A singularly perturbed reaction diffusion problem for the nonlinear boundary condition with two parameters*

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A class of singularly perturbed initial boundary value problems of reaction diffusion equations for the nonlinear boundary condition with two parameters is considered. Under suitable conditions, by using the theory of differential inequalities, the existence and the asymptotic behaviour of the solution for the initial boundary value problem are studied. The obtained solution indicates that there are initial and boundary layers and the thickness of the boundary layer is less than the thickness of the initial layer.

Keywords: nonlinear, two parameters, singular perturbation

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1. Introduction

The investigation of the solutions for nonlinear equations plays an important role in the study of reaction diffusion theory. Many natural sciences such as chemistry, biology, mathematics, communication and particularly almost all branches of physics like fluid dynamics, plasma physics, field theory, optics, condensed matter physics etc. are interested in the reaction diffusion equation. Recently, many approximate methods for the nonlinear singularly perturbed problem have been developed and refined,[1] and many scholars have done a great deal of work.[2−6] Using the differential inequalities and other methods Mo *et al.* also considered a class of singularly perturbed nonlinear boundary value problems for ordinary differential equations,[7] the reaction diffusion equations,[8−10] the boundary value problems of the elliptic equations,[11] the shock layer and soliton solution of nonlinear equations,[12−15] the singularly perturbed ecological model,[16] the activator inhibitor system[17] and the problems of atmospheric physics.[18−21] In this paper, using a special singularly perturbed method, we study a class of the initial boundary value problem for the reaction equations with two parameters. The investigation of the nonlinear reaction equations with two parameters plays an important role in the study of some specific branches of physics such as fluid dynamics, condensed matter physics, atmospheric physics and ecology and so on.[14,22−25]

Now we consider the following nonlinear problem:[8,9]

$$\mu\frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial x}=f(x,u),\quad t>0,\ \ x\in(0,1), \tag{1}$$

$$u(0)=A, \tag{2}$$

$$g\left(u(1),\ \frac{\partial u}{\partial x}(1)\right)=0, \tag{3}$$

$$u=h(x),\quad t=0, \tag{4}$$

where ε, μ are positive small parameters. The problem (1)–(4) is a singularly perturbed problem for the nonlinear boundary condition with two small parameters.

We need the following hypotheses:

[H$_1$] $\varepsilon/\mu^2\to 0$ as $\mu\to 0$;

[H$_2$] The functions f, g, and h are sufficiently smooth functions with respect to their arguments in corresponding regions and $g_u\neq 0$, $A=h(0)$, $g(h(1),h_x(1))=0$;

[H$_3$] There is a positive constant δ, such that $\partial f/\partial u(x,u)\leq-\delta$.

We first let

$$\xi=\mu,\ \ \eta=(\varepsilon/\mu^2)^{1/2}. \tag{5}$$

From hypothesis [H$_1$], ξ and η are small parameters too. Then the equation (1) is

$$\xi\frac{\partial u}{\partial t}-(\xi\eta)^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial x}=f(x,u). \tag{6}$$

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We now construct the formal asymptotic solution of problem (1)–(4).

2. Outer solution

Let the outer solution U of problem (1)–(4) be

$$U(x,\xi,\eta)=\sum_{i,j=0}^{\infty}U_{ij}(x)\xi^i\eta^j. \tag{7}$$

Substituting Eq. (7) into Eqs. (1) and (3), developing f in $\xi,\ \eta$ and equating coefficients of the same powers of $\xi,\ \eta$, for the coefficients of $\xi^0\eta^0$ and $\xi^i\eta^j,\ i+j\neq 0$, we obtain

$$\frac{\partial U_{00}}{\partial x}=-f(x,U_{00}), \tag{8}$$

$$g\left(U_{00}(1),\frac{\partial U_{00}}{\partial x}(1)\right)=0, \tag{9}$$

$$\frac{\partial U_{ij}}{\partial x}+f(x,U_{00})U_{ij}=F_{ij},\quad i+j\neq 0, \tag{10}$$

$$g_u\left(U_{00}(1),\frac{\partial U_{00}}{\partial x}(1)\right)U_{ij}(1) +g_{u_x}\left(U_{00}(1),\frac{\partial U_{00}}{\partial x}(1)\right)\frac{\partial U_{ij}}{\partial x}(1)=\bar{F}_{ij},\quad i+j\neq 0. \tag{11}$$

where $F_{ij},\ \bar{F}_{ij}$ are determined, whose constructions are omitted. From Eqs. (8) and (10), we have

$$\frac{\partial U_{00}}{\partial x}(1)=-f(1,U_{00}(1)), \tag{12}$$

$$\frac{\partial U_{ij}}{\partial x}(1)=-f(1,U_{00}(1))U_{ij}(1) +F_{ij}\,|_{x=1}\ ,\quad i+j\neq 0. \tag{13}$$

Substituting Eqs. (12) and (13) into Eqs. (9) and (11), we obtain

$$g(U_{00}(1),-f(1,U_{00}(1))=0, \tag{14}$$

$$\left[g_u(U_{00}(1),\frac{\partial U_{00}}{\partial x}(1)) -g_{u_x}(U_{00}(1),\frac{\partial U_{00}}{\partial x}(1))f_u(1,U_{00}(1))\right]U_{ij}(1) = -g_{u_x}\left(U_{00}(1),\frac{\partial U_{00}}{\partial x}(1)\right)F_{ij}\,|_{x=1}+\bar{F}_{ij},\quad i+j\neq 0. \tag{15}$$

We need to also assume that

[H$_4$] There exists a unique simple root $U_{00}(1)=\alpha$ of equation (14).

Thus from the hypothesis [H$_4$], we yield

$$P(\alpha)\equiv g(\alpha,-f(1,\alpha))=0, \tag{16}$$

$$P'(\alpha)\equiv\left[g_u\left(U_{00}(1),\frac{\partial U_{00}}{\partial x}(1)\right) -g_{u_x}\left(U_{00}(1),\frac{\partial U_{00}}{\partial x}(1)\right)f_u(1,U_{00}(1))\right]\neq 0. \tag{17}$$

From Eqs. (14)–(17), we have

$$U_{00}(1)=\alpha, \tag{18}$$

$$U_{ij}(1)=[P'(\alpha)]^{-1}\left[-g_{u_x}\left(U_{00}(1),\frac{\partial U_{00}}{\partial x}(1)\right) \times F_{ij}\,|_{x=1}+\bar{F}_{ij}\right],\quad i+j\neq 0. \tag{19}$$

From Eqs. (8), (18) and (10), (19), we can solve solutions $U_{ij}(x),\ i,k=0,1,\ldots$. Thus we obtain the outer solution (7) for the original problem (1)–(4). But it may not satisfy the initial condition (4) and boundary condition (2), so that we need to construct the initial layer and boundary layer corrective terms near $t=0$ and $x=0$ respectively.

3. Initial layer correction

We introduce a stretched variable[1] $\tau=t/\mu$, and let the solution u be

$$u=U(x,\xi,\eta)+V(\tau,x,\xi,\eta). \tag{20}$$

Substituting Eq. (20) into Eqs. (6) and (4), we have

$$\frac{\partial V}{\partial\tau}-(\xi\eta)^2\frac{\partial^2V}{\partial x^2}-\frac{\partial V}{\partial x} = f(x,U+V)-f(x,U), \tag{21}$$

$$V\,|_{\tau=0}=h(x)-U(x,\xi,\eta)\ . \tag{22}$$

Let

$$V=\sum_{i,j=0}^{\infty}v_{ij}(\tau,x)\xi^i\eta^j. \tag{23}$$

Substituting Eqs. (7), (23) and (20) into Eqs. (21)–(22), developing f in $\xi,\ \eta$ and equating coefficients of the same powers of $\xi,\ \eta$, for the coefficients of $\xi^0\eta^0$ and $\xi^i\eta^j,\ i+j\neq 0$, we obtain

$$\frac{\partial v_{00}}{\partial\tau}-\frac{\partial v_{00}}{\partial x}=f(x,U_{00}+v_{00})-f(x,U_{00}), \tag{24}$$

$$v_{00}\,|_{\tau=0}=h(x)-U_{00}(x)\ , \tag{25}$$

$$\frac{\partial v_{ij}}{\partial\tau}-\frac{\partial v_{ij}}{\partial x}-f_u(x,U_{00}+v_{00})v_{ij}=G_{ij},\quad i+j\neq 0, \tag{26}$$

$$v_{ij}\,|_{\tau=0}=-U_{ij}(x),\quad i+j\neq 0\ , \tag{27}$$

where $G_{ij},\ i+j\neq 0$, are determined functions, whose constructions are also omitted. From the hypotheses, we can obtain $v_{ij},\ i,j=0,1,2,\ldots$, and

$$v_{ij}=O(\exp(-k_{ij}\tau))$$

$$= O(\exp(-k_{ij}\frac{t}{\mu})), \quad i,j = 0,1,2,\ldots, \quad (28)$$

where $k_{ij},\ i,j = 0,1,2,\ldots$, are positive constants. Substituting v_{ij} into Eq. (23), we then have the initial layer corrective term V near $t = 0$.

4. Boundary layer correction

We lead into a stretched variable

$$\sigma = x/(\xi\eta)^2. \quad (29)$$

Let the solution of the original problem (1)–(4) be

$$u = U(x,\xi,\eta) + V(\tau,x,\xi,\eta) + W(t,\sigma,\xi,\eta). \quad (30)$$

Substituting Eqs. (29) and (30) into problems (6), (2), we have

$$\frac{\partial^2 W}{\partial\sigma^2} + \frac{\partial W}{\partial\sigma} - \xi^3\eta^2\frac{\partial W}{\partial t} + \xi^2\eta^2[f(\xi^2\eta^2\sigma, U + V + W) - f(\xi^2\eta^2,\sigma,U+V)] = 0, \quad (31)$$

$$W(t,0,\xi,\eta) = A - U(0,\xi,\eta) - V(\tau,0,\xi,\eta). \quad (32)$$

Let

$$W = \sum_{i,j=0}^{\infty} w_{ij}(t,\sigma)\xi^i\eta^j. \quad (33)$$

Substituting Eqs. (7), (23), (33) and (30) into Eqs. (31) and (32), developing f in $\xi,\ \eta$ and equating coefficients of the same powers of $\xi,\ \eta$, for the coefficients of $\xi^0\eta^0$ and $\xi^i\eta^j,\ i+j\neq 0$, we obtain

$$\frac{\partial^2 w_{00}}{\partial\sigma^2} + \frac{\partial w_{00}}{\partial\sigma} = 0, \quad (34)$$

$$w_{00}(t,0) = A - U_{00}(0) - v_{00}(\tau,0), \quad (35)$$

$$\frac{\partial^2 w_{ij}}{\partial\sigma^2} + \frac{\partial w_{ij}}{\partial\sigma} = H_{ij}, \quad i+j\neq 0, \quad (36)$$

$$w_{ij}(t,0) = -U_{ij}(0) - v_{ij}(\tau,0), \quad i+j\neq 0, \quad (37)$$

where $H_{ij},\ i+j\neq 0$, are determined functions, whose constructions are omitted too. From Eqs. (34)–(35) and (36)–(37), we can obtain $w_{ij},\ i,j = 0,1,2,\ldots$, and

$$\frac{\partial w_{ij}}{\partial\sigma}$$

$$= O(\exp(-\bar{k}_{ij}\sigma))$$

$$= O\left(\exp\left(-\bar{k}_{ij}\frac{x}{(\xi\eta)^2}\right)\right), \quad i,j = 0,1,2,\ldots, \quad (38)$$

where $\bar{k}_{ij},\ i,j = 0,1,2,\ldots$, are positive constants. Substituting w_{ij} into (33), we then have the boundary layer corrective term W near $x = 0$.

Thus we can construct the following formal asymptotic expansion of the solution u for the original problem (1)–(4):

$$u = \sum_{i,j=0}^{m}(U_{ij} + v_{ij})\xi^i\eta^j + \sum_{i,j=0}^{m+2} w_{ij}\xi^i\eta^j + O(\zeta), \quad 0 < \xi = \mu, \eta = \varepsilon^{1/2}/\mu \ll 1, \quad (39)$$

where $\zeta = \max(\xi^{m+1},\ \eta^{m+1})$.

Remark Noting Eqs. (28), (38), (39) and $\varepsilon/\mu^2 \to 0$ as $\mu \to 0$, it is easy to see that there are two thin layers for the solution, and the thickness of the boundary layer for W is less than the thickness of the initial layer for V.

5. Uniform validity

We have the following theorem:

Theorem Under the hypotheses [H$_1$]–[H$_4$], there exists a solution u of the singularly perturbed reaction diffusion problem (1)–(4) with two parameters and the solution may be expanded into the uniformly valid asymptotic expansion (39) in $t \geq 0,\ x \in \bar{\Omega}$.

Proof We construct the auxiliary functions α_m and β_m:

$$\alpha_m = Z_m - r\zeta, \quad \beta_m = Z_m + r\zeta, \quad (40)$$

where r is a positive constant large enough to be chosen below, and

$$Z_m \equiv \sum_{i,j=0}^{m}(U_{ij} + v_{ij})\xi^i\eta^j + \sum_{i,j=0}^{m+2} w_{ij}\xi^i\eta^j.$$

Obviously, we have

$$\alpha_m \leq \beta_m, \quad t \geq 0,\ \ x \in \bar{\Omega}, \quad (41)$$

$$\alpha_m|_{x=0} \leq A \leq \beta_m|_{x=0},\ \left[\alpha_m - \bar{g}\left(\frac{\partial\alpha_m}{\partial x}\right)\right]|_{x=1} \leq 0 \leq \left[\beta_m - \bar{g}\left(\frac{\partial\beta_m}{\partial x}\right)\right]|_{x=1}, \quad (42)$$

$$\alpha_m|_{t=0} \leq h(x) \leq \beta_m|_{t=0}, \quad (43)$$

where function $y = \bar{g}(z)$ is explicit for the implicit function $g(y,z) = 0$.

Now we prove that

$$\mu\frac{\partial\alpha_m}{\partial t}-\varepsilon\frac{\partial^2\alpha_m}{\partial x^2}-\frac{\partial\alpha_m}{\partial x}-f(x,\alpha_m)\leq 0,\quad t>0,\ \ x\in(0,1), \tag{44}$$

$$\mu\frac{\partial\beta_m}{\partial t}-\varepsilon\frac{\partial^2\beta_m}{\partial x^2}-\frac{\partial\beta_m}{\partial x}-f(x,\beta_m)\geq 0,\quad t>0,\ \ x\in(0,1). \tag{45}$$

From the hypotheses [H$_3$], it follows that for $\xi=\mu,\ \eta=\varepsilon^{1/2}/\mu$ small enough, there exists a positive constant M, such that

$$\begin{aligned}
&\mu\frac{\partial\alpha_m}{\partial t}-\varepsilon\frac{\partial^2\alpha_m}{\partial x^2}-\frac{\partial\alpha_m}{\partial x}-f(x,\alpha_m)\\
&=\mu\frac{\partial Z_m}{\partial t}-\varepsilon\frac{\partial^2 Z_m}{\partial x^2}-\frac{\partial Z_m}{\partial x}-f(x,Z_m)+[f(x,Z_m)-f(x,Z_m-r\zeta)]\\
&\leq -\frac{\partial U_{00}}{\partial x}-f(x,U_{00})-\sum_{\substack{i,j=0\\ i+j\neq 0}}^{m}\left[\frac{\partial U_{ij}}{\partial x}+f(x,U_{00})U_{ij}-F_{ij}\right]\xi^i\eta^j+\frac{\partial v_{00}}{\partial\tau}-\frac{\partial v_{00}}{\partial x}-f(x,U_{00}+v_{00})+f(x,U_{00})\\
&\quad+\sum_{\substack{i,j=0\\ i+j\neq 0}}^{m}\left[\frac{\partial v_{ij}}{\partial\tau}-\frac{\partial v_{ij}}{\partial x}-f_u(x,U_{00}+v_{00})v_{ij}-G_{ij}\right]\xi^i\eta^j-\left[\frac{\partial^2 w_{00}}{\partial\sigma^2}+\frac{\partial w_{00}}{\partial\sigma}\right]\\
&\quad-\sum_{\substack{i,j=0\\ i+j\neq 0}}^{m+2}\left[\frac{\partial^2 w_{ij}}{\partial\sigma^2}+\frac{\partial w_{ij}}{\partial\sigma}-H_{ij}\right]\xi^i\eta^j+M\zeta-r\delta\zeta\\
&=(M-r\delta)\zeta.
\end{aligned} \tag{46}$$

Selecting $r\geq M/\delta$, we have proved the inequality (44). Similarly, we can prove the inequality (45) too. Thus from Eqs. (41)–(45), by using the theory of differential inequalities, there is a solution u of the problem (1)–(4), such that

$$\alpha_m\leq u\leq\beta_m,\quad t\geq 0,\ \ x\in\bar{\Omega}.$$

Thus from Eq. (40), we obtain the final result (39). The proof of the Theorem is completed.

References

[1] de Jager E M and Jiang Furu 1996 *The Theory of Singular Perturbation* (Amsterdam: North-Holland Publishing Co.)
[2] Ni W M and Wei J C 2006 *J. Diff. Eqns.* **221** 158
[3] Bartier J P 2006 *Asymptotic Anal.* **46** 325
[4] Dibre J, da Silva P R and Teixeira M A 2007 *J. Dyn. Differ. Equs.* **19** 309
[5] Duehring D and Huang W Z 2007 *J. Dyn. Differ. Eqns.* **19** 457
[6] Guarguaglini F R and Natalini R 2007 *Partial Differ. Equations* **32** 163
[7] Mo J Q 1993 *J. Math. Anal. Appl.* **178** 289
[8] Mo J Q 1989 *Science in China* Ser. A **32** 1306
[9] Mo J Q and Lin W T 2005 *Acta Math. Appl. Sinica* **21** 101 (in Chinese)
[10] Mo J Q, Zhang W J and He M 2007 *Acta Math. Sci.* **27** 777
[11] Mo J Q 2006 *Advances in Math.* **35** 75
[12] Mo J Q, Zhu J and Wang H 2003 *Prog. Nat. Sci.* **13** 768
[13] Mo J Q 2009 *Chin. Phys. Lett.* **26** 010204-1
[14] Mo J Q 2009 *Acta Phys. Sin.* **58** 693 (in Chinese)
[15] Mo J Q 2009 *Acta Phys. Sin.* **58** 2930 (in Chinese)
[16] Mo J Q and Wang H 2007 *Acta Ecologica Sin.* **27** 4366
[17] Mo J Q and Lin W T 2008 *J. Sys. Sci.* & *Complexity* **20** 119
[18] Mo J Q, Lin W T and Wang H 2007 *Prog. Nat. Sci.* **17** 230
[19] Mo J Q, Lin W T and Wang H 2007 *Chin. Phys.* **16** 951
[20] Mo J Q and Lin W T 2008 *Chin. Phys.* B **17** 370
[21] Mo J Q and Lin W T 2008 *Chin. Phys.* B **17** 743
[22] Mo J Q and Lin W T 2007 *Acta Phys. Sin.* **56** 3662 (in Chinese)
[23] Mo J Q and Lin W T 2008 *Appl. Math. J. Chin. Univ.* **23** 251
[24] Mo J Q 2008 *Adv. In Math.* **37** 83
[25] Mo J Q, Lin W T and Wang H 2009 *Acta Math. Sci.* **29** 101