Nonperturbative solutions to cylindrical resonant cavities with dissipative medium and imperfectly conducting walls*

Lin Qiong-Gui(林琼桂)†

School of Physics and Engineering, Sun Yat-Sen University, Guangzhou 510275*, China*

(Received 12 December 2008; revised manuscript received 28 April 2009)

Cylindrical waveguides without end surfaces can serve as two-dimensional resonant cavities. In such cavities the electromagnetic oscillations corresponding to an eigenfrequency can always be taken as TM or TE modes even when the walls have a finite conductivity and the medium is absorptive. This paper obtains analytic solutions to the field equations when the cylinder has a circular cross section. Some nonperturbative conclusions are drawn from the eigenvalue equation. Approximate analytic results for the resonant frequencies are obtained when the absorption of the medium is small and the walls are good conductors. Stability of the eigen modes is discussed. Similar results for the coaxial line are presented.

Keywords: resonant cavity, nonperturbative solution, dissipative medium, finite conductivity
PACC: 0350D, 4110H

1. Introduction

In classical electrodynamics, the subject of waveguides and resonant cavities is of practical importance. The physical theory for the subject has been well developed and their applications have also been widely studied.[1−5] Yet it remains an active subject of current researches.[6−12]

A waveguide is a cylindrical hollow pipe with conducting walls which may be filled with a dielectric medium. Let the z axis of the coordinate system be parallel to the generatrix of the cylinder; the electric field inside the pipe is of the form $\boldsymbol{E}(\boldsymbol{x},t) = \boldsymbol{E}(\boldsymbol{\rho})\mathrm{e}^{\mathrm{i}(kz-\omega t)}$, where $\boldsymbol{\rho}$ stands for the transverse coordinates on the xy plane, and similarly for the magnetic field. The frequency ω is determined by the external source which excites the electromagnetic fields, the propagating constant k is related to ω and the relation is determined by the field equations and boundary conditions. An important class of resonant cavities is produced by placing two conducting end surfaces at $z = 0$ and $z = d$, which are perpendicular to the z axis. These are cylindrical resonant cavities. However, the endless pipe itself can serve as a two-dimensional resonant cavity. In fact, if the exciting source has a frequency ω such that $k = 0$, the above electric field becomes $\boldsymbol{E}(\boldsymbol{x},t) = \boldsymbol{E}(\boldsymbol{\rho})\mathrm{e}^{-\mathrm{i}\,\omega t}$, and similarly for the magnetic field. The fields are independent of z, in the transverse directions they are mainly confined to a finite region, and the possible frequency is determined by the field equations and boundary conditions. Therefore they are typically electromagnetic oscillations in a two-dimensional cavity. Here 'two-dimensional' means that all fields are independent of z, but it does not mean that they have only transverse components. It appears that little attention has been paid to this case. Though the application of such two-dimensional resonant cavities is not obvious, the problem is of theoretical interest. The purpose of this paper is to study this subject.

In Section 2, we deal with a general cylinder with an arbitrary cross section. Similar to the propagating waves, only the longitudinal fields have to be solved, and the transverse ones can be obtained by simple differentiation. It turns out that the fields with a specific eigenvalue can always be taken as TM or TE modes. This is simpler than the propagating waves, where both TM and TE modes become hybrid ones when the walls have a finite conductivity.

In Section 3, we study two examples, the circular waveguide and the coaxial line. In a general waveguide or resonant cavity, solutions to the field equations are obtained by first imposing the ideal boundary conditions for perfectly conducting walls, and the

*Project supported by the National Natural Science Foundation of China (Grant No. 10675174).
†Corresponding author. E-mail: stsqgl@mail.sysu.edu.cn
© 2010 Chinese Physical Society and IOP Publishing Ltd

effect of a finite conductivity is then calculated by the method of perturbation of boundary conditions.[1,3,4] The circular waveguide is one of the few cases which allow nonperturbative solutions. Such solutions for propagating fields in the circular waveguide were previously studied.[2,6] Here we extend them to resonant oscillations, both in the circular waveguide and in the coaxial line. Approximate eigenfrequencies in analytic form are obtained and the stability of the eigen modes is discussed. Though the transcendental equation for the eigenfrequencies can be solved only approximately, the nonperturbative approach possesses several advantages in comparison with the perturbative one. First, some nonperturbative conclusions can be drawn from the eigenvalue equation before it is solved. Second, though the eigenvalue equation is solved to the first order, it is easy to get higher-order corrections. Third, corrected eigenfunctions can be easily obtained. However, it should be emphasized that even the nonperturbative formalism is valid only when the losses of the dielectric and the conductor are small. Only under such assumptions do we have quasi-monochromatic waves and thus simple constitutive relations (for a linear medium).

We use MKS units throughout this paper.

2. Two-dimensional resonant cavity

Consider an endless cylindrical waveguide as a two-dimensional resonant cavity. It has an arbitrary cross section which occupies a two-dimensional region D on the xy plane. The boundary of D is denoted by ∂D. The region D may be doubly connected (more complicated cases seem unpractical), for which the coaxial line is a typical example. The cavity is filled with a dielectric medium whose permeability μ is assumed to be real but the dielectric constant $\epsilon = \epsilon_1 + \mathrm{i}\,\epsilon_2$ may be complex where ϵ_2 is nonnegative. All other parts of the space are filled with a conductor which has a permeability μ_c, dielectric constant ϵ_c and conductivity σ. The conductivity σ is regarded as a constant, while the other parameters may explicitly depend on the frequency ω. But since ϵ_c will finally disappear in the approximate calculations, its frequency dependence will be suppressed. The following requirements will be assumed:

$$\epsilon_2 \ll \epsilon_1, \quad \omega\epsilon_\mathrm{c} \ll \sigma. \tag{1}$$

The first means that any resonant frequency of the cavity is not close to any frequency of resonant absorption of the dielectric so that the dissipation is small. The second means that we are considering low frequency so that the conductor is good. Within and below the microwave band, for many metals σ is substantially independent of ω and the second assumption above is well satisfied.

The field configurations inside the cylinder are assumed to be of the form

$$\boldsymbol{E}(\boldsymbol{x},t) = \boldsymbol{E}(\boldsymbol{\rho})\mathrm{e}^{-\mathrm{i}\,\omega t}, \quad \boldsymbol{H}(\boldsymbol{x},t) = \boldsymbol{H}(\boldsymbol{\rho})\mathrm{e}^{-\mathrm{i}\,\omega t}. \tag{2}$$

We use the same symbol $\boldsymbol{E}$ for the field $\boldsymbol{E}(\boldsymbol{x},t)$ and its space factor $\boldsymbol{E}(\boldsymbol{\rho})$, and similarly for the magnetic field; this would not lead to confusion in the following. Because of the dissipation of the dielectric and the conductor, ω cannot be a real quantity and these are not monochromatic waves. However, the energy loss is small as assumed above, so that the imaginary part of ω is small and the above fields are quasi-monochromatic. Then we have approximately the constitutive relations $\boldsymbol{D}(\boldsymbol{x},t) = \epsilon(\omega)\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{B}(\boldsymbol{x},t) = \mu(\omega)\boldsymbol{H}(\boldsymbol{x},t)$. These allow us to reduce the source-free Maxwell equations to the following equations:

$$\nabla_\mathrm{t} \times \boldsymbol{E} = \mathrm{i}\,\omega\mu(\omega)\boldsymbol{H}, \quad \nabla_\mathrm{t} \times \boldsymbol{H} = -\mathrm{i}\,\omega\epsilon(\omega)\boldsymbol{E}. \tag{3}$$

Here and below $\boldsymbol{E}$ and $\boldsymbol{H}$ stand for $\boldsymbol{E}(\boldsymbol{\rho})$ and $\boldsymbol{H}(\boldsymbol{\rho})$, ∇_t is the transverse gradient operator acting on the transverse coordinates $\boldsymbol{\rho}$. It is understood that ω stands for $\mathrm{Re}\,\omega$ in $\mu(\omega)$ and $\epsilon(\omega)$, and in Eq. (1) $\omega\epsilon_\mathrm{c}$ stands for $|\omega|\epsilon_\mathrm{c}$.

The fields have both transverse and longitudinal components:

$$\boldsymbol{E} = \boldsymbol{E}_\mathrm{t} + E_z\boldsymbol{e}_z, \quad \boldsymbol{H} = \boldsymbol{H}_\mathrm{t} + H_z\boldsymbol{e}_z, \tag{4}$$

where all components are functions of $\boldsymbol{\rho}$, and $\boldsymbol{e}_z$ is an unit vector along the z axis. With this decomposition Eq. (3) yields

$$\begin{aligned} \boldsymbol{E}_\mathrm{t} &= -\frac{\mathrm{i}}{\omega\epsilon(\omega)}\boldsymbol{e}_z \times \nabla_\mathrm{t} H_z, \\ \boldsymbol{H}_\mathrm{t} &= \frac{\mathrm{i}}{\omega\mu(\omega)}\boldsymbol{e}_z \times \nabla_\mathrm{t} E_z, \end{aligned} \tag{5}$$

and

$$\nabla_\mathrm{t}^2 E_z + \lambda E_z = 0, \quad \nabla_\mathrm{t}^2 H_z + \lambda H_z = 0, \tag{6}$$

where

$$\lambda = \omega^2\mu(\omega)\epsilon(\omega). \tag{7}$$

Equation (5) gives the transverse fields in terms of the longitudinal ones. The latter are obtained by solving Eq. (6) with appropriate boundary conditions.

Let us pause to discuss the ideal case where the walls of the waveguide are perfect conductors. Then we have the boundary conditions

$$E_z|_{\partial D} = 0, \quad \left.\frac{\partial H_z}{\partial n}\right|_{\partial D} = 0, \tag{8}$$

where $\boldsymbol{n}$ is a unit vector normal to ∂D. These and Eq. (6) are identical to those for propagating waves in the waveguide. We make some remarks. First, like the case for propagating waves, we have TM and TE modes separately, though some of them may be degenerate. However, the field configurations for the present case are simpler. For TM modes only the components $(E_z, \boldsymbol{H}_\mathrm{t})$ are nonvanishing, while for TE modes we have $(H_z, \boldsymbol{E}_\mathrm{t})$. It is easy to show that there is no TEM mode even when D is doubly connected. Second, though ω is complex, λ is real for both TM and TE modes and is the same as that for nondissipative dielectrics. Therefore, for ideal walls the dissipative effect of the dielectric lies only in Eq. (7). This is also true for ordinary cavities. Third, the orthogonal relations among the eigenfunctions can be easily established, and the effect of imperfectly conducting walls can be calculated by the perturbation theory of waveguides.

We now return to the general formalism, since we are more interested in nonperturbative solutions. To satisfy the boundary conditions at ∂D, the fields in the conducting region should have the form

$$\begin{aligned} \boldsymbol{E}_\mathrm{c}(\boldsymbol{x}, t) &= \boldsymbol{E}_\mathrm{c}(\boldsymbol{\rho})\mathrm{e}^{-\mathrm{i}\,\omega t}, \\ \boldsymbol{H}_\mathrm{c}(\boldsymbol{x}, t) &= \boldsymbol{H}_\mathrm{c}(\boldsymbol{\rho})\mathrm{e}^{-\mathrm{i}\,\omega t}. \end{aligned} \tag{9}$$

Similar to the above, we have (all fields below are functions of $\boldsymbol{\rho}$)

$$\begin{aligned} \boldsymbol{E}_\mathrm{ct} &= -\frac{\mathrm{i}}{\omega\epsilon_\mathrm{c} + \mathrm{i}\,\sigma}\boldsymbol{e}_z \times \nabla_\mathrm{t} H_{\mathrm{c}z}, \\ \boldsymbol{H}_\mathrm{ct} &= \frac{\mathrm{i}}{\omega\mu_\mathrm{c}(\omega)}\boldsymbol{e}_z \times \nabla_\mathrm{t} E_{\mathrm{c}z}, \end{aligned} \tag{10}$$

and

$$\nabla_\mathrm{t}^2 E_{\mathrm{c}z} + \lambda_\mathrm{c} E_{\mathrm{c}z} = 0, \quad \nabla_\mathrm{t}^2 H_{\mathrm{c}z} + \lambda_\mathrm{c} H_{\mathrm{c}z} = 0, \tag{11}$$

where

$$\lambda_\mathrm{c} = \omega^2\mu_\mathrm{c}(\omega)\epsilon_\mathrm{c} + \mathrm{i}\,\omega\mu_\mathrm{c}(\omega)\sigma. \tag{12}$$

It should be noted that λ_c is not a new eigenvalue. It is determined by ω and thus by λ.

On account of the transverse–longitudinal relations, the boundary conditions $\boldsymbol{n}\times\boldsymbol{E}|_{\partial D} = \boldsymbol{n}\times\boldsymbol{E}_\mathrm{c}|_{\partial D}$ and $\boldsymbol{n}\times\boldsymbol{H}|_{\partial D} = \boldsymbol{n}\times\boldsymbol{H}_\mathrm{c}|_{\partial D}$ can be expressed solely in terms of the longitudinal fields as

$$\begin{aligned} &E_z|_{\partial D} = E_{\mathrm{c}z}|_{\partial D}, \\ &\mu_\mathrm{c}(\omega)\left.\frac{\partial E_z}{\partial n}\right|_{\partial D} = \mu(\omega)\left.\frac{\partial E_{\mathrm{c}z}}{\partial n}\right|_{\partial D}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} &H_z|_{\partial D} = H_{\mathrm{c}z}|_{\partial D}, \\ &(\omega\epsilon_\mathrm{c} + \mathrm{i}\,\sigma)\left.\frac{\partial H_z}{\partial n}\right|_{\partial D} = \omega\epsilon(\omega)\left.\frac{\partial H_{\mathrm{c}z}}{\partial n}\right|_{\partial D}. \end{aligned} \tag{14}$$

It is clear that there is no coupling between the longitudinal electric fields and the magnetic ones. Therefore the fields corresponding to a specific eigenvalue can always be taken as TM or TE ones. This is rather different from the propagating fields in the waveguide, where only hybrid modes exist when the walls have a finite conductivity.

Analytic solutions to the above eigenvalue problems are available for a circular waveguide or a coaxial line. These are studied in the next section.

3. Circular waveguide and coaxial line as resonant cavities

First consider a circular waveguide of radius a. We use the polar coordinates (ρ, ϕ) for $\boldsymbol{\rho}$.

For the TM modes, the solutions which are regular at the origin and at infinity are

$$\begin{aligned} &E_z(\boldsymbol{\rho}) = E_m^\pm \mathrm{J}_m(\kappa\rho)\mathrm{e}^{\pm\mathrm{i}\,m\phi}, \\ &E_{\mathrm{c}z}(\boldsymbol{\rho}) = F_m^\pm \mathrm{H}_m^{(1)}(\kappa_\mathrm{c}\rho)\mathrm{e}^{\pm\mathrm{i}\,m\phi}, \quad m = 0, 1, 2, \ldots, \end{aligned} \tag{15a}$$

$$H_z(\boldsymbol{\rho}) = H_{\mathrm{c}z}(\boldsymbol{\rho}) = 0, \tag{15b}$$

where $E_m^\pm$ and $F_m^\pm$ are arbitrary constants, $\mathrm{J}_m(x)$ are Bessel functions and $\mathrm{H}_m^{(1)}(x)$ Hankel functions of the first kind, and

$$\begin{aligned} \kappa &= \omega\sqrt{\mu(\omega)\epsilon(\omega)}, \\ \kappa_\mathrm{c} &= \sqrt{\omega^2\mu_\mathrm{c}(\omega)\epsilon_\mathrm{c} + \mathrm{i}\,\omega\mu_\mathrm{c}(\omega)\sigma}, \end{aligned} \tag{16}$$

where we only take the root with positive real and imaginary parts. The boundary condition leads to $F_m^\pm = E_m^\pm \mathrm{J}_m(\kappa a)/\mathrm{H}_m^{(1)}(\kappa_\mathrm{c} a)$ and the eigenvalue equation

$$\frac{\mathrm{J}_m(\kappa a)}{\kappa a \mathrm{J}_m'(\kappa a)} = \frac{\mu_\mathrm{c}(\omega)}{\mu(\omega)}\frac{\mathrm{H}_m^{(1)}(\kappa_\mathrm{c} a)}{\kappa_\mathrm{c} a \mathrm{H}_m^{(1)\prime}(\kappa_\mathrm{c} a)}, \tag{17}$$

where the prime denotes derivative with respect to the argument.

Similarly, the solutions for the TE modes are

$$H_z(\boldsymbol{\rho}) = H_m^{\pm}\mathrm{J}_m(\kappa\rho)\mathrm{e}^{\pm\mathrm{i}\,m\phi},$$
$$H_{cz}(\boldsymbol{\rho}) = G_m^{\pm}\mathrm{H}_m^{(1)}(\kappa_c\rho)\mathrm{e}^{\pm\mathrm{i}\,m\phi}, \quad m = 0, 1, 2, \ldots, \tag{18a}$$
$$E_z(\boldsymbol{\rho}) = E_{cz}(\boldsymbol{\rho}) = 0. \tag{18b}$$

The boundary condition leads to

$$G_m^{\pm} = H_m^{\pm}\mathrm{J}_m(\kappa a)/\mathrm{H}_m^{(1)}(\kappa_c a)$$

and the eigenvalue equation

$$\frac{\mathrm{J}_m'(\kappa a)}{\kappa a\mathrm{J}_m(\kappa a)} = \frac{\mu_c(\omega)}{\mu(\omega)}\frac{\mathrm{H}_m^{(1)\prime}(\kappa_c a)}{\kappa_c a\mathrm{H}_m^{(1)}(\kappa_c a)}. \tag{19}$$

Though exact solutions in analytic form to the above eigenvalue equations are not available, we can easily see that the two modes with $\pm m$ correspond to the same eigenvalue since the $\pm$ sign does not appear in these equations. In other words, this degeneracy is not removed by the finite conductivity. On the other hand, the degeneracy between TM_{1n} and TE_{0n} in the ideal case is obviously removed. Though perturbative calculations can give the same results, the conclusion obtained here is stronger since it is nonperturbative.

One may try to solve the above transcendental equations by numerical methods. However, under the assumption of Eq. (1) and the additional assumption

$$\delta(\omega) = \sqrt{\frac{2}{\omega\mu_c(\omega)\sigma}} \ll a, \tag{20}$$

approximate solutions in analytic forms can be easily obtained. Here $\delta(\omega)$ is the skin depth of the conductor at ω (to be understood as $\mathrm{Re}\,\omega$ as before). If a is of the scale of a centimeter, and ω is not far below the microwave band, the above assumption is well satisfied. Under these conditions, we have

$$\kappa_c \approx \sqrt{\mathrm{i}\,\omega\mu_c(\omega)\sigma} = \mathrm{e}^{\mathrm{i}\,\pi/4}\sqrt{\omega\mu_c(\omega)\sigma} \approx \frac{1+\mathrm{i}}{\delta(\omega)},$$
$$|\kappa_c a| \approx \frac{\sqrt{2}a}{\delta(\omega)} \gg 1. \tag{21}$$

Therefore, if m is not very large, the Hankel functions can be safely approximated by their asymptotic expression, and we have

$$\frac{\mathrm{H}_m^{(1)\prime}(\kappa_c a)}{\mathrm{H}_m^{(1)}(\kappa_c a)} \approx \mathrm{i}\,. \tag{22}$$

Then Eq. (17) for TM modes is simplified as

$$\frac{\mathrm{J}_m(x)}{x\mathrm{J}_m'(x)} = -\frac{1+\mathrm{i}}{2}\frac{\mu_c(\omega)}{\mu(\omega)}\frac{\delta(\omega)}{a}, \tag{23}$$

where $x = \kappa a = \omega a\sqrt{\mu(\omega)\epsilon(\omega)}$. Because the right-hand side (rhs) is very small, the zero-order approximation is determined by

$$\mathrm{J}_m(x) = 0. \tag{24}$$

This is the eigenvalue equation in the ideal case, whose positive roots are denoted by x_{mn}, $n = 1, 2, \ldots$. Then we have $\omega a\sqrt{\mu(\omega)\epsilon(\omega)} = x_{mn}$. Omitting the imaginary part of $\epsilon(\omega)$, we have

$$\omega a\sqrt{\mu(\omega)\epsilon_1(\omega)} = x_{mn}. \tag{25}$$

The solution of this equation, denoted by ω_{mn}^0, is the zero-order approximation for the eigenfrequencies. Given $\mu(\omega)$ and $\epsilon_1(\omega)$, we can solve ω_{mn}^0 by numerical or approximate methods. In particular, if $\mu(\omega)$ and $\epsilon_1(\omega)$ are approximately independent of ω, the solution is just $\omega_{mn}^0 = x_{mn}/a\sqrt{\mu\epsilon_1}$. We replace ω in the rhs of Eq. (23) by ω_{mn}^0, let $x = x_{mn} + \Delta x$ where Δx is a correction, and expand the left-hand side (lhs) to the first order in Δx; it is easy to find Δx, and we obtain

$$\omega a\sqrt{\mu(\omega)\epsilon(\omega)} = x_{mn}\left[1 - \frac{1+\mathrm{i}}{2}\frac{\mu_c(\omega_{mn}^0)}{\mu(\omega_{mn}^0)}\frac{\delta(\omega_{mn}^0)}{a}\right]. \tag{26}$$

Now replacing ω in the square root on the lhs by ω_{mn}^0, taking into account Eq. (25) for ω_{mn}^0 and the assumption in Eq. (1), we finally arrive at the eigenfrequencies with first-order correction:

$$\omega_{mn} = \omega_{mn}^0\left[1 - \frac{1+\mathrm{i}}{2}\frac{\mu_c(\omega_{mn}^0)}{\mu(\omega_{mn}^0)}\frac{\delta(\omega_{mn}^0)}{a} - \mathrm{i}\,\frac{\epsilon_2(\omega_{mn}^0)}{2\epsilon_1(\omega_{mn}^0)}\right]. \tag{27}$$

We see that the dissipation of the dielectric leads to energy loss but not frequency shift. On the other hand, a finite conductivity of the walls has an effect on both aspects. The same results for the latter effect can also be obtained by the perturbative method.[1,3] However, it is convenient to get higher-order corrections here, while in the perturbative method this is much more difficult.

In the circular waveguide, the hybrid propagating EH_{mn} mode (which reduces to TM_{mn} mode in the ideal case) has the propagating constant k_{mn} given by (this can be obtained by the perturbative method, or the nonperturbative one[6])

$$k_{mn}^2 = \omega^2\mu\epsilon - \left(\frac{x_{mn}}{a}\right)^2 + (1+\mathrm{i})\omega^2\mu_c\epsilon\frac{\delta}{a}, \tag{28}$$

where ω is the frequency of the exciting source, and all parameters are evaluated at this frequency. Now if we take $\omega = \mathrm{Re}\,\omega_{mn}$, it is easy to show that

$$k_{mn} = \mathrm{e}^{\mathrm{i}\,\pi/4}\frac{x_{mn}}{a}\sqrt{\frac{\epsilon_2(\omega_{mn}^0)}{\epsilon_1(\omega_{mn}^0)} + \frac{\mu_{\mathrm{c}}(\omega_{mn}^0)}{\mu(\omega_{mn}^0)}\frac{\delta(\omega_{mn}^0)}{a}}. \tag{29}$$

This means that at the resonant frequency $\mathrm{Re}\,\omega_{mn}$, the hybrid EH_{mn} mode is a cutoff mode. Therefore, small perturbations cannot drive the oscillation of the above TM_{mn} mode into a propagating wave of the corresponding EH_{mn} mode. Similar conclusions can be achieved for the TE_{mn} modes. Stability against evolution into lower propagating modes needs further study. However, we can at least conclude that the lowest mode of resonant oscillation (the TE_{11} for the current case) is stable, in spite of the absence of end surfaces.

A similar analysis for Eq. (19) can be carried out. The eigenfrequencies for the TE_{mn} modes with shifts and losses turn out to be

$$\bar{\omega}_{mn} = \bar{\omega}_{mn}^0\left[1 - \frac{1+\mathrm{i}}{2}\frac{\mu_{\mathrm{c}}(\bar{\omega}_{mn}^0)}{\mu(\bar{\omega}_{mn}^0)} \times \frac{\delta(\bar{\omega}_{mn}^0)}{a}\frac{\bar{x}_{mn}^2}{\bar{x}_{mn}^2 - m^2} - \mathrm{i}\,\frac{\epsilon_2(\bar{\omega}_{mn}^0)}{2\epsilon_1(\bar{\omega}_{mn}^0)}\right], \tag{30}$$

where $\bar{x}_{mn}$ are positive roots of $\mathrm{J}'_m(x) = 0$ which is the eigenvalue equation for the TE_{mn} modes in the ideal case, and the relation between $\bar{\omega}_{mn}^0$ and $\bar{x}_{mn}$ is determined by an equation like Eq. (25).

Substituting the eigenvalues into Eqs. (15), (18) and the transverse–longitudinal relations, we can easily find the eigenfields with corrections. In the perturbative approach, however, this is rather inconvenient, especially when the eigenvalue is degenerate ($m \neq 0$).

Next we consider a coaxial line with inner radius a and outer radius b.

For the TM modes, the solutions which are regular at the origin and at infinity are

$$E_z(\boldsymbol{\rho}) = [E_m^{\pm}\mathrm{J}_m(\kappa\rho) + F_m^{\pm}\mathrm{N}_m(\kappa\rho)]\mathrm{e}^{\pm\mathrm{i}\,m\phi},$$
$$H_z(\boldsymbol{\rho}) = 0, \quad a < \rho < b, \tag{31a}$$
$$E_z(\boldsymbol{\rho}) = A_m^{\pm}\mathrm{J}_m(\kappa_{\mathrm{c}}\rho)\mathrm{e}^{\pm\mathrm{i}\,m\phi},$$
$$H_z(\boldsymbol{\rho}) = 0, \quad \rho < a, \tag{31b}$$
$$E_z(\boldsymbol{\rho}) = C_m^{\pm}\mathrm{H}_m^{(1)}(\kappa_{\mathrm{c}}\rho)\mathrm{e}^{\pm\mathrm{i}\,m\phi},$$
$$H_z(\boldsymbol{\rho}) = 0, \quad \rho > b. \tag{31c}$$

where $\mathrm{N}_m(x)$ are Neumann functions. The solutions for the TE modes can be written down similarly. The boundary conditions lead to relations among the coefficients (such that for one eigen mode only one coefficient is arbitrary) and eigenvalue equations. The eigenvalue equations are much more complicated than the above ones. However, the same nonperturbative conclusions can be drawn from them. Approximate eigenfrequencies can be obtained under the same assumptions as before. We only give the results below.

The eigenfrequency for the TM_{mn} mode is

$$\omega_{mn} = \omega_{mn}^0\left\{1 - \frac{1+\mathrm{i}}{2}\frac{\mu_{\mathrm{c}}(\omega_{mn}^0)}{\mu(\omega_{mn}^0)}\frac{\delta(\omega_{mn}^0)[a\mathrm{J}_m^2(\kappa_{mn}^0 a) + b\mathrm{J}_m^2(\kappa_{mn}^0 b)]}{ab[\mathrm{J}_m^2(\kappa_{mn}^0 a) - \mathrm{J}_m^2(\kappa_{mn}^0 b)]} - \mathrm{i}\,\frac{\epsilon_2(\omega_{mn}^0)}{2\epsilon_1(\omega_{mn}^0)}\right\}, \tag{32}$$

where κ_{mn}^0 are positive roots of

$$\mathrm{J}_m(\kappa a)\mathrm{N}_m(\kappa b) - \mathrm{N}_m(\kappa a)\mathrm{J}_m(\kappa b) = 0, \tag{33}$$

which is the eigenvalue equation for the ideal case, and ω_{mn}^0 is determined by

$$\omega\sqrt{\mu(\omega)\epsilon_1(\omega)} = \kappa_{mn}^0. \tag{34}$$

The eigenfrequency for the TE_{mn} mode is

$$\bar{\omega}_{mn} = \bar{\omega}_{mn}^0\left\{1 - \frac{1+\mathrm{i}}{2}\frac{\mu_{\mathrm{c}}(\bar{\omega}_{mn}^0)}{\mu(\bar{\omega}_{mn}^0)}\frac{\delta(\bar{\omega}_{mn}^0)}{ab}f(a,b,m,\bar{\kappa}_{mn}^0) - \mathrm{i}\,\frac{\epsilon_2(\bar{\omega}_{mn}^0)}{2\epsilon_1(\bar{\omega}_{mn}^0)}\right\}, \tag{35}$$

where $f(a,b,m,\bar{\kappa}_{mn}^0)$ has the dimension of length and is given by

$$f(a,b,m,\bar{\kappa}_{mn}^0) = \frac{a\mathrm{J}_m'^2(\bar{\kappa}_{mn}^0 a) + b\mathrm{J}_m'^2(\bar{\kappa}_{mn}^0 b)}{[1-(m/\bar{\kappa}_{mn}^0 b)^2]\mathrm{J}_m'^2(\bar{\kappa}_{mn}^0 a) - [1-(m/\bar{\kappa}_{mn}^0 a)^2]\mathrm{J}_m'^2(\bar{\kappa}_{mn}^0 b)}, \tag{36}$$

$\bar{\kappa}_{mn}^0$ are positive roots of

$$\mathrm{J}'_m(\kappa a)\mathrm{N}'_m(\kappa b) - \mathrm{N}'_m(\kappa a)\mathrm{J}'_m(\kappa b) = 0, \tag{37}$$

which is the eigenvalue equation for the ideal case, and $\bar{\omega}^0_{mn}$ is related to $\bar{\kappa}^0_{mn}$ by an equation like Eq. (34). Note that in the ideal case the TE_{0n} mode and the TM_{1n} mode are degenerate.

Finally we point out that for propagating fields in the coaxial line, because of the coupling between the longitudinal electric fields and the magnetic fields, the eigenvalue equations become unexpectedly complicated. To our knowledge, solutions to this problem are available only in the axially symmetric case $m = 0$,[2] where one has TM and TE modes separately. In particular, one of the TM modes reduces to the TEM mode in the ideal case.

References

[1] Jackson J D 1998 *Classical Electrodynamics* (3rd ed) (New York: Wiley)

[2] Stratton J A 1941 *Electromagnetic Theory* (New York: McGraw-Hill)

[3] Collin R E 1960 *Field Theory of Guided Waves* (New York: McGraw-Hill)

[4] Goubau G 1961 *Electromagnetic Waveguides and Cavities* (New York: Pergamon)

[5] Argence E and Kahan T 1967 *Theory of Waveguides and Cavity Resonators* (London: Blackie and Son)

[6] Abe T and Yamaguchi Y 1981 *IEEE Trans. Microw. Theory Tech.* **29** 707

[7] Yu G X and Cui T J 2008 *Chin. Phys.* B **17** 164

[8] Chen Y G, Wang Y H, Zhang Y and Liu S T 2007 *Chin. Phys.* **16** 1315

[9] Xu A, Wang W X, Wei Y Y and Gong Y B 2009 *Chin. Phys.* B **18** 810

[10] Tian H, Zhang Y D, Wang H, Ouyang Q Y, Wang N and Yuan P 2009 *Chin. Phys.* B **18** 221

[11] Zhang H X, Gu Y and Gong Q H 2008 *Chin. Phys.* B **17** 2567

[12] Xu A, Wang W X, Wei Y Y and Gong Y B 2009 *Chin. Phys.* B **18** 1270