A new eight-dimensional Lie superalgebra and two corresponding hierarchies of evolution equations*

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A new eight-dimensional Lie superalgebra is constructed and two isospectral problems with six potentials are designed. Corresponding hierarchies of nonlinear evolution equations, as well as super-AKNS and super-Levi, are derived. Their super-Hamiltonian structures are established by making use of the supertrace identity, and they are integrable in the sense of Liouville.

Keywords: Lie superalgebra, supertrace identity, superintegrable system, super-Hamiltonian structure

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1. Introduction

It is of both theoretical and practical value to find as many new integrable systems as possible and to elucidate in depth their algebraic and geometric properties. In practice, it will provide us with dozens of nonlinear evolution equations, which are of potential value in physical applications.[1−14] Recently, Professor Ma Wen-Xiu proposed a method to obtain a superintegrable system in Ref. [15]. The main ideas are to construct a Lie superalgebra and obtain a super hierarchy and its Hamiltonian structure by making using of super-identity. In this paper, we want to construct another Lie superalgebra which is used to present another two isospectral problems with six potentials and obtain super-AKNS hierarchy and super-Levi hierarchy. In each case a relevant super-Hamiltonian structure is established by making use of the supertrace identity. Finally, we illustrate that the two integrable couplings are Liouville integrable.

Let us recall the general definition of Lie superalgebra.[16] Let $G = G_0 \uplus G_1$ be a linear space; we call G a Lie superalgebra after a super commutator $\langle\cdot,\cdot\rangle \in G$. For arbitrary $\boldsymbol{x}, \boldsymbol{y} \in G$, satisfies the following:

(i) Super skew-symmetry: $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = -(-1)^{\alpha\beta}\langle \boldsymbol{y}, \boldsymbol{x}\rangle$;

(ii) Bilinear: $\langle a\boldsymbol{x}+b\boldsymbol{y}, \boldsymbol{z}\rangle = a\langle \boldsymbol{x}, \boldsymbol{z}\rangle + b\langle \boldsymbol{y}, \boldsymbol{z}\rangle, \forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in G$;

(iii) Super Jacobi identity: $\langle\langle \boldsymbol{x}, \boldsymbol{y}\rangle, \boldsymbol{z}\rangle = \langle \boldsymbol{x}, \langle \boldsymbol{y}, \boldsymbol{z}\rangle\rangle + (-1)^{\alpha\beta}\langle \boldsymbol{y}, \langle \boldsymbol{x}, \boldsymbol{z}\rangle\rangle$.

Where $\begin{cases} \alpha = 0, \text{ if } \boldsymbol{x} \in G_0, \\ \alpha = 1, \text{ if } \boldsymbol{x} \in G_1, \end{cases}$ and $\begin{cases} \beta = 0, \text{ if } \boldsymbol{y} \in G_0, \\ \beta = 1, \text{ if } \boldsymbol{y} \in G_1. \end{cases}$

Now we first construct the following linear space $G = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7, \boldsymbol{e}_8\}$,

$$
\begin{aligned}
&[\boldsymbol{e}_1, \boldsymbol{e}_2] = [\boldsymbol{e}_2, \boldsymbol{e}_4] = \boldsymbol{e}_2, [\boldsymbol{e}_1, \boldsymbol{e}_3] = [\boldsymbol{e}_3, \boldsymbol{e}_4] = -\boldsymbol{e}_3, [\boldsymbol{e}_2, \boldsymbol{e}_3] = \boldsymbol{e}_1 - \boldsymbol{e}_4, [\boldsymbol{e}_1, \boldsymbol{e}_4] = 0,\\
&[\boldsymbol{e}_1, \boldsymbol{e}_5] = [\boldsymbol{e}_1, \boldsymbol{e}_6] = [\boldsymbol{e}_2, \boldsymbol{e}_5] = [\boldsymbol{e}_2, \boldsymbol{e}_7] = [\boldsymbol{e}_3, \boldsymbol{e}_6] = [\boldsymbol{e}_3, \boldsymbol{e}_8] = [\boldsymbol{e}_4, \boldsymbol{e}_7] = [\boldsymbol{e}_4, \boldsymbol{e}_8] = 0,\\
&[\boldsymbol{e}_1, \boldsymbol{e}_7] = \boldsymbol{e}_7, [\boldsymbol{e}_1, \boldsymbol{e}_8] = -\boldsymbol{e}_8, [\boldsymbol{e}_2, \boldsymbol{e}_6] = -\boldsymbol{e}_7, [\boldsymbol{e}_2, \boldsymbol{e}_8] = \boldsymbol{e}_5,\\
&[\boldsymbol{e}_3, \boldsymbol{e}_5] = \boldsymbol{e}_8, [\boldsymbol{e}_3, \boldsymbol{e}_7] = -\boldsymbol{e}_6, [\boldsymbol{e}_4, \boldsymbol{e}_5] = -\boldsymbol{e}_5, [\boldsymbol{e}_4, \boldsymbol{e}_6] = \boldsymbol{e}_6,\\
&[\boldsymbol{e}_5, \boldsymbol{e}_5]_+ = [\boldsymbol{e}_6, \boldsymbol{e}_6]_+ = [\boldsymbol{e}_7, \boldsymbol{e}_7]_+ = [\boldsymbol{e}_8, \boldsymbol{e}_8]_+ = [\boldsymbol{e}_5, \boldsymbol{e}_8]_+ = [\boldsymbol{e}_6, \boldsymbol{e}_7]_+ = 0,\\
&[\boldsymbol{e}_5, \boldsymbol{e}_6]_+ = \boldsymbol{e}_1, [\boldsymbol{e}_5, \boldsymbol{e}_7]_+ = \boldsymbol{e}_2, [\boldsymbol{e}_6, \boldsymbol{e}_8]_+ = \boldsymbol{e}_3, [\boldsymbol{e}_7, \boldsymbol{e}_8]_+ = \boldsymbol{e}_4. \qquad (1)
\end{aligned}
$$

It is easy to prove that the linear space is a Lie superalgebra, where $G_0 = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ are even and $G_1 = \{\boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7, \boldsymbol{e}_8\}$ are odd, and $[\cdot,\cdot]$ and $[\cdot,\cdot]_+$ denote the commutator and the anticommutator. The corresponding loop superalgebra $\tilde{G}$ is given as

$$\tilde{G} = \tilde{B}(0,1) = B(0,1) \otimes C[\lambda, \lambda^{-1}]. \qquad (2)$$

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Based on the Lie superalgebra G in Eq. (1) and associated corresponding loop superalgebra $\tilde{G}$, a direct calculation gives

$$ad_{\boldsymbol{a}} = \begin{pmatrix} 0 & -a_3 & a_2 & 0 & a_6 & a_5 & 0 & 0 \\ -a_2 & a_1-a_4 & 0 & a_2 & a_7 & 0 & a_5 & 0 \\ a_3 & 0 & a_4-a_1 & -a_3 & 0 & a_8 & 0 & a_6 \\ 0 & a_3 & -a_2 & 0 & 0 & 0 & a_8 & a_7 \\ 0 & -a_8 & 0 & a_5 & -a_4 & 0 & 0 & a_2 \\ 0 & 0 & a_7 & -a_6 & 0 & a_4 & -a_3 & 0 \\ -a_7 & a_6 & 0 & 0 & 0 & -a_2 & a_1 & 0 \\ a_8 & 0 & -a_5 & 0 & a_3 & 0 & 0 & -a_1 \end{pmatrix} \tag{3}$$

for $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4 + a_5\boldsymbol{e}_5 + a_6\boldsymbol{e}_6 + a_7\boldsymbol{e}_7 + a_8\boldsymbol{e}_8 \in \tilde{G}$, where $ad_{\boldsymbol{a}}\boldsymbol{b} = [\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{a}, \boldsymbol{b} \in G$, the bracket $[\cdot, \cdot]$ is the Lie superbracket of G. Then we have

$$\mathrm{str}(ad_{\boldsymbol{a}}ad_{\boldsymbol{b}}) = 2(-a_1b_4 + a_2b_3 + a_3b_2 - a_4b_1 + a_5b_6 - a_6b_5 - a_7b_8 + a_8b_7). \tag{4}$$

2. The super-AKNS hierarchy and its super-Hamiltonian structure

Considering the super-isospectral problem as follows

$$\boldsymbol{\varphi}_x = [\boldsymbol{U}, \boldsymbol{\varphi}], \quad \boldsymbol{U} = \lambda\boldsymbol{e}_1 + u_1\boldsymbol{e}_2 + u_2\boldsymbol{e}_3 - \lambda\boldsymbol{e}_4 + u_3\boldsymbol{e}_5 + u_4\boldsymbol{e}_6 + u_5\boldsymbol{e}_7 + u_6\boldsymbol{e}_8, \lambda_t = 0. \tag{5}$$

Taking

$$\boldsymbol{V} = V_1\boldsymbol{e}_1 + V_2\boldsymbol{e}_2 + V_3\boldsymbol{e}_3 + V_4\boldsymbol{e}_4 + V_5\boldsymbol{e}_5 + V_6\boldsymbol{e}_6 + V_7\boldsymbol{e}_7 + V_8\boldsymbol{e}_8, V_i = \sum_{m\geq 0} V_i^m\lambda^{-m}, i = 1, 2, \ldots, 8.$$

Solving the stationary zero curvature equation $\boldsymbol{V}_x = [\boldsymbol{U}, \boldsymbol{V}]$, gives rise to a solution for V as follows:

$$\begin{cases} V_{1,x}^n = u_1V_3^n - u_2V_2^n + u_3V_6^n + u_4V_5^n, \\ V_2^{n+1} = \frac{1}{2}(\partial V_2^n + u_1V_1^n - u_1V_4^n - u_3V_7^n - u_5V_5^n), \\ V_3^{n+1} = \frac{1}{2}(u_2V_1^n - u_2V_4^n + u_4V_8^n + u_6V_6^n - \partial V_3^n), \\ V_{4,x}^n = u_2V_2^n - u_1V_3^n + u_5V_8^n + u_6V_7^n, \\ V_5^{n+1} = \partial V_5^n - u_1V_8^n + u_6V_2^n - u_3V_4^n, \\ V_6^{n+1} = u_5V_3^n - u_2V_7^n - u_4V_4^n - \partial V_6^n, \\ V_7^{n+1} = \partial V_7^n - u_4V_2^n + u_1V_6^n + u_5V_1^n, \\ V_8^{n+1} = u_6V_1^n + u_2V_5^n - u_3V_3^n - \partial V_8^n, \\ V_1^0 = -V_4^0 = 1, V_2^0 = V_3^0 = V_5^0 = V_6^0 = V_7^0 = V_8^0 = 0, \\ V_2^1 = u_1, V_3^1 = u_2, V_5^1 = u_3, V_6^1 = u_4, V_7^1 = u_5, \\ V_8^1 = u_6, V_1^1 = 2\partial^{-1}u_3u_4, V_4^1 = 2\partial^{-1}u_5u_6. \end{cases} \tag{6}$$

Denoting

$$\boldsymbol{V}_+^{(n)} = \sum_{m=0}^{n}(V_1^m\boldsymbol{e}_1 + V_2^m\boldsymbol{e}_2 + V_3^m\boldsymbol{e}_3 + V_4^m\boldsymbol{e}_4 + V_5^m\boldsymbol{e}_5 + V_6^m\boldsymbol{e}_6 + V_7^m\boldsymbol{e}_7 + V_8^m\boldsymbol{e}_8)\lambda^{n-m}, \quad \boldsymbol{V}_-^{(n)} = \lambda^n\boldsymbol{V} - \boldsymbol{V}_+^{(n)}, \tag{7}$$

a direct calculation reads

$$\boldsymbol{V}_{+,x}^{(n)} - [\boldsymbol{U}, \boldsymbol{V}_+^{(n)}] = -\boldsymbol{V}_{-,x}^{(n)} + [\boldsymbol{U}, \boldsymbol{V}_-^{(n)}] = 2V_2^{n+1}\boldsymbol{e}_2 - 2V_3^{n+1}\boldsymbol{e}_3 + V_5^{n+1}\boldsymbol{e}_5 - V_6^{n+1}\boldsymbol{e}_6 + V_7^{n+1}\boldsymbol{e}_7 - V_8^{n+1}\boldsymbol{e}_8. \tag{8}$$

Denote $\boldsymbol{V}^{(n)} = \boldsymbol{V}_+^{(n)}$, then the zero curvature equation

$$\boldsymbol{U}_t - \boldsymbol{V}_x^{(n)} + [\boldsymbol{U}, \boldsymbol{V}^{(n)}] = 0 \tag{9}$$

admits the following superintegrable system

$$\boldsymbol{u}_t=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\end{pmatrix}_t=\begin{pmatrix}2V_2^{n+1}\\-2V_3^{n+1}\\V_5^{n+1}\\-V_6^{n+1}\\V_7^{n+1}\\-V_8^{n+1}\end{pmatrix}=\begin{pmatrix}0&2&0&0&0&0\\-2&0&0&0&0&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&-1&0\end{pmatrix}\begin{pmatrix}V_3^{n+1}\\V_2^{n+1}\\-V_6^{n+1}\\V_5^{n+1}\\V_8^{n+1}\\-V_7^{n+1}\end{pmatrix}=\boldsymbol{J}P_{n+1}. \tag{10}$$

From the recursion relations in Eqs. (6), a recurrence operation $\boldsymbol{L}$ is given by

$$\boldsymbol{L}=\begin{pmatrix}-\frac{1}{2}\partial+u_2\partial^{-1}u_1&-u_2\partial^{-1}u_2&-\frac{1}{2}u_6-\frac{1}{2}u_2\partial^{-1}u_3&\frac{1}{2}u_2\partial^{-1}u_4&\frac{1}{2}u_4-\frac{1}{2}u_2\partial^{-1}u_5&\frac{1}{2}u_2\partial^{-1}u_6\\u_1\partial^{-1}u_1&\frac{1}{2}\partial-u_1\partial^{-1}u_2&-\frac{1}{2}u_1\partial^{-1}u_3&\frac{1}{2}u_1\partial^{-1}u_4-\frac{1}{2}u_5&-\frac{1}{2}u_1\partial^{-1}u_5&\frac{1}{2}u_1\partial^{-1}u_6+\frac{1}{2}u_3\\-u_5-u_4\partial^{-1}u_1&u_4\partial^{-1}u_2&-\partial&0&u_4\partial^{-1}u_5&-u_2-u_4\partial^{-1}u_6\\u_3\partial^{-1}u_1&u_6-u_3\partial^{-1}u_2&0&\partial&-u_1-u_3\partial^{-1}u_5&u_3\partial^{-1}u_6\\u_6\partial^{-1}u_1-u_3&-u_6\partial^{-1}u_2&-u_6\partial^{-1}u_3&u_2+u_6\partial^{-1}u_4&-\partial&0\\-u_5\partial^{-1}u_1&u_4+u_5\partial^{-1}u_2&u_1+u_5\partial^{-1}u_3&-u_5\partial^{-1}u_4&0&\partial\end{pmatrix}.$$

Taking $u_3=u_4=u_5=u_6=0$ in Eqs. (6), we can conclude $V_1^n=-V_4^n$, then the system reduces to the AKNS hierarchy. Therefore Eqs. (10) are a super-AKNS hierarchy.

It is easy to compute that

$$\begin{aligned}&\mathrm{str}(ad_{\boldsymbol{V}}ad_{\boldsymbol{U}_\lambda})=2(V_4-V_1),\\&\mathrm{str}(ad_{\boldsymbol{V}}ad_{\partial\boldsymbol{U}/\partial u_1})=2V_3,\qquad\mathrm{str}(ad_{\boldsymbol{V}}ad_{\partial\boldsymbol{U}/\partial u_2})=2V_2,\\&\mathrm{str}(ad_{\boldsymbol{V}}ad_{\partial\boldsymbol{U}/\partial u_3})=-2V_6,\qquad\mathrm{str}(ad_{\boldsymbol{V}}ad_{\partial\boldsymbol{U}/\partial u_4})=2V_5,\\&\mathrm{str}(ad_{\boldsymbol{V}}ad_{\partial\boldsymbol{U}/\partial u_5})=2V_8,\qquad\mathrm{str}(ad_{\boldsymbol{V}}ad_{\partial\boldsymbol{U}/\partial u_6})=-2V_7.\end{aligned} \tag{11}$$

According to the supertrace identity,[15] we have

$$\frac{\delta}{\delta\boldsymbol{u}}\int 2(V_4-V_1)\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left(2V_3,2V_2,-2V_6,2V_5,2V_8,-2V_7\right)^{\mathrm{T}}. \tag{12}$$

Comparing the coefficient of λ^{-n-2} yields

$$\frac{\delta}{\delta\boldsymbol{u}}\int(V_4^{n+2}-V_1^{n+2})\mathrm{d}x=(\gamma-n-1)\left(V_3^{n+1},V_2^{n+1},-V_6^{n+1},V_5^{n+1},V_8^{n+1},-V_7^{n+1}\right)^{\mathrm{T}}. \tag{13}$$

Since $\mathrm{str}(ad_{\boldsymbol{V}}ad_{\boldsymbol{V}})=4\neq0$, we have $\gamma=0$. Therefore,

$$\boldsymbol{P}_{n+1}=\frac{\delta H_{n+1}}{\delta\boldsymbol{u}},\quad H_{n+1}=-\int\frac{V_4^{n+2}-V_1^{n+2}}{n+1}\mathrm{d}x,n\geq0. \tag{14}$$

Hence, the super-AKNS hierarchy has the following super-bi-Hamiltonian structure

$$\boldsymbol{u}_t=\boldsymbol{J}\boldsymbol{P}_{n+1}=\boldsymbol{J}\frac{\delta H_{n+1}}{\delta\boldsymbol{u}}=\boldsymbol{M}\frac{\delta H_n}{\delta\boldsymbol{u}},n\geq0 \tag{15}$$

with the super-Hamiltonian operator $\boldsymbol{M}$ defined by

$$\boldsymbol{M}=\boldsymbol{J}\boldsymbol{L}=\begin{pmatrix}2u_1\partial^{-1}u_1&\partial-2u_1\partial^{-1}u_2&-u_1\partial^{-1}u_3&u_1\partial^{-1}u_4-u_5&-u_1\partial^{-1}u_5&u_1\partial^{-1}u_6+u_3\\\partial-2u_2\partial^{-1}u_1&2u_2\partial^{-1}u_2&u_6+u_2\partial^{-1}u_3&-u_2\partial^{-1}u_4&-u_4+u_2\partial^{-1}u_5&-u_2\partial^{-1}u_6\\u_3\partial^{-1}u_1&u_6-u_3\partial^{-1}u_2&0&\partial&-u_1-u_3\partial^{-1}u_5&u_3\partial^{-1}u_6\\-u_5-u_4\partial^{-1}u_1&u_4\partial^{-1}u_2&-\partial&0&u_4\partial^{-1}u_5&-u_2-u_4\partial^{-1}u_6\\u_5\partial^{-1}u_1&-u_4-u_5\partial^{-1}u_2&-u_1-u_5\partial^{-1}u_3&u_5\partial^{-1}u_4&0&-\partial\\u_3-u_6\partial^{-1}u_1&u_6\partial^{-1}u_2&u_6\partial^{-1}u_3&-u_2-u_6\partial^{-1}u_4&\partial&0\end{pmatrix}.$$

3. The super-Levi hierarchy and its super-Hamiltonian structure

Now considering the super-isospectral problem

$$\boldsymbol{\varphi}_x=[\boldsymbol{U},\boldsymbol{\varphi}],\quad\boldsymbol{U}=q\boldsymbol{e}_2+r\boldsymbol{e}_3+(\lambda+r-q)\boldsymbol{e}_4+u_1\boldsymbol{e}_5+u_2\boldsymbol{e}_6+u_3\boldsymbol{e}_7+u_4\boldsymbol{e}_8,\lambda_t=0. \tag{16}$$

Taking

$$\boldsymbol{V}=V_1\boldsymbol{e}_1+V_2\boldsymbol{e}_2+V_3\boldsymbol{e}_3+V_4\boldsymbol{e}_4+V_5\boldsymbol{e}_5+V_6\boldsymbol{e}_6+V_7\boldsymbol{e}_7+V_8\boldsymbol{e}_8,\quad V_i=\sum_{m\geq 0}V_i^m\lambda^{-m},i=1,2,\ldots,8.$$

Solving the stationary zero curvature equation $\boldsymbol{V}_x=[\boldsymbol{U},\boldsymbol{V}]$, gives rise to a solution for $\boldsymbol{V}$ as follows:

$$\begin{cases}
V_{1,x}^n=qV_3^n-rV_2^n+u_1V_6^n+u_2V_5^n,\\
V_{2,x}^n=-qV_1^n+qV_4^n-(\lambda+r-q)V_2^n+u_1V_7^n+u_3V_5^n,\\
V_{3,x}^n=rV_1^n+(\lambda+r-q)V_3^n-rV_4^n+u_4V_6^n+u_2V_8^n,\\
V_{4,x}^n=rV_2^n-qV_3^n+u_3V_8^n+u_4V_7^n,\\
V_{5,x}^n=qV_8^n-u_4V_2^n+u_1V_4^n-(\lambda+r-q)V_5^n,\\
V_{6,x}^n=u_3V_3^n-rV_7^n+(\lambda+r-q)V_6^n-u_2V_4^n,\\
V_{7,x}^n=u_2V_2^n-qV_6^n-u_3V_1^n,\\
V_{8,x}^n=u_4V_1^n+rV_5^n-u_1V_3^n,\\
V_1^0=-V_4^0=1,V_2^0=V_3^0=V_5^0=V_6^0=V_7^0=V_8^0=0,\\
V_2^1=2q,V_3^1=2r,V_5^1=u_1,V_6^1=u_2,V_1^1=\partial^{-1}u_1u_2,\\
V_7^1=\partial^{-1}(u_3\partial^{-1}u_1u_2-2qu_2),V_8^1=\partial^{-1}(u_4\partial^{-1}u_1u_2-2u_1r),\\
V_4^1=\partial^{-1}[u_3\partial^{-1}(u_4\partial^{-1}u_1u_2-2u_1r)+u_4\partial^{-1}(u_3\partial^{-1}u_1u_2-2qu_2)].
\end{cases}\tag{17}$$

Denoting

$$\boldsymbol{V}_+^{(n)}=\sum_{m=0}^{n}(V_1^m\boldsymbol{e}_1+V_2^m\boldsymbol{e}_2+V_3^m\boldsymbol{e}_3+V_4^m\boldsymbol{e}_4+V_5^m\boldsymbol{e}_5+V_6^m\boldsymbol{e}_6+V_7^m\boldsymbol{e}_7+V_8^m\boldsymbol{e}_8)\lambda^{n-m},\quad \boldsymbol{V}_-^{(n)}=\lambda^n\boldsymbol{V}-\boldsymbol{V}_+^{(n)},$$

a direct calculation reads

$$\boldsymbol{V}_{+,x}^{(n)}-[\boldsymbol{U},\boldsymbol{V}_+^{(n)}]=-\boldsymbol{V}_{-,x}^{(n)}+[\boldsymbol{U},\boldsymbol{V}_-^{(n)}]=-V_2^{n+1}\boldsymbol{e}_2+V_3^{n+1}\boldsymbol{e}_3-V_5^{n+1}\boldsymbol{e}_5+V_6^{n+1}\boldsymbol{e}_6.\tag{18}$$

Taking $\boldsymbol{\Delta}_n=(V_1^n-V_4^n+V_3^n-V_2^n)\boldsymbol{e}_4+(V_6^n-V_7^n)\boldsymbol{e}_7-(V_5^n+V_8^n)\boldsymbol{e}_8$, $\boldsymbol{V}^{(n)}=\boldsymbol{V}_+^{(n)}+\boldsymbol{\Delta}_n$, then we admit the following superintegrable system

$$\boldsymbol{u}_t=\begin{pmatrix}q\\r\\u_1\\u_2\\u_3\\u_4\end{pmatrix}_t=\begin{pmatrix}-V_2^{n+1}-q(V_1^n-V_4^n+V_3^n-V_2^n)-u_1V_6+u_1V_7\\V_3^{n+1}+r(V_1^n-V_4^n+V_3^n-V_2^n)+u_2V_5+u_2V_8\\-v_5^{n+1}-u_1(V_1^n-V_4^n+V_3^n-V_2^n)+q(V_5+V_8)\\v_6^{n+1}+u_2(V_1^n-V_4^n+V_3^n-V_2^n)+r(V_6-V_7)\\(V_6-V_7)_x\\(V_5+V_8)_x\end{pmatrix}=\boldsymbol{J}\begin{pmatrix}V_3^n+V_1^n\\V_2^n-V_1^n\\-V_6^n\\V_5^n\\V_8^n\\-V_7^n\end{pmatrix}=\boldsymbol{J}\boldsymbol{P}_n,\tag{19}$$

where

$$\boldsymbol{J}=\begin{pmatrix}0&\partial&0&u_3-u_2&0&0\\\partial&0&-u_1-u_4&0&0&0\\0&u_1+u_4&0&\partial&\partial&0\\u_2-u_3&0&-\partial&0&0&\partial\\0&0&-\partial&0&0&\partial\\0&0&0&-\partial&-\partial&0\end{pmatrix}.\tag{20}$$

From the recursion relations in Eqs. (17), a recurrence operation $\boldsymbol{L}$ meets as follows:

$$\boldsymbol{L}=\begin{pmatrix}L_{11}&L_{12}&L_{13}&L_{14}&(r+2\partial^{-1}u_1u_2)\partial^{-1}u_3-u_2&-(r+2\partial^{-1}u_1u_2)\partial^{-1}u_4\\L_{21}&L_{22}&L_{23}&L_{24}&(q-2\partial^{-1}u_1u_2)\partial^{-1}u_3&-(q+2\partial^{-1}u_1u_2)\partial^{-1}u_4-u_1\\L_{31}&L_{32}&L_{33}&L_{34}&-u_2\partial^{-1}u_3&u_2\partial^{-1}u_4+r\\L_{41}&L_{42}&L_{43}&L_{44}&u_1\partial^{-1}u_3+q&-u_1\partial^{-1}u_4\\L_{51}&L_{52}&L_{53}&L_{54}&L_{55}&-2\partial^{-1}u_4\partial^{-1}u_1u_2\partial^{-1}u_4\\L_{61}&L_{62}&L_{63}&L_{64}&2\partial^{-1}u_3\partial^{-1}u_1u_2\partial^{-1}u_3&L_{66}\end{pmatrix},\tag{21}$$

where

$$
\begin{aligned}
L_{11} &= (\partial - r + \partial^{-1}q\partial) + (r + \partial^{-1}u_1u_4 - \partial^{-1}u_1u_3)(q + r + \partial)^{-1}q + \partial^{-1}u_1u_3 \\
&\quad + (r + 2\partial^{-1}u_1u_2)\partial^{-1}(q + r)(q + r + \partial)^{-1}q - (r + 2\partial^{-1}u_1u_2)\partial^{-1}q, \\
L_{12} &= (r + \partial^{-1}r\partial) + r(q + r + \partial)^{-1}(q + \partial + 1) + \partial^{-1}u_1u_4(q + r + \partial)^{-1}(q + \partial) \\
&\quad + \partial^{-1}u_1u_3(q + r + \partial)^{-1}r + (r + 2\partial^{-1}u_1u_2)\partial^{-1}[r - (q + r)(q + r + \partial)^{-1}r, \\
L_{13} &= u_1 + u_4 + \partial^{-1}qu_4 + \partial^{-1}u_1\partial - [r + \partial^{-1}u_1(u_4 - u_3) + (r + 2\partial^{-1}u_1u_2)(q + r)](q + r + \partial)^{-1}u_1, \\
L_{14} &= -u_2 - \partial^{-1}ru_4 - \partial^{-1}u_2\partial + [r + \partial^{-1}u_1(u_4 - u_3) + (r + 2\partial^{-1}u_1u_2)(q + r)](q + r + \partial)^{-1}u_2, \\
L_{21} &= (-q - \partial^{-1}q\partial) + (q - r + \partial r + \partial^{-1}u_2u_4)(q + r + \partial)^{-1}q \\
&\quad + (q - \partial q + \partial^{-1}u_1u_3)[1 - (q + r + \partial)^{-1}q] - (q - 2\partial^{-1}u_1u_2)^{-1}\partial(q + r + \partial)^{-1}q, \\
L_{22} &= (-\partial + q - \partial^{-1}r\partial) + (q - r + \partial r + \partial^{-1}u_2u_4)(q + r + \partial)^{-1}(q + \partial) \\
&\quad - (q - \partial q + \partial^{-1}u_1u_3)(q + r + \partial)^{-1}\partial - (q - 2\partial^{-1}u_1u_2)^{-1}[(q + r)(q + r + \partial)^{-1}(q + \partial) - q], \\
L_{23} &= -[(\partial r - r + \partial^{-1}u_2u_4) - (-\partial q + \partial^{-1}u_1u_4) + (q - 2\partial^{-1}u_1u_2)\partial^{-1}(q + r)](q + r + \partial)^{-1}u_1 \\
&\quad + \partial^{-1}u_1\partial - \partial^{-1}qu_4 + \partial^{-1}u_1, \\
L_{24} &= [(\partial r - r + \partial^{-1}u_2u_4) - (-\partial q + \partial^{-1}u_1u_4) + (q - 2\partial^{-1}u_1u_2)\partial^{-1}(q + r)](q + r + \partial)^{-1}u_2 \\
&\quad + u_3 + \partial^{-1}u_3\partial + \partial^{-1}u_2\partial - \partial u_2, \\
L_{31} &= u_3 + u_2\partial^{-1}q - [u_3 + u_2\partial^{-1}(q + r)](q + r + \partial)^{-1}q, \\
L_{32} &= u_3 + u_2\partial^{-1}q - [u_3 + u_2\partial^{-1}(q + r)](q + r + \partial)^{-1}(q + \partial), \\
L_{33} &= \partial - r + q + [u_3 + u_2\partial^{-1}(q + r)](q + r + \partial)^{-1}u_1, \\
L_{34} &= -[u_3 + u_2\partial^{-1}(q + r)](q + r + \partial)^{-1}u_2, \\
L_{41} &= [-u_4 + u_1\partial^{-1}(q + r)](q + r + \partial)^{-1}q - u_1\partial^{-1}q, \\
L_{42} &= [-u_4 + u_1\partial^{-1}(q + r)](q + r + \partial)^{-1}(q + \partial) - u_1\partial^{-1}q, \\
L_{43} &= [u_4 - u_1\partial^{-1}(q + r)](q + r + \partial)^{-1}u_1, \\
L_{44} &= [-u_4 + u_1\partial^{-1}(q + r)](q + r + \partial)^{-1}u_2 + q - r - \partial, \\
L_{51} &= \partial^{-1}\{u_4\partial^{-1}q(q + \partial - r) - u_1(q + \partial - r) - u_4\partial^{-1}u_1u_3 - 2u_4\partial^{-1}u_1u_2(q + r + \partial)^{-1} \\
&\quad + [u_4\partial^{-1}(-q^2 - q\partial + qr + u_1u_3 + r^2 + r\partial - u_2u_4) + (u_1\partial + u_1q - ru_4)](q + r + \partial)^{-1}q\}, \\
L_{52} &= \partial^{-1}\{u_4\partial^{-1}q(q + \partial - r) - u_1(q + \partial - r) + 2u_4\partial^{-1}u_1u_2[\partial^{-1}r - \partial^{-1}(q + r)(q + r + \partial)^{-1}r] \\
&\quad + [u_4\partial^{-1}(-q^2 - q\partial + qr + u_1u_3 + r^2 + r\partial - u_2u_4) + (u_1\partial + u_1q - ru_4)](q + r + \partial)^{-1}r\}, \\
L_{53} &= -\partial^{-1}\{[u_4\partial^{-1}(-q^2 - q\partial + qr + u_1u_3 + r^2 + r\partial - u_2u_4) + (u_1\partial + u_1q - ru_4)](q + r + \partial)^{-1}u_1 \\
&\quad + u_4\partial^{-1}(-qu_4 + u_1\partial - u_1r + u_1q) + u_1u_4 + 2u_4\partial^{-1}u_1u_2\partial^{-1}(q + r)(q + r + \partial)^{-1}u_1\}, \\
L_{54} &= \partial^{-1}\{[u_4\partial^{-1}(-q^2 - q\partial + qr + u_1u_3 + r^2 + r\partial - u_2u_4) + (u_1\partial + u_1q - ru_4)](q + r + \partial)^{-1}u_2 \\
&\quad + u_4\partial^{-1}(-ru_3 - ru_2 + u_2q - u_2\partial) - r(r - q + \partial) + 2u_4\partial^{-1}u_1u_2\partial^{-1}(q + r)(q + r + \partial)^{-1}u_2\}, \\
L_{55} &= \partial^{-1}(u_1u_2 + qr + 2u_4\partial^{-1}u_1u_2\partial^{-1}u_3), \\
L_{61} &= \partial^{-1}\{-qu_3 + u_3\partial^{-1}[q(q + \partial - r) - u_1u_3 + 2u_1u_2\partial^{-1}q]\} \\
&\quad + \partial^{-1}\{u_2(r + \partial) - qu_3 + u_3\partial^{-1}[2u_1u_2\partial^{-1}(r - q) + r^2 + r\partial - u_2u_4 - q^2 - q\partial + u_1u_3]\}(q + r + \partial)^{-1}q, \\
L_{62} &= \partial^{-1}\{u_3\partial^{-1}[-qr - u_1u_3 + 2u_1u_2\partial^{-1}r - u_2u_4 - q^2 - q\partial + u_1u_3] + u_2(r + \partial - q)\} \\
&\quad - \partial^{-1}\{u_2(r + \partial) - qu_3 + u_3\partial^{-1}[2u_1u_2\partial^{-1}(r - q) + r^2 + r\partial - u_2u_4 - q^2 - q\partial + u_1u_3]\}(q + r + \partial)^{-1}r, \\
L_{63} &= -\partial^{-1}[q(\partial - r + q) + u_3\partial^{-1}(-qu_4 + u_1\partial - u_1r + u_1q)] \\
&\quad - \partial^{-1}\{u_2(r + \partial) - qu_3 + u_3\partial^{-1}[2u_1u_2\partial^{-1}(r - q) + r^2 + r\partial - u_2u_4 - q^2 - q\partial + u_1u_3]\}(q + r + \partial)^{-1}u_1, \\
L_{64} &= \partial^{-1}[u_3\partial^{-1}(u_2q - ru_3 - u_2r - u_2\partial) - u_2u_3] \\
&\quad + \partial^{-1}\{u_2(r + \partial) - qu_3 + u_3\partial^{-1}[2u_1u_2\partial^{-1}(r - q) + r^2 + r\partial - u_2u_4 - q^2 - q\partial + u_1u_3]\}(q + r + \partial)^{-1}u_2, \\
L_{66} &= -\partial^{-1}(qr + 2u_3\partial^{-1}u_1u_2\partial^{-1}u_4 - u_1u_2).
\end{aligned}
$$

According to the supertrace identity,[10] we have

$$-\frac{\delta}{\delta \boldsymbol{u}}\int 2V_1\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left(2(V_3+V_1),2(V_2-V_1),-2V_6,2V_5,2V_8,-2V_7\right)^{\mathrm{T}}. \tag{22}$$

Comparing the coefficient of λ^{-n-1} yields

$$-\frac{\delta}{\delta \boldsymbol{u}}\int V_1^{n+1}\mathrm{d}x=(\gamma-n)\left(V_3^n+V_1^n,V_2^n-V_1^n,-V_6^n,V_5^n,V_8^n,-V_7^n\right)^{\mathrm{T}}. \tag{23}$$

Since $\mathrm{str}(ad_{\boldsymbol{V}}ad_{\boldsymbol{V}})=2\neq 0$, we have $\gamma=0$. Therefore,

$$\boldsymbol{P}_n=\frac{\delta H_n}{\delta \boldsymbol{u}},\quad H_n=\int\frac{V_1^{n+1}}{n}\mathrm{d}x,\quad n\geq 0. \tag{24}$$

Taking $u_3=u_4=u_5=u_6=0$ in recursion relations (17) we can conclude $V_1^n=-V_4^n$, then Eqs. (19) degrade into the Levi hierarchy

$$\boldsymbol{u}_t=\begin{pmatrix}q\\r\end{pmatrix}_t=\begin{pmatrix}0&\partial\\\partial&0\end{pmatrix}\begin{pmatrix}V_3^n+V_1^n\\V_2^n-V_1^n\end{pmatrix}=\begin{pmatrix}0&\partial\\\partial&0\end{pmatrix}\begin{pmatrix}\partial-r+\partial^{-1}q\partial&r+\partial^{-1}r\partial\\-q-\partial^{-1}q\partial&-\partial+q-\partial^{-1}r\partial\end{pmatrix}\begin{pmatrix}V_3^{n-1}+V_1^{n-1}\\V_2^{n-1}-V_1^{n-1}\end{pmatrix}.$$

Therefore Eqs. (19) are a super-Levi hierarchy.

4. Concluding remarks

In this paper, we have obtained a Lie superalgebra, super-AKNS hierarchy and super-Levi hierarchy, as well as their Hamiltonian structures. The obtained systems are integrable in the Liouville sense. Here, the super-AKNS hierarchy possessed bi-Hamiltonian structure is different from that Ma presented in Ref. [10]. The method used in this paper can be generalized to other integrable hierarchies. We will discuss these problems in the future.

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