Critical entanglement and geometric phase of a two-qubit model with Dzyaloshinski–Moriya anisotropic interaction*

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(Received 6 December 2008; revised manuscript received 15 May 2009)

We consider a two-qubit system described by the Heisenberg XY model with Dzyaloshinski–Moriya (DM) anisotropic interaction in a perpendicular magnetic field to investigate the relation between entanglement, geometric phase and quantum phase transition (QPT). It is shown that the DM interaction has an effect on the critical boundary. The combination of entanglement and geometric phase may characterize QPT completely. Their jumps mean that the occurrence of QPT and inversely the QPT at the critical point at least corresponds to a jump of one of them.

Keywords: entanglement, geometric phase, quantum phase transition

PACC: 0365, 0570J

Entanglement and Berry phase are unique properties of quantum systems and these pure quantum phenomena have been investigated extensively. Recently, these quantities have been adopted as new tools to scale the quantum phase transition (QPT).[1−8] QPT describes the fundamental changes of the macroscopic properties of systems at zero temperature due to small variations of a given external parameter. One kind of QPT is characterized by the structural change of the groundstate energy-spectrum.[9] The interchange of excited-state and ground-state levels at the critical point may exist and create QPT.[9,10] Meanwhile, a quantum state travelling adiabatically around a level-crossing point accumulates a geometric phase in addition to a dynamical phase.[11,12] Berry put a beautiful interpretation on the geometric phase as the magnetic flux due to magnetic monopoles located at a degenerate point. Since the geometric phase and QPT are associated with level crossing, the geometric phase might be a good indicator of QPTs.[7,8] In addition, entanglement, as an important resource central to the development of quantum information, is a special quantum correlation between subsystems. Concurrence or von Neumann entropy is a good entanglement measure for a bipartite state. Because at quantum critical points the correlation length becomes divergent, much attention has been paid to the relation between entanglement and QPT.[1,2,13] The relation between entanglement, Berry phases and level crossings for a two-qubit system with XY-type interaction was investigated and it was found that the level crossing is not always accompanied by an abrupt change in entanglement.[14] In this paper we consider Dzyaloshinski–Moriya (DM) anisotropic antisymmetric interaction, a combined effect of spin–orbit coupling and exchange interaction discussed by several authors[15−18] to expand the model in Ref. [14] and give a partial answer to the question of whether the entanglement and geometric phases are always good indicators of quantum transitions or not. It is shown that DM interaction affects critical points of QPT where either entanglement or geometric phase may have abrupt changes.

We consider a system of two qubits described by the Heisenberg XY model with DM anisotropic antisymmetric interaction in a perpendicular magnetic field. The corresponding Hamiltonian is

$$\begin{aligned}\boldsymbol{H} = & \frac{J}{2}\left[(1+\gamma)\,\boldsymbol{\sigma}_{1x}\boldsymbol{\sigma}_{2x}+(1-\gamma)\,\boldsymbol{\sigma}_{1y}\boldsymbol{\sigma}_{2y}\right] \\ & +\frac{D}{2}\left(\boldsymbol{\sigma}_{1x}\boldsymbol{\sigma}_{2y}-\boldsymbol{\sigma}_{1y}\boldsymbol{\sigma}_{2x}\right)+\frac{B}{2}\left(\boldsymbol{\sigma}_{1z}+\boldsymbol{\sigma}_{2z}\right), \quad (1)\end{aligned}$$

where J is the strength of the Heisenberg exchange in-

*Project supported by the Natural Science Foundation for Young Scientists of Shanxi Province of China (Grant No. 2007021001), the Science and Technology Key Item of Chinese Ministry of Education (Grant No. 207017), National Fundamental Fund of Personnel Training (Grant No. J0730317) and the National Natural Science Foundation of China (Grant No. 10774094).

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teraction, and γ is an anisotropy factor. The magnetic field B is along the z axis. $\boldsymbol{\sigma}_{i\alpha}$ are the Pauli matrices of the i-th qubit with $\alpha = x, y, z$. DM anisotropic interaction has the form $\boldsymbol{D}\cdot(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)$; here we choose the DM vector coupling factor $\boldsymbol{D} = D\hat{z}$ for simplicity. In the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, the Hamiltonian (1) has the matrix form

$$\boldsymbol{H} = \begin{pmatrix} B & 0 & 0 & J\gamma \\ 0 & 0 & J + \mathrm{i}D & 0 \\ 0 & J - \mathrm{i}D & 0 & 0 \\ J\gamma & 0 & 0 & -B \end{pmatrix} \tag{2}$$

with the eigenvalues

$$E_{1,2} = \pm\sqrt{(B^2 + J^2\gamma^2)}, \tag{3}$$

$$E_{3,4} = \pm\sqrt{(J^2 + D^2)}, \tag{4}$$

and the corresponding eigenvectors

$$|\phi_1\rangle = \cos\frac{\theta}{2}|11\rangle + \sin\frac{\theta}{2}|00\rangle, \tag{5}$$

$$|\phi_2\rangle = \sin\frac{\theta}{2}|11\rangle - \cos\frac{\theta}{2}|00\rangle, \tag{6}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\eta}|10\rangle + |01\rangle\right), \tag{7}$$

$$|\phi_4\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\eta}|10\rangle - |01\rangle\right), \tag{8}$$

where $\tan\theta = J\gamma/|B|$, $\tan\eta = D/J$. Two lowest energy levels E_2 and E_4 are crossing when $B^2 = \left(J\sqrt{1-\gamma^2}\right)^2 + D^2$. That is to say, when $B^2 > \left(J\sqrt{1-\gamma^2}\right)^2 + D^2$, $|\phi_2\rangle$ is the groundstate, otherwise, $|\phi_4\rangle$ is the groundstate. The DM interaction affects the critical boundary of QPT.

For a pure two-qubit state $|\Psi\rangle = a|11\rangle + b|10\rangle + c|01\rangle + d|00\rangle$ with $|a|^2+|b|^2+|c|^2+|d|^2 = 1$, the degree of entanglement measured by the concurrence C can be calculated by the expression[19] $C(|\Psi\rangle) = 2|ad-bc|$. For the groundstate $|\Psi_0\rangle$ given by $|\phi_2\rangle$ and $|\phi_4\rangle$, it is written as

$$C(|\Psi_0\rangle) = \begin{cases} |\sin\theta| = \left|\dfrac{J\gamma}{\sqrt{(B^2 + J^2\gamma^2)}}\right|, & B^2 > \left(J\sqrt{1-\gamma^2}\right)^2 + D^2, \\ 1, & B^2 < \left(J\sqrt{1-\gamma^2}\right)^2 + D^2. \end{cases} \tag{9}$$

From the above equation, it is clearly shown that, when $B \neq 0$, the entanglement always undergoes an abrupt change at the critical boundary of QPT, especially when $\gamma = 0$. But for $B = 0$, if $|\gamma| \leq 1$, the groundstate is always $|\phi_4\rangle$, there is no QPT to occur. If $|\gamma| > 1$ and the condition $J^2\left(1-\gamma^2\right) + D^2 < 0$ may come into existence, the QPT can occur, but the entanglement of the groundstate is a constant, i.e., $C(|\Psi_0\rangle) = 1$, in all regimes, so it is not a good indicator of QPT. Furthermore, when $D = 0$, all results reduce to those of Ref. [14].

Next, we turn to the relation between geometric phases and QPTs. In order to generate a Berry phase we perform a cyclic loop in the Poincaré sphere, rotating adiabatically the system around the direction of the magnetic field (z axis) by an angle ϕ. The transformed Hamiltonian by means of the unitary transformation $\boldsymbol{U}_z(\phi) = \exp\left[-\mathrm{i}\frac{\phi}{2}\left(\boldsymbol{\sigma}_1^z + \boldsymbol{\sigma}_2^z\right)\right]$ has the corresponding matrix form

$$\tilde{\boldsymbol{H}} = \boldsymbol{U}_z(\phi)\,\boldsymbol{H}\boldsymbol{U}_z^{\dagger}(\phi) = \begin{pmatrix} \cos\theta & 0 & 0 & \sin\theta\,\mathrm{e}^{-\mathrm{i}2\phi} \\ 0 & 0 & \sqrt{J^2 + D^2}\,\mathrm{e}^{\mathrm{i}\eta} & 0 \\ 0 & \sqrt{J^2 + D^2}\,\mathrm{e}^{-\mathrm{i}\eta} & 0 & 0 \\ \sin\theta\,\mathrm{e}^{\mathrm{i}2\phi} & 0 & 0 & -\cos\theta \end{pmatrix}, \tag{10}$$

where ϕ is a slowly varying parameter. Notice that the rotating action only has an effect on the subspace $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, while the other subspace $\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$ is invariable. The rotating azimuth angle is 2ϕ even if the system is rotated by ϕ. This is due to the bilinear form of Hamiltonian (1). It is instructive to compare it with the rotation of a single spin about the z-axis by ϕ. The single-spin-rotated Hamiltonian $\tilde{\boldsymbol{H}}_s = \boldsymbol{R}_z^{\dagger}\boldsymbol{H}_s\boldsymbol{R}_z = \begin{pmatrix} \cos\theta & \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} \\ \sin\theta\,\mathrm{e}^{\mathrm{i}\phi} & -\cos\theta \end{pmatrix}$ with $\boldsymbol{H}_s = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and $\boldsymbol{R}_z = \exp\left[\mathrm{i}\frac{\phi}{2}\boldsymbol{\sigma}_z\right]$. It is obvious that $\tilde{\boldsymbol{H}}_s$ is 2π periodic in θ and ϕ while the Hamiltonian $\tilde{\boldsymbol{H}}$ is π periodic in ϕ, $\tilde{\boldsymbol{H}}(\theta, \phi) = \tilde{\boldsymbol{H}}(\theta + 2\pi, \phi + \pi)$.

When the groundstate of the system is changed adiabatically through the path $\mathcal{C}$ in the parameter space $\boldsymbol{R} = (\sin\theta\cos(2\phi), \sin\theta\cos(2\phi), \cos\theta)$, the instantaneous groundstate $|\Psi_0(\boldsymbol{R})\rangle$ reads

$$\left|\tilde{\Psi}_0(\boldsymbol{R})\right\rangle = \begin{cases} |\tilde{\phi}_2\rangle = \sin\dfrac{\theta}{2}\,\mathrm{e}^{\mathrm{i}2\phi}|11\rangle - \cos\dfrac{\theta}{2}|00\rangle, & B^2 > \left(J\sqrt{1-\gamma^2}\right)^2 + D^2, \\ |\tilde{\phi}_4\rangle = \dfrac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\eta}|10\rangle - |01\rangle\right), & B^2 < \left(J\sqrt{1-\gamma^2}\right)^2 + D^2, \end{cases} \tag{11}$$

which satisfies the instantaneous eigen equation $\tilde{H}(\boldsymbol{R})\left|\tilde{\Psi}_0(\boldsymbol{R})\right\rangle = E_0(\boldsymbol{R})\left|\tilde{\Psi}_0(\boldsymbol{R})\right\rangle$. With Eq. (11), it is easy to calculate the Berry connection or vector potential $\boldsymbol{A}_g$,

$$\boldsymbol{A}_g \equiv \mathrm{i}\left\langle\tilde{\Psi}_0(\boldsymbol{R})\right|\nabla\left|\tilde{\Psi}_0(\boldsymbol{R})\right\rangle = \begin{cases} -\dfrac{2}{r\sin\theta}\sin^2\dfrac{\theta}{2}\widehat{\boldsymbol{\phi}}, & B^2 > \left(J\sqrt{1-\gamma^2}\right)^2 + D^2, \\ 0, & B^2 < \left(J\sqrt{1-\gamma^2}\right)^2 + D^2, \end{cases} \tag{12}$$

and the accumulating geometric phase Γ of the groundstate evolving adiabatically in a period,

$$\Gamma = \oint_{\mathcal{C}} \boldsymbol{A}_g \cdot \mathrm{d}\boldsymbol{R} = \begin{cases} -\pi(1-\cos\theta) = -\pi\left(1 - \dfrac{|B|}{\sqrt{(B^2 + J^2\gamma^2)}}\right), & B^2 > \left(J\sqrt{1-\gamma^2}\right)^2 + D^2, \\ 0, & B^2 < \left(J\sqrt{1-\gamma^2}\right)^2 + D^2. \end{cases} \tag{13}$$

To describe the relation between entanglement, geometric phase and the QPT in detail, we select several special cases. One case is making $\gamma = 0$, then the system reduces to the isotropic XY model with DM interaction. As shown by the solid line in Fig. 1, the critical boundary of QPT, that consists of the crossing points of energy level E_2 and energy level E_4 in J–D plane, is a circle with radius $|B|$. Within the circle the groundstate is $|\phi_2\rangle$, while out of the circle the groundstate is $|\phi_4\rangle$. The geometric phase in all regimes is zero for this case, so it is completely failing to characterize the QPT. However, as stated above, the entanglement has an abrupt variation from 0 to 1 at the boundary and so can indicate the QPT very well.

The other case is making $\gamma = 1$, then the system reduces to the Ising-like model with DM interaction. The critical boundary of QPT becomes two parallel lines with $D = \pm|B|$ in the J–D plane (see the two dot lines in Fig. 1). The variation of geometric phase and concurrence versus the strength of interactions J and D are shown in Fig. 2. For $J = 0$, the entanglement has a jump at the critical point but the geometric phase is invariable. With the increase of $|J/B|$, the jump of entanglement becomes smaller but the geometric phase becomes larger. When $|J/B| \gg 1$, the entanglement tends to 1 while the geometric phase has the largest jump from 0 to $-\pi$. As a result, the entanglement and the geometric phase are complementary to characterize QPT.

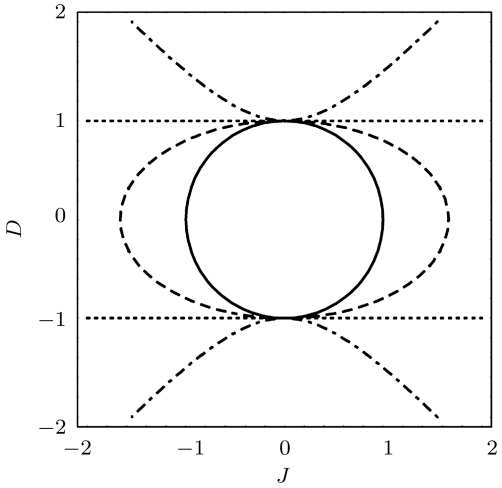


Fig. 1. Critical boundary of groundstates $|\phi_2\rangle$ and $|\phi_4\rangle$ in J–D plane with axial unit $|B|$ for different γ. Solid, dash, dot, dash-dot lines correspond to $\gamma = 0$, 0.5, 1.0, 1.5, respectively.

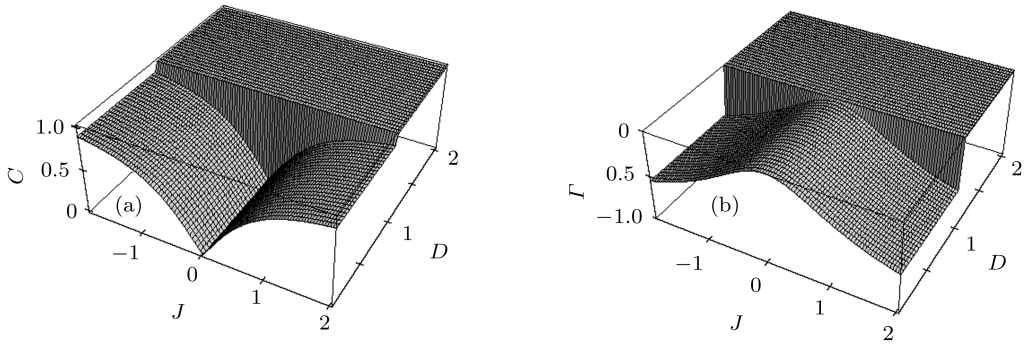


Fig. 2. The variations of concurrence (a) and geometric phase with unit π (b) versus the interactions J and D with unit $|B|$ for $\gamma = 1$. The part for $D < 0$ is a mirror image with respect to the plane of $D = 0$.

The third case is making $\gamma = 1.5$, so the system has anisotropic exchange interaction and the signs in the x direction and the y direction are opposite. The crossing line of two lowest energy levels is a hyperbola in the J–D plane (see the two dash-dot lines in Fig. 1). The variation of geometric phase and concurrence versus the strength of interactions J and D are shown in Fig. 3 and the same conclusion can be obtained as that in Fig. 2. When $B = 0$ in this case, the critical boundary of the hyperbola degenerates to two lines which cross at the origin and we have $C\left(|\phi_2\rangle\right) = C\left(|\phi_4\rangle\right) = 1$ while $\Gamma\left(|\phi_2\rangle\right) = -\pi$ and $\Gamma\left(|\phi_4\rangle\right) = 0$. So the entanglement fails to characterize QPT but the geometric phase is a very good indicator.

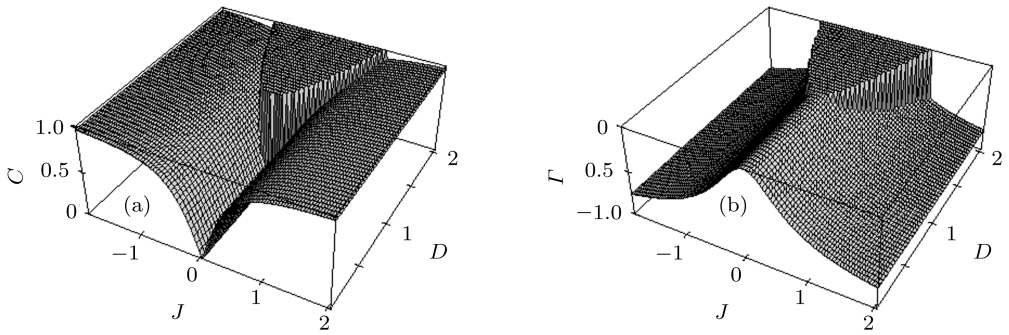


Fig. 3. The variations of concurrence (a) and geometric phase with unit π (b) versus the strength of interactions J and D with unit $|B|$ for $\gamma = 1.5$. The part for $D < 0$ is a mirror image with respect to the plane of $D = 0$.

To summarize, we choose a two-qubit Heisenberg XY model with DM anisotropic interaction in a perpendicular magnetic field to investigate the relation between the entanglement, geometric phase and the QPT (more exactly, the quantum critical phenomena). We show that DM interaction can affect the critical boundary. Only entanglement or geometric phase cannot characterize QPT at some critical points, but their combination can do so at the whole critical boundary. That is to say, QPT must correspond to a jump of either entanglement or geometric phase in this model. Here, we must mention that the universality of this conclusion is still an open question for other models and we will investigate this in succession.

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