based on the LR invariant operator is trouble-free in this situation. The class of damped light is indeed a good example of a TDHSs whose quantum properties have been investigated on the basis of the LR invariant operator theory by theoretical physics researchers.

If we consider the fact that the boson commutation relation is satisfied in this development, the eigenstates $|\phi_n(t)\rangle$ of the invariant operator may be obtained by means of the conventional method. The ground eigenstate is easily derived from $\hat{a}|\phi_0(t)\rangle = 0$. The arbitrary order excited eigenstates can also be evaluated by operating the creation operator at appropriate times to the ground eigenstate. Through these processes, we get complete eigenstates in q-space in the form[20]

$$\begin{aligned}\langle q|\phi_n(t)\rangle = &\left(\frac{\epsilon_0\dot{\gamma}(t)}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\\ &\times H_n\left(\sqrt{\frac{\epsilon_0\dot{\gamma}(t)}{\hbar}}\mathrm{e}^{\Lambda(t)/2}q\right)\\ &\times\exp\left[\frac{\Lambda(t)}{4} - \frac{\epsilon_0}{2\hbar}\mathrm{e}^{\Lambda(t)}\right.\\ &\left.\times\left(\dot{\gamma}(t) - \mathrm{i}\,\frac{\dot{M}(t)}{M(t)}\right)q^2\right].\end{aligned} \tag{18}$$

The basic idea of the invariant operator method in investigating the quantum solutions of the TDHS is that the wave functions $|\psi_n(t)\rangle$ which satisfy Schrödinger equation are different from the eigenstates $|\phi_n(t)\rangle$ of $\hat{I}$ by only some time-dependent phase factors.[4] Thus, we can write

$$|\psi_n(t)\rangle = |\phi_n(t)\rangle\exp\left[\mathrm{i}\,\delta_n(t)\right], \tag{19}$$

where $\delta_n(t)$ are time-dependent phases given by $\delta_n(t) = -\gamma(t)(n + 1/2)$.

3. Analysis of single-photon energy

In the case where the Hamiltonian is explicitly dependent on time, the role of the Hamiltonian is limited only to the generator of the classical equation of motion. Hence, it is clear that the energy operator for the TDHS is different from the Hamiltonian.[7,30] From the fundamentals of mechanics, the electromagnetic energy in time-varying media is given by[20]

$$U = \frac{1}{2}\epsilon(t)\dot{q}^2 + \frac{1}{2}\epsilon(t)\omega^2(t)q^2. \tag{20}$$

After converting the variable $\dot{q}$ in the above equation into p using the relation $\dot{q} = \mathrm{e}^{-\Lambda(t)}p/\epsilon_0$, one can replace q and p by operators $\hat{q}$ and $\hat{p}$ in order to establish the energy operator. Thus, we have

$$\hat{U} = \frac{1}{2}\epsilon(t)\mathrm{e}^{-2\Lambda(t)}\frac{\hat{p}^2}{\epsilon_0^2} + \frac{1}{2}\epsilon(t)\omega^2(t)\hat{q}^2. \tag{21}$$

By using Eq. (19) with Eq. (18), we can readily evaluate the expectation value of the above energy operator in the number state:

$$\begin{aligned}\langle\psi_n|\hat{U}|\psi_n\rangle = &\frac{\hbar\epsilon(t)}{2\dot{\gamma}(t)\epsilon_0}\mathrm{e}^{-\Lambda(t)}\left(\dot{\gamma}^2(t) + \frac{\dot{M}^2(t)}{M^2(t)}\right.\\ &\left.+\omega^2(t)\right)\left(n + \frac{1}{2}\right).\end{aligned} \tag{22}$$

For $n = 1$, without consideration of the ground state energy, the above equation reduces to single-photon

energy:

$$U_{\text{single-photon}} = \frac{\hbar\epsilon(t)}{2\dot{\gamma}(t)\epsilon_0}\mathrm{e}^{-\Lambda(t)} \times \left(\dot{\gamma}^2(t) + \frac{\dot{M}^2(t)}{M^2(t)} + \omega^2(t)\right). \quad (23)$$

Remarkably, we see that the single-photon energy depends explicitly on time.

One of the typical examples of an optical phenomenon that can be described by a time-dependent Hamiltonian may be the widely observed dissipation of biophoton signals that are emitted spontaneously from various living systems ranging from bacteria to human tissues.[21,34] There are many unusual features associated with biophoton signals. Among them, the most unusual one known from numerous observations is that the emitted photon signal exhibits a non-exponential but hyperbolic decay character in the semiclassical framework. Popp and Li proposed a frequency stable damped harmonic oscillator model in order to describe the relaxation behaviour of biophoton fields.[34] Based on this model, the three time-dependent parameters in Eq. (1) are given by

$$\epsilon(t) = \epsilon_0, \quad \mu(t) = \mu(0), \quad \sigma(t) = \frac{2\epsilon_0\lambda_0}{1+\lambda_0 t}, \quad (24)$$

where λ_0 is a positive real constant. Hence, the effective Hamiltonian is represented in the form

$$\hat{H}(\hat{q},\hat{p},t) = \frac{\hat{p}^2}{2\epsilon_0(1+\lambda_0 t)^2} + \frac{1}{2}\epsilon_0(1+\lambda_0 t)^2\omega_0^2\hat{q}^2, \quad (25)$$

where ω_0 stands for constant mode frequency. In this case, the time-variable functions in Eq. (7) can be written as[21]

$$M(t) = q_0/(1+\lambda_0 t), \quad \gamma(t) = \omega_0 t, \quad (26)$$

where q_0 is a real constant. By substitution of these representations into Eq. (23), the expectation value of the energy for a single photon becomes

$$U_{\text{single-photon}} = \hbar\omega_0\frac{1}{(1+\lambda_0 t)^2} \times \left(1 + \frac{\lambda_0^2}{2\omega_0^2(1+\lambda_0 t)^2}\right). \quad (27)$$

For $\lambda_0 = 0$, the above equation reduces to a familiar form $\hbar\omega_0$. If we analyse the signal on the basis of conventional theory, the energy of the single photon should be constant over time since the frequency of the model is stable. However, from Fig. 1, we can acknowledge that the corresponding single-photon energy based on the LR invariant operator theory decreases continuously with time. In this system, the damping term (second term) in Eq. (5) is proportional to λ_0 which is directly related to conductivity. Therefore, the decay rate of single-photon energy grows as λ_0 increases.

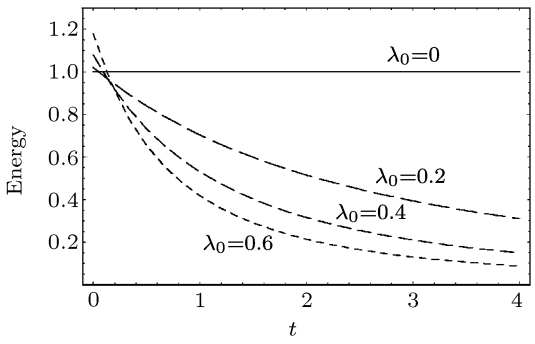


Fig. 1. The expectation value of the single-photon energy for a biophoton signal, represented in Eq. (27), as a function of t for various values of λ_0, indicated by lines. We have taken $\omega_0 = 1$ and $\hbar = 1$.

Now let us consider a simpler case. Even if the time dependence of the parameters disappears, i.e., $\epsilon(t) = \epsilon_0$, $\mu(t) = \mu(0)$, and $\sigma(t) = \sigma(0)$, the Hamiltonian of Eq. (8) still depends on time so long as the conductivity exists. We can confirm this fact by substituting the resultant time function of Eq. (9), $\Lambda(t) = \sigma t/\epsilon$, into Eq.(8). In this case, the Hamiltonian becomes that of the Caldirola–Kanai oscillator[35,36] which corresponds to an ordinary damped harmonic oscillator. Since the classical solutions associated with the Caldirola–Kanai oscillator are well known, we can easily show that the time-variable functions in Eq. (7) are given by

$$M(t) = q_0\mathrm{e}^{-\sigma t/(2\epsilon)}, \quad \gamma(t) = \omega_1 t, \quad (28)$$

where ω_1 is a modified frequency of the form

$$\omega_1 = [\omega_0^2 - \sigma^2/(4\epsilon^2)]^{1/2}. \quad (29)$$

In this case, the time constant is nothing but $\Omega = \omega_1$. Then, the energy of a single photon reduces to

$$U_{\text{single-photon}} = \hbar\frac{\omega_0^2}{\omega_1}\mathrm{e}^{-\sigma t/\epsilon}. \quad (30)$$

Note that this is exactly the same as that in Ref. [2] and also recovers $\hbar\omega_0$ for $\sigma = 0$. We can confirm that Eq. (30) decreases exponentially and continuously with time. The exponential decay character for the quantum energy of a damped harmonic oscillator

has been generally presented several times.[2,7,37,38] Although there is a possibility that the results of this analysis would leave room for some controversy among theoretical researchers in quantum optics, the exponential decay of (quantum) energy is in good agreement with the relevant theory of classical mechanics. Of course, the lost energy is converted to heat energy in the media.

4. Conclusion

The concept that the energy of a single photon is always $\hbar\omega$ has been generally recognized in quantum optics for a long time. However, under the assumption that LR invariant operator theory is admissible in quantum optics, this well known concept is valid only for light described by a time-independent Hamiltonian, i.e., for light in media satisfying the conditions $\epsilon(t) = \epsilon_0$, $\mu(t) = \mu(0)$, and $\sigma(t) = 0$ simultaneously. The energy of a single photon presented in Eq. (23) is explicitly dependent on time even when its frequency is constant. To promote understanding of this result, we considered two special systems, the so-called frequency stable damped harmonic oscillator and the Caldirola–Kanai oscillator. The time behaviour of the single-photon energy is given in Eq. (27) for the former system and in Eq. (30) for the latter system. These decrease continuously with time due to the presence of conductivity in the media. If the conductivity disappears ($\sigma \to 0$), both Eq. (27) and Eq. (30) become constant as expected.

Although lots of dynamical systems have been investigated using approximation and perturbation methods, it is not necessary to employ them when we formulate quantum dynamics on the basis of the invariant operator theory provided that the classical solution Eq. (7) is known or derivable. Therefore, our analysis of photon energy carried out here is mathematically precise. We showed in a previous paper[31] that the quantum electrodynamics developed via the LR invariant operator method is self-consistent, by verifying that the field commutation relations follow Maxwell's equations for the two special cases that we also present in Section 3 of this work. Since the main structure of statistical results for the quantum theory of a TDHS, constructed by the LR invariant operator method, is the same as that of the ordinary simple harmonic oscillator,[21] the present analysis does not give rise to the problem of ultraviolet catastrophe. If the time dependence of the Hamiltonian vanishes, the single-photon energy derived from the LR invariant operator theory exactly recovers to $\hbar\omega$ which is well known as the single-photon energy in free space.

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