Lie symmetries and conserved quantities for a two-dimensional nonlinear diffusion equation of concentration*

Zhao Li(赵 丽)[a], Fu Jing-Li(傅景礼)[a)†], and Chen Ben-Yong(陈本永)[b)]

[a)] *Institute of Mathematical Physics, Zhejiang Sci-Tech University, Hangzhou* 310018, *China*
[b)] *Institute of Mechanical and Automation Control Engineering, Zhejiang Sci-Tech University, Hangzhou* 310018, *China*

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The Lie symmetries and conserved quantities of a two-dimensional nonlinear diffusion equation of concentration are considered. Based on the invariance of the two-dimensional nonlinear diffusion equation of concentration under the infinitesimal transformation with respect to the generalized coordinates and time, the determining equations of Lie symmetries are presented. The Lie groups of transformation and infinitesimal generators of this equation are obtained. The conserved quantities associated with the nonlinear diffusion equation of concentration are derived by integrating the characteristic equations. Also, the solutions of the two-dimensional nonlinear diffusion equation of concentration can be obtained.

Keywords: Lie symmetry, conserved quantity, nonlinear, diffusion equation of concentration
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1. Introduction

Lie group theory has been used to study differential equations. It has developed into a powerful tool to solve differential equations, to classify them and to establish properties of their solution space. These aspects of Lie group theory have been described in many books and lecture notes.[1−27]

Also, Lie group theory has many applications in partial differential equations, such as one-dimensional diffusion-convection equations[28] which are applied to fluid in a porous medium, transport of water in homogeneous non-deformable porous media and so on.[29−31] Recently, Lie group theory for the general form of the two-dimensional heat equation was intensively investigated in Ref. [32].

In the present paper, we study the Lie symmetries and conserved quantities of a two-dimensional nonlinear diffusion equation of concentration. This work has good significance in solving the problem of the diffusion of concentration of chemical and physical substances.

We show that the infinitesimal criterion for invariance of the two-dimensional nonlinear diffusion equation of concentration leads directly to an algorithm to determine infinitesimal generators X admitted by given partial differential equations (PDEs). Invariant surfaces of the corresponding Lie group of point transformations lead to equations of conserved quantities (similarity solutions). These solutions are obtained by solving the two-dimensional nonlinear diffusion equation of concentration with fewer independent variables than the given PDEs.

2. General method for a PDE

2.1. Invariance of a PDE

First, we consider a kth scalar PDE in the general form

$$F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u}) = 0, \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n)$ denotes n independent variables, u represents the coordinate corresponding to the dependent variable, and $\underset{j}{u}$ refers to the set of coordinates corresponding to all jth order partial derivatives of u with respect to x; the coordinates of $\underset{j}{u}$ corresponding to $\dfrac{\partial^j u}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_j}}$ is denoted by $u_{i_1}u_{i_2}\ldots u_{i_j}$, $i_j = 1, 2, \ldots, n$, for $j = 1, 2, \ldots, k$.

We consider a one-parameter group of point-like transformations acting on the space of the independent variables $x = (x_1, x_2, \ldots, x_n)$ and on the space of

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†Corresponding author. E-mail: sqfujingli@163.com
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the dependent variable of Eq. (1). The one-parameter Lie group of transformations may be expressed as

$$x^* = X(x, u; \varepsilon), \quad u^* = U(x, u; \varepsilon), \tag{2}$$

then

$$\mathrm{X} = \xi_i(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u} \tag{3}$$

is the infinitesimal generator of Eq. (2). Let

$$\begin{aligned}\mathrm{X}^{(k)} = {} & \xi_i(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u} + \eta_i^{(1)}(x, u, \underset{1}{u})\frac{\partial}{\partial u_i} \\ & + \eta_{i_1 i_2}^{(2)}(x, u, \underset{1}{u}, \underset{2}{u})\frac{\partial}{\partial u_{i_1 i_2}} \\ & + \ldots + \eta_{i_1 i_2 \ldots i_k}^{(k)}(x, u, \underset{1}{u}\,\underset{2}{u}\ldots\underset{k}{u})\frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}}\end{aligned} \tag{4}$$

be the kth extended infinitesimal generator of Eq. (3).

Equation (2) is admitted by PDE (1) if and only if

$$\mathrm{X}^{(k)} F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u}) = 0, \tag{5}$$

when $F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u}) = 0$.

2.2. Determining equation

From Eqs. (3)–(5) we can see that the kth order PDE ($k \geq 2$)

$$u_{i_1 i_2 \ldots i_l} = f(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u}), \tag{6}$$

where $f(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u})$ does not depend on $u_{i_1 i_2 \ldots i_l}$, and admits Eq. (3) with kth extension given by Eq. (4), if and only if

$$\begin{aligned}\eta_{i_1 i_2 \ldots i_l}^{(l)} = {} & \xi_j \frac{\partial f}{\partial x_j} + \eta\frac{\partial f}{\partial u} + \eta_j^{(1)}\frac{\partial f}{\partial u_j} + \ldots \\ & + \eta_{j_1 j_2 \ldots j_k}^{(k)}\frac{\partial f}{\partial u_{j_1 j_2 \ldots j_k}},\end{aligned} \tag{7}$$

when $u_{i_1 i_2 \ldots i_l} = f(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u})$.

Explicit formulas for the extended infinitesimals $\left\{\eta_{i_1 i_2 \ldots i_l}^{(l)}\right\}$ result from the following equations:

$$\begin{aligned}& \eta_i^{(1)} = D_i\eta - (D_i\xi_j)u_j, \quad i = 1, 2, \ldots, n\,; \\ & \eta_{i_1 i_2 \ldots i_k}^{(k)} = D_{i_k}\eta_{i_1 i_2 \ldots i_{k-1}}^{(k-1)} - (D_{i_k}\xi_j)u_{i_1 i_2 \ldots i_{k-1} j}, \\ & i_l = 1, 2, \ldots, n \ \text{ for} \\ & l = 1, 2, \ldots, k \ \text{ with } \ k = 2, 3 \ldots .\end{aligned} \tag{8}$$

2.3. Determining equations and their solutions

The Lie symmetries for a PDE are the classical ones which keep the equation invariant under a Lie group of local infinitesimal transformations. In this section we study the Lie symmetries for an equation in the form of

$$\rho_t = (\rho^2\rho_x)_y, \tag{9}$$

where ρ denotes the concentration of the substance; we consider it as a dependent variable $\rho = \rho(x, y, t)$, the ρ_x , ρ_y are partial derivatives.

Diffusion will take place when chemical substances (or physical substances) in space have the concentration of non-uniform distribution, and the diffusion of substances admits the Fourier law.

The equivalent form of Eq. (9) is

$$\rho_t - 2\rho\rho_x\rho_y - \rho^2\rho_{xy} = 0. \tag{10}$$

Considering a one-parameter group of point-like transformations acting on the space of the independent variables (x, y, t) and on the space of the dependent variable of Eq. (10), according to Eq. (3) the associated infinitesimal generator given by

$$\begin{aligned}\mathrm{X}(x, y, t, \rho) = {} & \xi^1(x, y, t, \rho)\frac{\partial}{\partial x} + \xi^2(x, y, t, \rho)\frac{\partial}{\partial y} \\ & + \varphi(x, y, t, \rho)\frac{\partial}{\partial t} + \eta(x, y, t, \rho)\frac{\partial}{\partial \rho},\end{aligned} \tag{11}$$

and following Eq. (4) the second extension of operator X has the general form

$$\begin{aligned}\mathrm{X}^{(2)} = {} & \mathrm{X} + \eta^x\frac{\partial}{\partial \rho_x} + \eta^y\frac{\partial}{\partial \rho_y} + \eta^t\frac{\partial}{\partial \rho_t} \\ & + \eta^{xx}\frac{\partial}{\partial \rho_{xx}} + \eta^{xy}\frac{\partial}{\partial \rho_{xy}} + \eta^{xt}\frac{\partial}{\partial \rho_{xt}} \\ & + \eta^{yy}\frac{\partial}{\partial \rho_{yy}} + \eta^{yt}\frac{\partial}{\partial \rho_{yt}} + \eta^{tt}\frac{\partial}{\partial \rho_{tt}}.\end{aligned} \tag{12}$$

Equation (12) is admitted by PDE (10) if and only if

$$\mathrm{X}^{(2)}[\rho_t - 2\rho\rho_x\rho_y - \rho^2\rho_{xy}] = 0, \tag{13}$$

when $\rho_t - 2\rho\rho_x\rho_y - \rho^2\rho_{xy} = 0$.

Then Eq. (13) becomes

$$\begin{aligned}& \eta(-2\rho_x\rho_y - 2\rho\rho_{xy}) + \eta^x(-2\rho\rho_y) \\ & + \eta^y(-2\rho\rho_x) + \eta^t - \rho^2\eta^{xy} = 0.\end{aligned} \tag{14}$$

Using the formulas of Eq. (8) and substituting ρ_t obtained from Eq. (10) and its partial derivatives ρ_{tx}, ρ_{ty} into their expressions, we can obtain the η^x, η^y, η^t, η^{xy} from these equations

$$\eta^x = \frac{D\eta}{Dx} - \rho_x\frac{D\xi^1}{Dx} - \rho_y\frac{D\xi^2}{Dx} - \rho_t\frac{D\varphi}{Dx}, \quad \eta^y = \frac{D\eta}{Dy} - \rho_x\frac{D\xi^1}{Dy} - \rho_y\frac{D\xi^2}{Dy} - \rho_t\frac{D\varphi}{Dy},$$
$$\eta^t = \frac{D\eta}{Dt} - \rho_x\frac{D\xi^1}{Dt} - \rho_y\frac{D\xi^2}{Dt} - \rho_t\frac{D\varphi}{Dt}, \quad \eta^{xy} = \frac{D\eta^x}{Dy} - \rho_{xx}\frac{D\xi^1}{Dy} - \rho_{xy}\frac{D\xi^2}{Dy} - \rho_{xt}\frac{D\varphi}{Dy}.$$

They may have the forms

$$\begin{aligned}
\eta^x &= \eta_x + (\eta_\rho - \xi^1_x)\rho_x - \xi^1_\rho\rho^2_x - \xi^2_x\rho_y - (2\rho\varphi_x + \xi^2_\rho)\rho_x\rho_y \\
&\quad - 2\rho\varphi_\rho\rho^2_x\rho_y - \rho^2\varphi_x\rho_{xy} - \rho^2\varphi_\rho\rho_x\rho_{xy}, \qquad (15)\\
\eta^y &= \eta_y + (\eta_\rho - \xi^2_y)\rho_y - \xi^2_\rho\rho^2_y - \xi^1_y\rho_x - (2\rho_u\varphi_y + \xi^1_\rho)\rho_x\rho_y \\
&\quad - 2\rho\varphi_\rho\rho^2_y\rho_x - \rho^2\varphi_y\rho_{xy} - \rho^2\varphi_\rho\rho_y\rho_{xy}, \qquad (16)\\
\eta^t &= \eta_t + (2\rho\eta_\rho - 2\rho\varphi_t)\rho_x\rho_y + \rho^2(\eta_\rho - \varphi_t)\rho_{xy} - 2\rho\xi^1_\rho\rho^2_x\rho_y \\
&\quad - \rho^2\xi^1_\rho\rho_x\rho_{xy} - 2\rho\xi^2_\rho\rho^2_y\rho_x - \rho^2\xi^2_\rho\rho_y\rho_{xy} - \xi^1_t\rho_x - \xi^2_t\rho_y \\
&\quad - 4\rho^2\varphi_\rho\rho^2_x\rho^2_y - \rho^4\varphi_\rho\rho^2_{xy} - 2\rho^2\varphi_\rho\eta_\rho\rho_x\rho_y\rho_{xy}, \qquad (17)\\
\eta^{xy} &= \eta_{xy} + (\eta_{y\rho} - \xi^1_{xy})\rho_x + (\eta_{x\rho} - \xi^2_{xy})\rho_y + (\eta_{\rho\rho} - \xi^1_{x\rho} - \xi^2_{y\rho} - 2\rho\varphi_{xy})\rho_x\rho_y \\
&\quad + (\eta_\rho - \xi^1_x - \xi^2_y - \rho^2\varphi_{xy})\rho_{xy} - \xi^1_{y\rho}\rho^2_x - \xi^2_{x\rho}\rho^2_y - \xi^1_y\rho_{xx} - \xi^2_x\rho_{yy} \\
&\quad - (\xi^1_{\rho\rho} + 2\rho\varphi_{y\rho} + 2\varphi_y)\rho^2_x\rho_y - (\xi^2_{\rho\rho} + 2\rho\varphi_{x\rho} + 2\varphi_x)\rho^2_y\rho_x - \rho^2\varphi_\rho\rho^2_{xy} \\
&\quad - (2\rho\varphi_{\rho\rho} + 4\varphi_\rho)\rho^2_x\rho^2_y - (\xi^1_\rho + \rho^2\varphi_{y\rho} + 4\rho\varphi_y)\rho_x\rho_{xy} - (\xi^2_\rho + \rho^2\varphi_{x\rho} \\
&\quad + 4\rho\varphi_x)\rho_y\rho_{xy} - (\xi^2_\rho + 2\rho\varphi_x)\rho_x\rho_{yy} - (\xi^1_\rho + 2\rho\varphi_y)\rho_y\rho_{xx} \\
&\quad - (\rho^2\varphi_{\rho\rho} + 10\rho\varphi_\rho)\rho_x\rho_y\rho_{xy} - \rho^2\varphi_x\rho_{xyy} - \rho^2\varphi_y\rho_{xxy} - 2\rho\varphi_\rho\rho_x\rho_{xyy} \\
&\quad - \rho^2\varphi_\rho\rho_y\rho_{xxy} - 2\rho\varphi_\rho(\rho^2_x\rho_{yy} + \rho^2_y\rho_{xx}). \qquad (18)
\end{aligned}$$

By substituting Eqs. (15)–(18) into Eq. (14) and then causing to vanish the coefficients of the various monomials in partial derivatives of the dependent variable $\rho(x,\ y,\ t)$, we can obtain a differential equation system for the unknown functions, i.e. $\xi^1(x,\ y,\ t,\ \rho)$, $\xi^2(x,\ y,\ t,\ \rho)$, $\varphi(x,\ y,\ t,\ \rho)$, and $\eta(x,\ y,\ t,\ \rho)$. It could be reduced into

$$\eta_t - \rho^2\eta_{xy} = 0, \qquad (19)$$
$$\xi^1_t + 2\rho\eta_y + \rho^2\eta_{\rho y} = 0, \qquad (20)$$
$$\xi^2_t + 2\rho\eta_x + \rho^2\eta_{\rho x} = 0, \qquad (21)$$
$$-2\rho\eta + \rho^2(\xi^1_x + \xi^2_y - \varphi_t) = 0, \qquad (22)$$
$$-2\eta + 2\rho(\xi^1_x + \xi^2_y - \varphi_t - \eta_\rho) - \rho^2\eta_{\rho\rho} = 0, \qquad (23)$$

where $\xi^1 = \xi^1(x,t)$, $\xi^2 = \xi^2(y,t)$, $\varphi = \varphi(t)$, $\eta = \eta(x,y,t,\rho)$.

Now we solve the determining equations. First, we take the derivative of Eq. (22) with respect to ρ, and use Eq. (23) to give

$$\rho^2\eta_{\rho\rho} = 0 \Rightarrow \eta_{\rho\rho} = 0. \qquad (24)$$

The solution of Eq. (24) is

$$\eta = \eta^1(x,y,t)\rho + \eta^0(x,y,t). \qquad (25)$$

Equations (22) and (23) can be reduced into

$$\xi^1_x + \xi^2_y - \varphi_t = \frac{2\eta}{\rho} = 2\eta^1 + \frac{2\eta^0}{\rho},$$
$$\xi^1_x + \xi^2_y - \varphi_t = 2\eta^1 + \frac{\eta^0}{\rho}. \qquad (26)$$

Equation (26) can give

$$\eta = \eta^1\rho. \qquad (27)$$

Combining the other determining equations and Eq. (27) yields the solution

$$\xi^1(x) = c_1x + c_2, \quad \xi^2(y) = c_3y + c_4,$$
$$\varphi(t) = c_5t + c_6, \quad \eta = \frac{1}{2}(c_1 + c_3 - c_5)\rho, \qquad (28)$$

where $c_i, i = 1, \ldots, 6$, are arbitrary constants.

In the next section, we consider the infinitesimal generators of the corresponding Lie algebra, and give the invariant solution by the characteristic equations.

3. Infinitesimal generators

For the general Eq. (1), each parameter of an r-parameter Lie group of transformations leads to an

infinitesimal generator. By solving differential equations (19)–(23), we can get the ξ^1, ξ^2, φ, η with arbitrary constants c_i. The Lie symmetry operator X_α can be obtained by choosing each value of parameters c_i to be the specific value 1, thereby taking out all the remaining arbitrary constants. The Lie symmetry generator for Eq. (10) is

$$\mathrm{X} = (c_1x + c_2)\frac{\partial}{\partial x} + (c_3x + c_4)\frac{\partial}{\partial y} + (c_5t + c_6)\frac{\partial}{\partial t} + \frac{1}{2}(c_1 + c_3 - c_5)\rho\frac{\partial}{\partial \rho}. \quad (29)$$

Under the theory we can obtain the Lie symmetry operators

$$\mathrm{X}_1 = x\frac{\partial}{\partial x} + \frac{1}{2}\rho\frac{\partial}{\partial \rho}; \quad \mathrm{X}_2 = \frac{\partial}{\partial x};$$
$$\mathrm{X}_3 = y\frac{\partial}{\partial y} + \frac{1}{2}\rho\frac{\partial}{\partial \rho}; \quad \mathrm{X}_4 = \frac{\partial}{\partial y};$$
$$\mathrm{X}_5 = t\frac{\partial}{\partial t} - \frac{1}{2}\rho\frac{\partial}{\partial \rho}; \quad \mathrm{X}_6 = \frac{\partial}{\partial t}\,. \quad (30)$$

Consider an r-parameter Lie group of transformations with infinitesimal generators $\{\mathrm{X}_\alpha\}, \alpha = 1, 2, \ldots, r$. The commutator of X_α and X_β is another first order operator

$$[\mathrm{X}_\alpha, \mathrm{X}_\beta] = \mathrm{X}_\alpha\mathrm{X}_\beta - \mathrm{X}_\beta\mathrm{X}_\alpha = \sum_{j=1}^{n}\left[\left(\eta_j(x)\frac{\partial}{\partial x_j}\right)\right], \quad (31)$$

where

$$\eta_j(x = \sum_{i=1}^{n}\left[\xi_{\alpha i}(x)\frac{\partial \xi_{\beta j}(x)}{\partial x_i} - \xi_{\beta i}(x)\frac{\partial \xi_{\alpha j}(x)}{\partial x_i}\right]. \quad (32)$$

After computing the Lie algebra of the operators Eq. (30) by Eqs. (31) and (32), we can have the following relations:

$$[\mathrm{X}_1, \mathrm{X}_2] = -\mathrm{X}_2, [\mathrm{X}_3, \mathrm{X}_4] = -\mathrm{X}_4, [\mathrm{X}_5, \mathrm{X}_6] = -\mathrm{X}_6. \quad (33)$$

4. Invariants

The invariants can be obtained by integrating the characteristic equations. The general form of the equation is

$$\frac{\mathrm{d}x_1}{\xi_1(x,u)} = \frac{\mathrm{d}x_2}{\xi_2(x,u)} = \ldots = \frac{\mathrm{d}x_n}{\xi_n(x,u)} = \frac{\mathrm{d}u}{\eta(x,u)}. \quad (34)$$

The invariants associated with the symmetry operator X_1 are obtained by integrating the characteristic equation

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}\rho}{\rho/2}, \quad (35)$$

and we have the invariants y, t, ρ^2/x. For the last invariant, we assume a similarity solution in the form of

$$\rho(x, y, t) = \sqrt{xg(y, t)}. \quad (36)$$

By substituting it into Eq. (10), we can have the form of function $g(y, t)$. It satisfies the following equation: $g_t = 3gg_y/2$, and it has the solution

$$g(y, t) = -\frac{a_1y + a_2}{a_3 + \frac{3}{2}a_1t}\,. \quad (37)$$

By doing the same to X_3, the invariants are obtained to be x, t, ρ^2/y, then the similarity solution is

$$\rho(x, y, t) = \sqrt{yh(x, t)}, \quad (38)$$

and it satisfies an equation similar to Eq. (37) in the form of

$$h(x, t) = -\frac{b_1x + b_2}{b_3 + \frac{3}{2}b_1t}. \quad (39)$$

By doing the same to X_5, the similarity solution is

$$\rho = \sqrt{\frac{2xy}{t}}\,. \quad (40)$$

For the remaining symmetry operators X_2, X_4, X_6, the associated invariants are the arbitrary functions $L(y, t, \rho)$, $M(x, y, \rho)$, $N(x, t, \rho)$, respectively.

Integrating the above solutions yields a generalized expression as follows:

$$\rho = \sqrt{\frac{axy}{b - \frac{3}{2}at}}\,. \quad (41)$$

5. Conclusions

We analyse the general form of the two-dimensional nonlinear diffusion equation of concentration and obtain its determining equations Eqs. (19)–(23) for ξ^1, ξ^2, φ, η. According to the Lie symmetry operators we can obtain the characteristic equations and, further, the invariants. Of course, the method is worth further study in more physical problems.

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