Identifying the temperature distribution in a parabolic equation with overspecified data using a multiquadric quasi-interpolation method*

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In this paper, we use a kind of univariate multiquadric quasi-interpolation to solve a parabolic equation with overspecified data, which has arisen in many physical phenomena. We obtain the numerical scheme by using the derivative of the quasi-interpolation to approximate the spatial derivative of the dependent variable and a simple forward difference to approximate the temporal derivative of the dependent variable. The advantage of the presented scheme is that the algorithm is very simple so it is very easy to implement. The results of the numerical experiment are presented and are compared with the exact solution to confirm the good accuracy of the presented scheme.

Keywords: control parameter, diffusion processes, temperature overspecification, quasi-interpolation

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1. Introduction

It is well known that the investigation of partial differential equations (PDEs) which can describe many important phenomena and dynamical processes in different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains and so on, has played an important role in the study of modern physics. Over the last few years, it has become increasingly apparent that many physical phenomena can be described in terms of parabolic PDEs with source control parameters. The parabolic PDEs with source control parameters usually take the form

$$w_t(x,t) = w_{xx}(x,t) + p(t)w(x,t) + \varphi(x,t).$$

This type of equations arises, for example, in the study of heat conduction processes, thermoelasticity, chemical diffusion and control theory.[1−4] Growing attention is being paid to the development, analysis and implementation of accurate methods for the numerical solution of parabolic equation with overspecified data, i.e. for the determination of the unknown distribution function and control parameter $p(t)$ in the parabolic PDEs.

In the past decades, many numerical methods have been developed successfully to solve virtually most well-posed PDEs which arise in many real-life physical problems.[5,6] These include effective and widely applicable techniques like finite difference (FD), finite elements (FE), boundary elements (BE) and finite volume (FV) methods. These methods, however, depend on a suitable generation of meshes, which is difficult for problems with very complicated and irregular geometries. Moreover, the effects are not as good as we expect when these schemes are applied to the initial conditions with scattered data. To avoid the generation of meshes, in recent years meshless techniques have attracted the attention of researchers. The aim of meshless methods is to eliminate the generation of meshes by constructing the approximation only in terms of nodes. That is to say, in a meshless method a set of nodes are used instead of meshing the domain of the problems. Recently, a truly meshless computational algorithm by using radial basis functions (RBFs) has been used for solving PDEs numerically. Unlike the FD method which approximates the derivatives for the solution by difference quotients on some nodes, the proposed RBF method provides an interpolation formula not only for the solution but also for its derivatives. The earlier approach of applying the RBF method to solve PDEs was proposed by Kansa[7] and solidified by Golberg *el al.*[8] The original idea of using RBFs for solving PDEs appeared in Ref. [9]. In most of the known methods of solving differential equations using the RBF

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interpolation method, one must solve a linear system of equations at each time step. There is also a serious drawback when using the RBF interpolation method for dense data sets. The condition number of the matrices will get extremely large. That is to say, the RBF interpolation method will face a serious ill-conditioning problem due to the use of RBFs as a global interpolation. Furthermore, as we know the accuracy of many schemes for interpolating scattered data with RBFs depends on a shape parameter c of the RBF, and how to select a good c is still an open problem. Moreover, if users expect noise in the data, an interpolatory recovery is not appropriate, because it treats noise as data. In all noisy situations, interpolation should be replaced by approximation. To avoid the problem of ill-condition of the matrices, many researchers have turned to quasi-interpolation. The most important advantage of quasi-interpolation is that one can evaluate the approximant directly and does not have to solve any linear system of equations. The multiquadric quasi-interpolation scheme, as one of meshless methods, possesses some advantages as compared with other approaches, such as less computation time, better shape-preserving properties, etc. With so many good properties, the multiquadric quasi-interpolation method has been extensively researched.[10,11] For a good performance of the multiquadric quasi-interpolation scheme, many researchers have employed the multiquadric quasi-interpolation scheme as a numerical method to solve PDEs.[12−14] In this paper we use the multiquadric quasi-interpolation scheme to solve a class of parabolic equations with overspecified data numerically. Therefore, we can save computation time and decrease the numerical error. It is a very difficult problem to deal with a boundary in a quasi-interpolation scheme. We handle this problem by introducing a polynomial, which is shown to be an effective technique in our multiquadric quasi-interpolation schemes. Furthermore, we develop a kind of univariate multiquadric quasi-interpolation scheme, whose second-order derivative converges to the corresponding derivative of the approximated functions. This scheme is employed here for the numerical solution of a parabolic equation with overspecified data.

This paper is organized as follows. Section 2 introduces the parabolic equation we studied with overspecified data; furthermore, this semi-linear parabolic equation is re-formulated into a linear parabolic equation. In Section 3, we develop a scheme to solve the parabolic equation with overspecified data using the multiquadric quasi-interpolation. To exhibit the efficiency and accuracy of this scheme, Section 4 gives some numerical examples. Section 5 ends this paper with a brief conclusion.

2. The parabolic equation with overspecified data

It is well known that the parabolic PDE with overspecified data

$$w_t(x,t) = w_{xx}(x,t) + p(t)w(x,t) + \varphi(x,t)$$

appears in the mathematical modeling of many phenomena.[15,16]

The parabolic problem with overspecified data is stated as follows:

$$w_t(x,t) = w_{xx}(x,t) + p(t)w(x,t) + \varphi(x,t), \quad 0 \le x \le 1, \quad 0 < t \le T \tag{1}$$

with initial condition

$$w(x,0) = f(x), \quad 0 \le x \le 1 \tag{2}$$

and boundary conditions

$$w(0,t) = g_0(t), \quad 0 \le t \le T, \tag{3}$$

$$w(1,t) = g_1(t), \quad 0 \le t \le T, \tag{4}$$

subject to either overspecification at a point in the spatial domain,

$$w(x^*,t) = E(t), \quad 0 \le t \le T, \tag{5}$$

or integral overspecification of the function $u(x,t)$ over the spatial domain,

$$\int_0^1 u(x,t)\,\mathrm{d}x = E(t), \quad 0 \le t \le T, \tag{6}$$

where $f(x)$, $g_0(t)$, $g_1(t)$, $\varphi(x,t)$ and $E(t)$ are given functions, $w(x,t)$ and $p(t)$ are unknowns. Many diffusion processes involving control parameters can be described by these two model equations. Equation (1) can be used to describe a heat transfer process in the presence of a source parameter. Equation (5) represents the temperature at a fixed point x^* in the spatial domain at time t, and Eq. (6) can then be interpreted as a weighted thermal energy contained in a portion of the spatial domain. Thus, the purpose of solving these parabolic problems with overspecified data is to identify the distribution function and the control parameters. The intent of this work is the development of a multiquadric quasi-interpolation algorithm to obtain the solution of the semi-linear time-dependent one-dimensional diffusion equation subject to temperature overspecification (5). The problem described by Eqs. (1)–(5) can be used to describe a heat

transfer process with a source control parameter $p(t)$, where Eq. (5) is the measured data, which represents the temperature at a given node x^* in a spatial domain at time t. The function $w(x,t)$ is the solution of the heat equation that represents the temperature distribution.

For the existence, uniqueness and some applications of this problem, readers may refer to Refs. [17] and [18].

Adopting the approach of Dehghan,[19] we eliminate the term $p(t)w(x,t)$ in Eq. (1) by introducing the following transformations

$$r(t) = \mathrm{e}^{-\int_0^t p(s)\,\mathrm{d}s}, \tag{7}$$

$$u(x,t) = r(t)w(x,t), \tag{8}$$

then Eqs. (1)–(5) will become

$$u_t(x,t) = u_{xx}(x,t) + r(t)\varphi(x,t), \quad 0 \le x \le 1, \quad 0 \le t \le T, \tag{9}$$

subject to

$$u(x,0) = f(x), \qquad 0 \le x \le 1, \tag{10}$$

$$u(0,t) = r(t)g_0(t), \qquad 0 \le t \le T, \tag{11}$$

$$u(1,t) = r(t)g_1(t), \qquad 0 \le t \le T. \tag{12}$$

It can be easily seen from Eq. (8) that if $E(t) \neq 0$, then $r(t)$ can be obtained by the following formula

$$r(t) = \frac{u(x^*,t)}{E(t)}, \tag{13}$$

furthermore one can obtain the control parameter $p(t)$. With this transformation the problem to find $p(t)$ is turned into finding $r(t)$ and can be determined step by step by discretizing Eqs. (9)–(12) in the time domain. If $\tilde{u}(x,t)$ is an approximation of $u(x,t)$, according to Eq.(13), an approximation of $r(t)$ can be obtained as follows:

$$\tilde{r}(t) = \frac{\tilde{u}(x^*,t)}{E(t)}. \tag{14}$$

Then the original problem described by Eqs. (1)–(5) possesses an approximate solution

$$\tilde{w}(x,t) = \frac{\tilde{u}(x,t)}{\tilde{r}(t)}, \tag{15}$$

therefore

$$\tilde{p}(t) = -\frac{\tilde{r}'(t)}{\tilde{r}(t)} \tag{16}$$

is an approximation of $p(t)$.

3. Multiquadric quasi-interpolation and the construction of the numerical scheme

Multiquadric functions were first proposed by Hardy,[20] and were performed very well in many calculations including the numerical experiments reported by Franke in Ref. [21]. For the good performance of multiquadric functions, multiquadric quasi-interpolation is one of the most often applied approaches in modern approximation theory when the task is to approximate scattered data in one-dimensional problems. Multiquadric quasi-interpolation of a function $u : [x_0, x_N] \mapsto R$ with multiquadric functions on the scattered points

$$a = x_0 < x_1 < \ldots < x_N = b$$

and

$$h := \max_{1 \le j \le N}(x_j - x_{j-1})$$

usually takes the form

$$(\mathcal{L}u)(x) = \sum_{j=0}^{N} u(x_j)\psi_j(x),$$

where $\psi_j(x)$ are linear combinations of the multiquadrics

$$\phi_j(x) = \sqrt{c^2 + (x - x_j)^2},$$

in which c is a positive constant. Beatson and Powell[10] proposed three univariate multiquadric quasi-interpolation schemes, namely, $\mathcal{L}_\mathcal{A}$, $\mathcal{L}_\mathcal{B}$, and $\mathcal{L}_\mathcal{C}$, to approximate a function $\{f(x), x_0 \le x \le x_N\}$ from the space that is spanned by the multiquadric functions $\phi_j(x) = \sqrt{c^2 + (x - x_j)^2}$ and a linear function, where c is a positive constant and $\{x_j : j = 0, ..., N\}$ are given distinct points in the interval $[x_0, x_N]$. Afterward, Wu and Schaback[11] proposed the univariate multiquadric quasi-interpolation scheme $\mathcal{L}_\mathcal{D}$ on $[x_0, x_N]$ and proved that the scheme is shape preserving and convergent. However, in these papers, multiquadric quasi-interpolation was only discussed for the function itself, not its derivatives. To solve PDEs numerically, we construct a new multiquadric quasi-interpolation scheme, named $\mathcal{L}_\mathcal{E}$, which possesses the property

$$u''(x) \approx (\mathcal{L}_\mathcal{E}u)''(x).$$

Since $\sqrt{c^2 + x^2}$ tends to $|x|$ as c tends to zero, and multiquadric quasi-interpolation (as well as multiquadric interpolation) based on $|x|$ is piecewise linear interpolation, the convergence property of derivatives can be expected to hold for quasi-interpolation with a multiquadric function, too. Similar to Refs. [10] and [11],

this paper defines the normalized second divided difference as

$$\psi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}, \quad x \in R,$$

for every integer j. Given $\{(x_j, u(x_j))\}_{j=0}^{N}$, where $x_0 < x_1 < \ldots < x_N$, we construct the multiquadric quasi-interpolation $\mathcal{L}_\mathcal{E}$. We start by considering the following multiquadric quasi-interpolation scheme

$$(\mathcal{L}u)(x) = \sum_{j=-\infty}^{\infty} u(x_j)\psi_j(x), \quad x \in R. \tag{17}$$

Buhmann[22] studied the accuracy of the multiquadric quasi-interpolation operator $\mathcal{L}$ of a function $f : R \mapsto R$ on infinite regular points in R and showed that this scheme reproduces linear polynomials. However, the ability to reproduce linear polynomials does not require the centres $\{x_j\}$ to be equally spaced when there are infinitely many of them, say $\{x_j : j \in Z\}$ that extend to both ends of the real line. With the centres being in a strictly increasing sequence, Powell[23] proved that the multiquadric quasi-interpolation operator $\mathcal{L}$ satisfies the identity $(\mathcal{L}f)(x) = f(x)$ whenever f is a linear polynomial. The convergence estimates, however, need a more difficult analysis than the familiar ones from spline theory for instance, because compact support of the basis functions makes the proof techniques much simpler. Note the asymptotic decay at an algebraic rate of the basis functions: not radial basis functions but linear combinations of the translations; it is important to distinguish carefully between them. The basis function we use in the approximation is $\psi_j(x)$. Using this decay in tandem with polynomial recovery of a nontrivial order, one can prove the convergence (for more details cf Ref. [22]).

In application, the data always come from a finite interval $[x_0, x_N]$ but not the infinite domain R. To handle this, we introduce a polynomial $P(x)$ satisfying

$$\begin{aligned}
&P(x_0) = u(x_0), \quad P(x_N) = u(x_N),\\
&P(x_1) = u(x_1), \quad P(x_{N-1}) = u(x_{N-1}),\\
&P(x_2) = u(x_2), \quad P(x_{N-2}) = u(x_{N-2}),\\
&P(x_3) = u(x_3), \quad P(x_{N-3}) = u(x_{N-3}),\\
&P(x_4) = u(x_4), \quad P(x_{N-4}) = u(x_{N-4}).
\end{aligned}$$

With polynomial $P(x)$ in hand, we extend the function $f(x)$ from $[x_0, x_N]$ to domain $(-\infty, \infty)$

$$u^*(x) = \begin{cases} P(x), & \text{if } x \in (-\infty, x_0),\\ u(x), & \text{if } x \in [x_0, x_N],\\ P(x), & \text{if } x \in (x_N, \infty). \end{cases} \tag{18}$$

We apply the presented operator $\mathcal{L}$ to function $u^*(x) - P(x)$. Note that function $u^*(x) - P(x)$ is a compactly supported function with $[x_4, x_{N-4}]$ as its support. Moreover, $u^*(x) - P(x) = 0$ when $x = x_4, x_{N-4}$. Now we present our multiquadric quasi-interpolation scheme $(\mathcal{L}_\mathcal{E}u)(x)$ as follows:

$$(\mathcal{L}_\mathcal{E}u)(x) = \sum_{j=5}^{N-5} (u(x_j) - P(x_j)) \cdot \psi_j(x) + P(x)$$

with $x \in [x_0, x_N]$. The scheme $\mathcal{L}_\mathcal{E}$ possesses the property that $\mathcal{L}_\mathcal{E}u$ converges to u with a satisfactory approximation order. We should mention that with the introduced polynomial $P(x)$ terms $u(x_0)$, ..., $u(x_4)$, $u(x_N)$, $\cdots$, $u(x_{N-4})$ disappear, and their roles are represented implicitly by $P(x)$. Apparently, in application one can use the second derivative of $(\mathcal{L}_\mathcal{E}u)(x)$

$$(\mathcal{L}_\mathcal{E}u)''(x) = \sum_{j=5}^{N-5} (u(x_j) - P(x_j)) \cdot \psi_j{}''(x) + P''(x)$$

with $x \in [x_0, x_N]$, to approximate $u''(x)$.

Now, we present the numerical scheme for the parabolic equation with overspecified data by using the multiquadric quasi-interpolation scheme. In our methods, we use the derivative of the multiquadric quasi-interpolation to approximate the spatial derivative of the differential equations and employ a simple forward difference to approach the temporal derivative.

Discretizing equation

$$u_t(x,t) = u_{xx}(x,t) + r(t)\varphi(x,t)$$

in time, we obtain

$$u_k^{n+1} = u_k^n + \Delta t \cdot [(u_{xx})_k^n + r^n\varphi_k^n],$$

where u_k^n is the approximation of the value of $u(x,t)$ at point (x_k, t), $t = n\Delta t$; Δt is the time step. Because Eq. (9) is a developmental equation, it is reasonable to employ a finite difference method in the time domain. Then, we use the derivatives of the multiquadric quasi-interpolation to approximate u_{xx}. Thus, we have the following iterative formula

$$\begin{aligned}
u_k^{n+1} = u_k^n + \Delta t \cdot \Bigg[&\sum_{j=4}^{N-3} (u_j^n - P_n(x_j)) \cdot \psi_k{}''(x)\\
&+ P_n{}''(x_k) + r^n\varphi_k^n \Bigg].
\end{aligned}$$

With u_k^n one can compute $\tilde{r}(t_n)$ and w_k^n by using formulas (13) and (15). Let $N = 1/\Delta x + 1$ be the number of data centres in the space domain, with Δx the space length, and $M = T/\Delta t$ be the number of the steps, with Δt being the step length. Set $s = (\Delta x)^2/\Delta t$.

Since $p(t) = -r'(t)/r(t)$, we use the following numerical formula to approximate the control parameter $p(t)$:

$$r'(t_n) = (r(t_{n+1}) - r(t_{n-1}))/2\Delta t,$$
$$\text{for} \quad n = 1, 2, \cdots, N-1,$$
$$r'(t_0) = r'(t_1), \quad r'(t_N) = r'(t_{N-1}),$$
$$p(t_n) = -r'(t_n)/r(t_n), \quad \text{for} \quad n = 0, 1, \cdots, N.$$

4. Numerical example

To illustrate the numerical accuracy of the proposed multiquadric quasi-interpolation method for the parabolic problems with overspecified data (1)–(5), it is applied to an example having an analytical solution. Consider Eqs. (1)–(5) with

$$\varphi(x,t) = (\pi^2 - (t+1)^2)\exp(-t^2) \times(\cos(\pi x) + \sin(\pi x)),$$
$$g_0(t) = \exp(-t^2), \quad g_1(t) = -\exp(-t^2),$$
$$x^* = 0.25, \quad E(t) = \sqrt{2}\exp(-t^2),$$
$$f(x) = \cos(\pi x) + \sin(\pi x),$$

for which the exact solution is

$$w(x,t) = \exp(-t^2)(\cos(\pi x) + \sin(\pi x))$$

and

$$p(t) = 1 + t^2.$$

In our computations, we assume $s = 4$ and choose $N = 21, 41, 81$ respectively.

The results obtained for w_j^N at $T = 1.0$ using the multiquadric quasi-interpolation method are shown in Table 1.

Table 1. Error between the exact solution and the numerical one obtained using the multiquadric quasi-interpolation method, with $c = 0.1(\Delta x)^{1/3}$, $T = 1$ and $N = 21, 41, 81$ respectively.

x	error$_{(N=21)}$	error$_{(N=41)}$	error$_{(N=81)}$
0.10	6.2259×10^{-6}	6.0230×10^{-6}	2.3067×10^{-6}
0.20	1.3385×10^{-5}	6.1218×10^{-6}	2.1726×10^{-6}
0.30	2.3231×10^{-5}	8.9086×10^{-6}	3.0929×10^{-6}
0.40	7.1178×10^{-5}	2.6836×10^{-5}	9.2235×10^{-6}
0.50	8.8479×10^{-5}	3.3061×10^{-5}	1.1302×10^{-5}
0.60	6.2033×10^{-5}	2.3221×10^{-5}	7.8969×10^{-6}
0.70	1.4981×10^{-5}	5.4907×10^{-6}	1.7838×10^{-6}
0.80	1.3701×10^{-5}	6.2995×10^{-6}	2.3410×10^{-6}
0.90	5.3568×10^{-6}	4.8033×10^{-6}	1.8896×10^{-6}

The results obtained for p^n using the multiquadric quasi-interpolation method are listed in Table 2.

From the tables, we conclude that the methods are feasible and valid.

Table 2. Error between the exact solution and the numerical one obtained using the multiquadric quasi-interpolation method, with $c = 0.1(\Delta x)^{1/3}$, $T = 1$ and $N = 21, 41, 81$ respectively.

t	error$_{(N=21)}$	error$_{(N=41)}$	error$_{(N=81)}$
0.10	1.6360×10^{-4}	5.8891×10^{-5}	2.9942×10^{-5}
0.20	1.5605×10^{-4}	5.1104×10^{-5}	2.5088×10^{-5}
0.30	3.0558×10^{-4}	8.5728×10^{-5}	3.2822×10^{-5}
0.40	5.2824×10^{-4}	1.3867×10^{-4}	4.5302×10^{-5}
0.50	8.2386×10^{-4}	2.0962×10^{-4}	6.2243×10^{-5}
0.60	1.2020×10^{-4}	3.0094×10^{-4}	8.4212×10^{-5}
0.70	1.6741×10^{-3}	4.1547×10^{-4}	1.1191×10^{-4}
0.80	2.2524×10^{-3}	5.5626×10^{-4}	1.4610×10^{-4}
0.90	2.9500×10^{-3}	7.2653×10^{-4}	1.8757×10^{-4}
1.00	2.5253×10^{-3}	6.1688×10^{-4}	1.5905×10^{-4}

5. Concluding remarks

In this paper, the multiquadric quasi-interpolation method is used for solving a parabolic equation with overspecified data. The meshless property of the multiquadric quasi-interpolation method is the most important advantage of this scheme over traditional mesh-dependent techniques such as finite difference, finite element and boundary element methods. At the same time, it still possesses high accuracy. Compared with the traditional meshless collocation method which interpolates the collocation nodes, the quasi-interpolation method does not have to solve a large scale linear system of equations. Furthermore, we do not meet the problem of the ill-condition of the matrices.

It is a very difficult problem to deal with a boundary in a quasi-interpolation scheme. We handle this problem by introducing a polynomial, which is shown to be an effective technique in our multiquadric quasi-interpolation scheme. From numerical examples, all in all, we conclude that our scheme gives good accuracy.

The techniques can be used for non-equidistant grids, for convenience, although we are using equidistant grids in our numerical experiments. We see that the present techniques require the calculation of

$$\psi_j{}''(x) = \frac{\phi''_{j+1}(x) - \phi_j{}''(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j{}''(x) - \phi''_{j-1}(x)}{2(x_j - x_{j-1})}$$

$j = 5, \ldots, N-5$ once and for all.

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