Dynamics of vector solitons in two-component Bose–Einstein condensates with time-dependent interactions and harmonic potential*

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We present two kinds of exact vector-soliton solutions for coupled nonlinear Schrödinger equations with time-varying interactions and time-varying harmonic potential. Using the variational approach, we investigate the dynamics of the vector solitons. It is found that the two bright solitons oscillate about slightly and pass through each other around the equilibration state which means that they are stable under our model. At the same time, we obtain the opposite situation for dark–dark solitons.

Keywords: two-component Bose–Einstein condensates, vector solitons, the variational approach

PACC: 0365, 0530J, 1110L

1. Introduction

The experimental realization of multicomponent Bose–Einstein condensates (BECs) has generated great interest in this interdisciplinary field.[1,2] Compared with single-component BECs, the multicomponent BECs possess more rich phenomena and complex dynamics,[3−5] in particular, vector solitons[6−8] and their interactions.[9] Vector solitons in the form of dark–dark,[10] bright–bright,[11] and bright–dark[12] solitons are studied. Recently, BEC mixtures of two hyperfine states of the same atom have been available for experiment,[13] which motivates further research around this problem. Within the mean-field approximation, these systems can be described by a coupled Gross–Pitaevskii (GP) equation.[14]

As we know, in the GP model, the properties of the nonlinear excitation are determined by the interaction between atoms, which is typically characterized by the s-wave scattering length. Recently experiments have demonstrated that the atomic s-wave scattering length can be changed in time by means of external magnetic fields, i.e. by using Feshbach resonances.[15,16] This exciting technical progress offers a great opportunity for the manipulation of atomic matter waves and nonlinear excitations in multicomponent BECs.

Here our aim is to study vector solitons and their dynamics in two-component BECs with tunable interaction parameters and harmonic trapping potential. We extend the study on single-component BECs to the two-component case and obtain exact vector-soliton solutions of the quasi-one-dimensional coupled GP equation in the case with equal interaction parameters,[17] including bright–bright and dark–dark vector solitons. In general, the interaction parameters are different, but we can use Feshbach resonances to modify them to keep exactly equal values. The parameters (amplitude and width) of vector solitons can be changed in a controllable manner by changing the scattering length and harmonic trapping potential. This is significant to help in devising experiments. We also investigate the dynamics of the vector solitons by using the variational approach.

2. The theoretical model

Our model is the quasi-one-dimensional coupled dimensionless nonlinear Schrödinger (NLS) equations[7,18]

$$\mathrm{i}\frac{\partial\psi_1}{\partial t} = \left(-\frac{\partial^2}{\partial x^2} + V_1 + g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2\right)\psi_1, \quad (1)$$

$$\mathrm{i}\frac{\partial\psi_2}{\partial t} = \left(-\frac{\partial^2}{\partial x^2} + V_2 + g_{21}|\psi_1|^2 + g_{22}|\psi_2|^2\right)\psi_2, \quad (2)$$

where the condensate wave functions are normalized by conditions: $\int|\psi_1|^2\,\mathrm{d}x = 1$ and $\int|\psi_2|^2\,\mathrm{d}x = N_2/N_1$, with N_j being the number of atoms for species j ($j = 1, 2$). The trapping potentials are assumed to be $V_j = k_j^2(t)x^2 = \omega_j^2(t)x^2/\omega_{j\perp}^2$, where ω_j and $\omega_{j\perp}$

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are angular frequencies in the axial and radial directions respectively. The coefficients of the nonlinear terms are $g_{jk} = 4a_{jk}N_1/a_\perp$, where a_{jk} is the scattering length between species j and k, $j, k = 1, 2$, m is the mass of an atom, and $a_\perp$ is defined as $a_\perp = \sqrt{\hbar/m\omega_\perp}$.

In this paper, we assume that $k_1(t) = k_2(t) \equiv k(t)$, and $g_{jk}(t) = \sigma g(t)$, $\sigma = \pm 1$. Equations (1) and (2) can be rewritten as

$$\mathrm{i}\frac{\partial \psi_j}{\partial t} = -\frac{\partial^2 \psi_j}{\partial x^2} + k(t)x^2\psi_j + \sigma g(t)(|\psi_1|^2 + |\psi_2|^2)\psi_j, \quad j = 1, 2, \tag{3}$$

where $\sigma = \pm 1$ corresponds to positive and negative scattering length, respectively. In order to obtain an exact solution of Eq. (3), we make use of the modified lens-type transformation as follows:[17,19]

$$\psi_j(x,t) = g^{-1/2}(t)\ell^{-1}(t)\phi_j(z,\tau) \times \exp\{\mathrm{i}[f(t)x^2]\}, \quad j = 1, 2, \tag{4}$$

$$z = x/\ell(t), \quad \tau(t) = \int_0^t \ell^{-2}(s)\,\mathrm{d}s, \tag{5}$$

where $f(t)$ and $\ell(t)$ are two real functions of time. Taking into account these transformation, we can reduce Eq. (3) to a standard coupled nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial \phi_j}{\partial \tau} = -\frac{\partial^2 \phi_j}{\partial z^2} + \sigma(|\phi_1|^2 + |\phi_2|^2)\phi_j, \quad j = 1, 2, \tag{6}$$

if $f(t)$, $\ell(t)$, $k(t)$, and $g(t)$ satisfy the following relations:

$$f_t + 4f^2 + k^2 = 0, \quad \ell_t - 4f\ell = 0, \quad g^{-1}g_t + 4f = 0. \tag{7}$$

It follows from Eq. (7) that

$$\ell(t) = \ell(0)g(0)/g(t), \tag{8}$$

$$-g^{-1}g_{tt} + 2g^{-2}g_t^2 + 4k^2 = 0. \tag{9}$$

For attractive inter-atomic interactions, i.e., $\sigma = -1$, Eq. (6) possesses an exact bright–bright vector-soliton solution in the form

$$\phi_{\mathrm{B}j} = \eta \mathrm{sech}[\eta(z - z_0)]\exp(\mathrm{i}\eta^2\tau), \quad j = 1, 2, \tag{10}$$

where η and z_0 are real constants which determine the amplitude (as well as width) and initial position of the soliton respectively. The two bright solitons in Eq. (11) overlap completely and keep rest. We call this state the equilibrium state.

Returning to the original variables x and t defined by Eq. (5), we obtain an exact bright–bright vector-soliton solution of the coupled GP equation (3) as

$$\psi_{\mathrm{B}j} = \tilde{\eta}\sqrt{g(t)}\ \mathrm{sech}[\tilde{\eta}g(t)(x - x_0)] \times \exp\left[\mathrm{i}\tilde{\eta}^2\int_0^t g^2(s)\,\mathrm{d}s + \mathrm{i}f(t)(x - x_0)^2\right], \quad j = 1, 2, \tag{11}$$

here $\tilde{\eta} = \eta/[\ell(0)g(0)]$.

Similarly, for repulsive inter-atomic interactions, i.e., $\sigma = 1$, Eq. (3) possesses an exact dark–dark vector-soliton solution in the form

$$\psi_{\mathrm{D}j} = \tilde{\eta}\sqrt{g(t)}\tanh[\tilde{\eta}g(t)(x - x_0)]\exp\left[-2\mathrm{i}\tilde{\eta}^2\int_0^t g^2(s)\,\mathrm{d}s + \mathrm{i}f(t)(x - x_0)^2\right], \quad j = 1, 2. \tag{12}$$

Equations (11) and (12) indicate that the amplitude and the width of the vector solitons vary with function $g(t)$. On the other hand, for a given variation of the scattering length $g(t)$, one can control the variation of the soliton parameter. This is significant to help in devising experiments.

3. The stability of the vector solitons

First, we discuss the interaction between components in a bright–bright solitons (in this case, $\sigma = -1$). For this purpose, we assume that the vector solitons deviate from their equilibrium state, and they are expressed by the following ansatz.

$$\phi_{\mathrm{B}j} = \eta(\tau)\ \mathrm{sech}[\eta(\tau)(z - z_j(\tau))]\exp\left[\mathrm{i}\int \eta(\tau)^2\,\mathrm{d}\tau + \mathrm{i}\mu_j(\tau)(z - z_j(\tau)) + \mathrm{i}\delta_j(\tau)\right], \quad j = 1, 2. \tag{13}$$

The Lagrangian density $\mathcal{L}$ from Eq. (6) can be derived as:

$$\mathcal{L} = \frac{\mathrm{i}}{2}(\phi_1^*\phi_{1\tau} - \phi_1\phi_{1\tau}^* + \phi_2^*\phi_{2\tau} - \phi_2\phi_{2\tau}^*) - (|\phi_{1z}|^2 + |\phi_{2z}|^2) + |\phi_1|^2|\phi_2|^2 + \frac{1}{2}(|\phi_1|^4 + |\phi_2|^4), \tag{14}$$

where the asterisk denotes the complex conjugate. Substituting Eq. (13) into the Lagrangian density Eq. (14) and integrating the result over z from $-\infty$ to ∞, we obtain the Lagrangian

$$L = \int_{-\infty}^{\infty} \mathcal{L}\mathrm{d}z = -2\eta(\eta^2 - \mu_1\dot{z}_1 + \dot{\delta}_1) - 2\eta(\eta^2 - \mu_2\dot{z}_2 + \dot{\delta}_2) - 2\mu_1^2\eta - 2\mu_2^2\eta + \eta^3 F(z_2 - z_1), \tag{15}$$

where overdot represents the derivative with respect to the time τ and

$$F(z_2 - z_1) = 4\left\{\frac{\eta(z_2 - z_1)\cosh[\eta(z_2 - z_1)]}{\sinh^3[\eta(z_2 - z_1)]} - \frac{1}{\sinh^2[\eta(z_2 - z_1)]}\right\}. \tag{16}$$

Now we can derive a set of differential equations for ansatz parameters from the least-action principle $\delta I = \delta\int_{\tau_1}^{\tau_2} L\mathrm{d}\tau = 0$:

$$\delta I/\delta\delta_1 = 0 \to \eta = \text{const.}, \tag{17}$$

$$\delta I/\delta\delta_2 = 0 \to \eta = \text{const.}, \tag{17}$$

$$\delta I/\delta\mu_1 = 0 \to \dot{z}_1 = 2\mu_1, \tag{18}$$

$$\delta I/\delta\mu_2 = 0 \to \dot{z}_2 = 2\mu_2, \tag{18}$$

$$\delta I/\delta_{z_1} = 0 \to 2\eta\dot{\mu}_1 - \eta^3\partial_{z_1}F(z_2 - z_1), \tag{19}$$

$$\delta I/\delta_{z_2} = 0 \to 2\eta\dot{\mu}_2 - \eta^3\partial_{z_2}F(z_2 - z_1). \tag{20}$$

When $z_2 - z_1$ is small, we expand $F(z_2 - z_1)$ as a power series of $z_2 - z_1$ up to the second-order, and obtain

$$F(z_2 - z_1) = \frac{4}{3} - \frac{8}{15}(z_2 - z_1)^2. \tag{21}$$

From Eqs. (18)–(21) we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(z_2 - z_1) + \frac{32}{15}\eta^4(z_2 - z_1) = 0, \tag{22}$$

so the solution of Eq. (22) is

$$z_2 - z_1 = A\sin(\omega_{\mathrm{BB}}\tau + \varphi), \tag{23}$$

where the both amplitude A and initial phase angle φ are decided by the initial conditions. The angular frequency is obtained from Eq. (22) as

$$\omega_{\mathrm{BB}} = \sqrt{\frac{32}{15}}\eta^2. \tag{24}$$

Returning to the original variables x and t, and noting that $\Delta_{\mathrm{BB}}(0) = 0$, we obtain from Eq. (23) that

$$\begin{aligned}\Delta_{\mathrm{BB}} &= A\ell(t)\sin\left[\frac{\sqrt{32}}{15}\eta^2\int_0^t \ell^{-2}(s)\mathrm{d}s\right] \\ &= \frac{A}{\tilde{g}(t)}\sin\left[\frac{\sqrt{32}}{15}\eta^2\int_0^t \tilde{g}^2(s)\mathrm{d}s\right], \\ \tilde{g}(t) &= \frac{g(t)}{\ell(0)g(0)},\end{aligned} \tag{25}$$

where $\Delta_{\mathrm{BB}} = x_2 - x_1$ represents the separation between the two solitons. For better study, as discussed in Ref. [20], a periodic scattering length is obtained by means of an ac magnetic field, we suppose that the interaction parameter $g(t)$ is modulated periodically as

$$g(t) = g_0[1 + m\sin(\omega t)], \tag{26}$$

where m and ω are the amplitude and frequency of the modulation. Substituting Eq. (26) into Eq. (25) yields

$$\begin{aligned}\Delta_{\mathrm{BB}} = &\frac{A}{\ell(0)[1 + m\sin(\omega t)]} \\ &\times\sin\left[\sqrt{\frac{32}{15}}\eta^2\int_0^t (1 + m\sin(\omega t))^2\mathrm{d}s\right]\end{aligned} \tag{27}$$

The curve Δ_{BB}–t is described in Fig. 1. Obviously, the two solitons oscillate about slightly and pass through each other around the equilibration state. This means that the system of bright–bright solitons is stable.

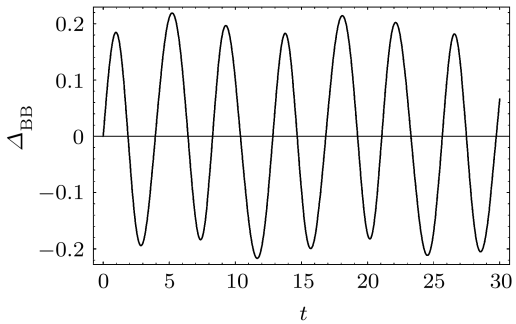


Fig. 1. The distance between the two bright–bright solitons Δ_{BB} versus time t. Here $A = 0.2$, $\ell(0) = 1$, $m = 0.1$, $\omega = 1$, and $\eta = 1$

Similarly, for repulsive inter-atomic interactions, ($\sigma = 1$), the ansatz for dark–dark vector soliton solutions is

$$\phi_{\mathrm{D}j} = \eta(\tau)\tanh[\eta(\tau)(z - z_j(\tau))]\exp\left[-2\mathrm{i}\int\eta(\tau)^2\mathrm{d}\tau + \mathrm{i}\mu_j(\tau)(z - z_j(\tau)) + \mathrm{i}\delta_j(\tau)\right], \quad j = 1, 2. \tag{28}$$

The appropriate Lagrangian density is

$$\mathcal{L} = \frac{\mathrm{i}}{2}(\phi_1^*\phi_{1\tau} - \phi_1\phi_{1\tau}^* + \phi_2^*\phi_{2\tau} - \phi_2\phi_{2\tau}^*) - (|\phi_{1z}|^2 + |\phi_{2z}|^2) - |\phi_1|^2|\phi_2|^2 - \frac{1}{2}(|\phi_1|^4 + |\phi_2|^4). \tag{29}$$

Integrating Eq. (29) over z, we obtain the Lagrangian L as follows:

$$L = \int_{-l/2}^{l/2} \mathcal{L}\,\mathrm{d}z = (\eta^2 l - 2\eta)(\eta^2 + \mu_1\dot{z}_1 - \dot{\delta}_1 - \mu_1^2) + (\eta^2 l - 2\eta)(\eta^2 + \mu_2\dot{z}_2 - \dot{\delta}_2 - \mu_2^2) + \eta^2 l\dot{\mu}_1 z_1 + \eta^2 l\dot{\mu}_2 z_2 - \eta^3 F(z_2 - z_1), \tag{30}$$

where l is the length of the cigar-shaped BEC. According to the least-action principle, we obtain

$$\delta I/\delta\delta_1 = 0 \to \eta = \text{const.},$$
$$\delta I/\delta\delta_2 = 0 \to \eta = \text{const.}, \tag{31}$$
$$\delta I/\delta\mu_1 = 0 \to \dot{z}_1 = (2 - \eta l)\mu_1,$$
$$\delta I/\delta\mu_2 = 0 \to \dot{z}_2 = (2 - \eta l)\mu_2, \tag{32}$$
$$\delta I/\delta_{z_1} = 0 \to 2\eta\dot{\mu}_1 - \eta^3\partial_{z_1}F(z_2 - z_1), \tag{33}$$
$$\delta I/\delta_{z_2} = 0 \to 2\eta\dot{\mu}_2 - \eta^3\partial_{z_2}F(z_2 - z_1). \tag{34}$$

When $z_2 - z_1$ is small, from Eqs. (33)–(35), and (21) we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(z_2 - z_1) - \lambda(z_2 - z_1) = 0, \tag{35}$$

where $\lambda = 16\eta^4(\eta l - 2)/15$. Because $|\phi_i|^2 > 0$, we obtain the condition $\eta l - 2 > 0$. The solution of Eq. (36) is

$$z_2 - z_1 = B\,\mathrm{e}^{\sqrt{\lambda}\tau} + C\,\mathrm{e}^{-\sqrt{\lambda}\tau}. \tag{36}$$

According to Eq. (5), changes to the original temporal and spatial coordinates, together with Δ_{DD}

$$\Delta_{\mathrm{DD}} = \frac{B}{\tilde{g}(t)}\left[\mathrm{e}^{\sqrt{\lambda}\int_0^t \tilde{g}^2(s)\,\mathrm{d}s} - \mathrm{e}^{-\sqrt{\lambda}\int_0^t \tilde{g}^2(s)\,\mathrm{d}s}\right],$$
$$\tilde{g}(t) = \frac{g(t)}{\ell(0)g(0)}. \tag{37}$$

Substituting Eq. (26) into Eq. (37)

$$\Delta_{\mathrm{DD}} = \frac{B}{\zeta(t)}\left[\mathrm{e}^{\sqrt{\lambda}\int_0^t \zeta^2(s)\,\mathrm{d}s} - \mathrm{e}^{-\sqrt{\lambda}\int_0^t \zeta^2(s)\,\mathrm{d}s}\right],$$
$$\zeta(t) = \frac{1 + m\sin(\omega t)}{\ell(0)}. \tag{38}$$

The curve Δ_{DD}–t is described in Fig. 2. Obviously, the distance between the two dark–dark solitons grows infinitely in time t, so the dark–dark solitons are unstable.

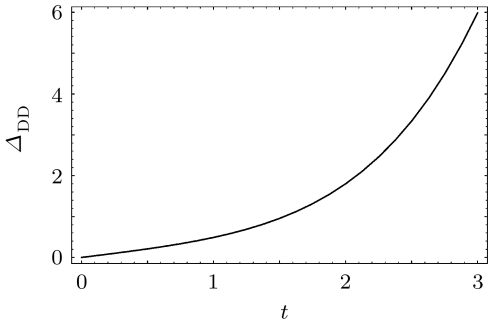


Fig. 2. The distance between the two dark–dark solitons Δ_{DD} versus time t with $\lambda = 1$ and $B = 1$. The other parameters are the same as in Fig. 1.

4. Conclusions

In summary, when the interactions take equal values, we have obtained exact bright–bright solitons and dark–dark solitons in the frame of the coupled GP mean-field theory. Based on this model, we investigate the dynamics of the vector solitons by using the variational approach. The two bright solitons oscillate about slightly and pass through each other around the equilibration state which means the bright–bright solitons are stable. However, supposing some perturbation appears, the distance between the two dark–dark solitons increases with time t, which means that the dark–dark solitons are unstable. We hope our theoretical results will stimulate experiments in this direction.

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