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Analytical solutions for the slow neutron capture process of heavy element nucleosynthesis*

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In this paper, the network equation for the slow neutron capture process (s-process) of heavy element nucleosynthesis is investigated. Dividing the s-process network reaction chains into two standard forms and using the technique of matrix decomposition, a group of analytical solutions for the network equation are obtained. With the analytical solutions, a calculation for heavy element abundance of the solar system is carried out and the results are in good agreement with the astrophysical measurements.

Keywords: s-process, network equation, analytical solution, element abundance
PACC: 9880F, 9530C, 2540L

1. Introduction

The nucleosynthesis of heavy elements is an important problem in nuclear astrophysics. Here, the heavy element refers to the element beyond the iron group.

According to the theory of Primordial Nucleosynthesis, only hydrogen, helium and a few light isotopes with $A < 12$ are produced in the Big Bang. These isotopes are stepwise converted to heavier elements by the energy-generating fusion reactions in stars. In successive stages of stellar evolution a sequence of burning processes converts H into He, He into C, and so on until the isotopes around iron are created in Si burning. Beyond the iron group, the fusion reaction of charged particles quickly ceases to contribute to the heavy element because of the increasing coulomb barriers and the decreasing binding energies per nucleon.[1]

In 1957, Burbidge *et al*[2] published their famous paper to provide a physical mechanism for the heavy element nucleosynthesis. In their pioneering investigation, Burbidge *et al* pointed out that neutron capture reactions played a main role in the heavy element nucleosythesis. In correspondence with different astrophysical sites, there are two different processes of neutron capture, i.e. the rapid neutron capture process (r-process) for the explosive astronomical site such as supernova and the slow neutron capture process (s-process) for helium burning in the red-giant stage of stellar evolution. In both the r-process and the s-process, heavy elements are produced by seed nuclei to capture neutrons. Here, the seed nuclei usually refer to the iron group, for instance, ^{56}Fe, Co, Ni *et al*.

Because it contributes to approximately 50% of the heavy elements of the solar system, the s-process has received more and more attention and made much progress[3.4] since Burgidge *et al* came out with their research. Clayton *et al*[5] employed the Laplace transformation method to obtain the solutions of network equations of the classical s-process in which it is supposed that in the s-process path, if some nuclei are unstable, the beta decay always takes place before the neutron capture. Subsequently, Seeger *et al*[6] introduced the exponential distribution of neutron exposure to fit the $\sigma_A N_A$-curve of solar s-nuclei.

Taking into account the branching ratio between beta decay and neutron capture, Ward *et al*[7] brought in the network equations and obtained the solutions for the branching s-process. The subsequent calculations of star evolution indicated that the neutron sources came forth in the pulse form, so Ward and Newman[8] investigated the single pulse s-process and gave the solutions for network equations. Another result[9] of star evolution showed that there were two neutron sources for the s-process, ^{13}C(α, n) ^{16}O and ^{22}Ne(α, n)^{25}Mg, and Beer[10] found out the solutions of the double pulse s-process.

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To understand the s-process of nucleosynthesis, especially the effects of different physical parameters on nucleosynthesis, analytical solutions of the s-process are advantageous and it is also convenient to calculate the abundance of s-process heavy elements with analytical solutions. For the branching s-process, however, the Laplace transformation method is impracticable, because the expression of the Laplace transformation is too intricate to obtain the analytical formulae of Laplace inverse transformation. So Ward *et al* used the matrix method to solve the branching network equation. But many trivial physical processes were taken into account in their investigation, for example, electron-capture, positron-emission, neutron capture and beta decay *et al*, thereby rendering the branching process main matrix of s-process network equations too intricate; therefore, they could not give every item in an expression of the analytic solution.

Relying on the up-to-date nuclear data,[11] however, we find out that there are only two branching point nuclei in the s-process path: ^{152}Eu and ^{204}Tl which can induce the intricacy of the branching s-process as a result of the coexistence of various physical processes. As for the other branching points, the main physical processes are neutron capture and beta decay. (A detailed and comprehensive analysis will be given in sub-section 3.2.) So, in the present paper, only neutron capture and beta decay are considered in the branching point of the s-process. Thus the main matrix of the branching s-process network equation will become simple and easy to solve analytically.

The network equation of the s-process is introduced in Section 2. In Section 3, the network equation of the branching s-process is divided into two standard forms and solved separately. In this section, we only comsider the most important factors – neutron capture and beta decay so that the main matrix of the network equation becomes simple. As an application of our analytical solutions, the solar system heavy element abundance is computed in Section 4. A summary is presented in Section 5.

2. Network equation

Each nucleus in the s-process reaction chain is created and destroyed by neutron captures and/or by weak decays. In the path of the s-process, if a nucleus is unstable, then it will keep on capturing neutrons with capturing time scale τ_n or it will perform beta decay with an average life time τ_β. In the classical s-process it is supposed that $\tau_\beta \ll \tau_n$, which means that the beta decay always takes place prior to the neutron capture except for some long-lifetime nuclei. So the differential equations for the classical s-process can be written as

$$\frac{\mathrm{d}N_A}{\mathrm{d}t} = \langle\sigma v\rangle_{A-1} n_n(t) N_{A-1} - \langle\sigma v\rangle_A n_n(t) N_A, \quad (1)$$

where N_A is the abundance of the unique isobar of atomic weight A on the s-process path. The time dependence enters explicitly through the free-neutron density $n_n(t)$ and through the temperature dependence of $\langle\sigma v\rangle$.

One idealization of the clear physical interest, however, is that of a constant- temperature s-process. Generally speaking, a given neutron source turns itself on at a certain temperature and exhausts itself at nearly the same temperature. Even if the temperature is not constant, moreover, the solution at a constant temperature is obviously a first approximation to a more general problem. A consequence of this assumption is that the weak rates are also constant (at values determined by T). For the unique chain, where the equation is applicable, Clayton *et al* showed the change of variable from time t to irradiation τ (or exposure)

$$\tau = \int_0^t n_n(t) v_T \mathrm{d}t. \quad (2)$$

Without any assumption about $n_n(t)$, it is convenient to reduce Eq.(1) into

$$\frac{\mathrm{d}N_A}{\mathrm{d}\tau} = \sigma_{A-1} N_{A-1} - \sigma_A N_A. \quad (3)$$

The solutions and their approximate forms have been discussed by Clayton *et al*,[4] Clayton and Newman,[12] and Clayton and Ward.[13] This simplification is impossible unfortunately at the branching point of the s-process, because the equation includes not only the neutron numerical density but also the beta decay rate. The neutron capture rate depends explicitly on $n_n(t)$ whereas the beta decay rate does not. Therefore time enters in an essential way as an independent variable. With regard to the general form for $n_n(t)$, moreover, the equations of the branching point of the s-process can be solved only by numerical integration. But just as we know, the s-process generally takes place in the He-burning shell of AGB stars. In the He-burning shell, He burns calmly and the temperature and the neutron numerical density $n_n(t)$ almost remain immovable. Letting the branching point nucleus be (Z, A), it becomes $(Z, A+1)$ by

neutron capture and becomes $(Z+1,\ A)$ by beta decay. With the approximate condition of temperature and neutron numerical density $n_n(t)$ constants, the network equation can be written as

$$\frac{\mathrm{d}N_A}{\mathrm{d}\tau} = \sigma_{A-1}N_{A-1} - (\sigma_A N_A + \lambda^- N_A), \qquad (4)$$

where λ^- is the beta decay rate.

3. Solutions of network equations

Now, we divide the network equations of the branching s-process into two standard forms: the no branching chain and the branching chain.

3.1. No branching chain

Let the standard no branching chain start from nucleus A_{k+1} and stop at A_{k+m}, i.e.

$$A_k + n \to A_{k+1} + n \to A_{k+2} + n \to \ldots$$
$$\to A_{k+m-1} + n \to A_{k+m}.$$

Using symbol N_{k+i} to denote the abundance of nucleus A_{k+i}, the no branching network equations can be written as

$$\begin{aligned}
&\frac{\mathrm{d}N_{k+1}}{\mathrm{d}\tau} = -\sigma_{k+1}N_{k+1} + \sigma_k N_k,\\
&\frac{\mathrm{d}N_{k+2}}{\mathrm{d}\tau} = -\sigma_{k+2}N_{k+2} + \sigma_{k+1}N_{k+1},\\
&\vdots\\
&\frac{\mathrm{d}N_{k+m}}{\mathrm{d}\tau} = -\sigma_{k+m}N_{k+m} + \sigma_{k+m-1}N_{k+m-1}.
\end{aligned} \qquad (5)$$

Let

$$N = \begin{bmatrix} N_{k+1}\\ N_{k+2}\\ \vdots\\ N_{k+m}\end{bmatrix},\ \mathrm{b} = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \qquad (6)$$

$$A = \begin{bmatrix}
-\sigma_{k+1} & & & & 0\\
\sigma_{k+1} & -\sigma_{k+2} & & & \\
0 & \sigma_{k+2} & -\sigma_{k+3} & & \\
\vdots & \vdots & \vdots & \ddots & \\
0 & 0 & 0 & \sigma_{k+m-1} & -\sigma_{k+m}
\end{bmatrix}. \qquad (7)$$

We refer to the matrix A as the main matrix of Eq.(5). Equation (5) can be written in the following form:

$$\frac{\mathrm{d}N}{\mathrm{d}\tau} = AN + \sigma_k \cdot N_k \cdot b. \qquad (8)$$

The eigenvalues of the matrix A are $-\sigma_{k+1}$,$-\sigma_{k+2}$, ..., $-\sigma_{k+m}$. Now, we compute the eigenvectors. The eigenequation of A is

$$(\lambda I - A)\alpha = \theta, \qquad (9)$$

where θ is the null vector and α is the eigenvector whose components are denoted by $x_1, x_2, \ldots, x_m$. While $\lambda = -\sigma_{k+1}$, Eq.(9) is

$$\begin{bmatrix}
0 & 0 & & & 0\\
-\sigma_{k+1} & \sigma_{k+2}-\sigma_{k+1} & & & \\
0 & -\sigma_{k+2} & \sigma_{k+3}-\sigma_{k+1} & & \\
\vdots & \vdots & \vdots & \ddots & \\
0 & 0 & 0 & -\sigma_{k+m-1} & \sigma_{k+m}-\sigma_{k+1}
\end{bmatrix} \cdot \begin{bmatrix}x_1\\x_2\\x_3\\\vdots\\x_m\end{bmatrix} = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} \qquad (10)$$

$$\begin{aligned}
&0 = 0,\\
&-\sigma_{k+1}x_1 + (\sigma_{k+2}-\sigma_{k+1})x_2 = 0,\\
&-\sigma_{k+2}x_2 + (\sigma_{k+3}-\sigma_{k+1})x_3 = 0,\\
&\vdots\\
&-\sigma_{k+m+1}x_{m-1} + (\sigma_{k+m}-\sigma_{k+1})x_m = 0.
\end{aligned} \qquad (11)$$

It is easy to obtain

$$x_1 = 1, \quad x_2 = \frac{\sigma_{k+1}}{\sigma_{k+2} - \sigma_{k+1}}, \quad x_3 = \prod_{i=2}^{3} \frac{\sigma_{k+i-1}}{\sigma_{k+i} - \sigma_{k+1}}, \ldots, \ x_m = \prod_{i=2}^{m} \frac{\sigma_{k+i-1}}{\sigma_{k+i} - \sigma_{k+1}}. \tag{12}$$

When $\lambda = -\sigma_{k+i}$, the eigenequation is

$$\begin{bmatrix} \sigma_{k+1} - \sigma_{k+i} & 0 & & & 0 \\ -\sigma_{k+1} & \sigma_{k+2} - \sigma_{k+i} & & & \\ & \ddots & & & \\ & & -\sigma_{k+i} & \sigma_{k+i-1} - \sigma_{k+i} & \\ & & & \ddots & \\ 0 & & & & -\sigma_{k+m-1} \ \ \sigma_{k+m} - \sigma_{k+i} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{i+1} \\ \vdots \\ x_m \end{bmatrix} \tag{13}$$

$$\begin{aligned} &(\sigma_{k+1} - \sigma_{k+i})x_1 = 0, \\ &-\sigma_{k+1}x_1 + (\sigma_{k+2} - \sigma_{k+i})x_2 = 0, \\ &\vdots \\ &-\sigma_{k+i-2}x_{i-2} + (\sigma_{k+i-1} - \sigma_{k+i})x_{i-1} = 0, \\ &-\sigma_{k+i-1}x_{i-1} = 0, \\ &-\sigma_{k+i}x_i + (\sigma_{k+i+1} - \sigma_{k+i})x_{i+1} = 0, \\ &\vdots \\ &-\sigma_{k+m-1}x_{m-1} + (\sigma_{k+m} - \sigma_{k+i})x_m = 0, \end{aligned} \tag{14}$$

$x_1 = x_2 = \ldots = x_{i-1} = 0$, x_i is independent, so we take $x_i = 1$, then

$$\begin{aligned} x_{i+1} &= \frac{\sigma_{k+1}}{\sigma_{k+i+1} - \sigma_{k+i}}, \\ x_{i+2} &= \frac{\sigma_{k+i+1}}{\sigma_{k+i+2} - \sigma_{k+i}} \cdot x_{i+1} \\ &= \frac{\sigma_{k+i+1}}{\sigma_{k+i+2} - \sigma_{k+i}} \cdot \frac{\sigma_{k+1}}{\sigma_{k+i+1} - \sigma_{k+i}}, \\ &\vdots \end{aligned}$$

$$\begin{aligned} x_m &= \prod_{j=0}^{m-i-1} \frac{\sigma_{k+i+j}}{\sigma_{k+i+j+1} - \sigma_{k+i}} \\ &= \prod_{j=i+1}^{m} \frac{\sigma_{k+j-1}}{\sigma_{k+j} - \sigma_{k+i}}. \end{aligned} \tag{15}$$

Now, matrix A can be reduced into

$$A = B \begin{bmatrix} \sigma_{k+1} & & & 0 \\ & -\sigma_{k+2} & & \\ & & \ddots & \\ 0 & & & -\sigma_{k+m} \end{bmatrix} B^{-1}, \tag{16}$$

$$B = \begin{bmatrix} \alpha_{k+1} \ \alpha_{k+2} \ \ldots \ \alpha_{k+m} \end{bmatrix}, \tag{17}$$

where $\alpha_{k+i} (i = 1, 2, \ldots, m)$ is the eigenvector of $A\alpha_{k+i} = -\sigma_{k+i}\alpha_{k+i}$. It is easy to obtain the form of matrix B

$$B = \begin{bmatrix} 1 & \ldots & \ldots & \ldots \ \ldots & 0 \\ \dfrac{\sigma_{k+1}}{\sigma_{k+2} - \sigma_{k+1}} & \ddots & & & \\ \prod\limits_{j=2}^{3} \dfrac{\sigma_{k+j-1}}{\sigma_j - \sigma_{k+1}} & \ddots & 1 & & \\ \vdots & & \dfrac{\sigma_{k+i}}{\sigma_{k+i+1} - \sigma_{k+i}} & \ldots & \\ \vdots & & & \ddots & \\ \prod\limits_{i=2}^{m} \dfrac{\sigma_{k+i-1}}{\sigma_{k+i} - \sigma_{k+1}} & & \prod\limits_{j=i+1}^{m} \dfrac{\sigma_{k+j-1}}{\sigma_{k+j} - \sigma_{k+i}} & \ldots & 1 \end{bmatrix}. \tag{18}$$

Let the inverse of B be

$$B^{-1}=\begin{bmatrix} 1 & & & \\ a_2 & 1 & & \\ a_3 & \vdots & \dots & \\ & \vdots & \dots & \\ a_m & \vdots & \dots & 1 \end{bmatrix},$$

and Eq.(8) be changed into

$$\frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}\tau}=B\mathrm{diag}(-\sigma_{k+1},-\sigma_{k+2},\dots,-\sigma_{k+m})B^{-1}\boldsymbol{N}+\sigma_k N_k\boldsymbol{b}. \tag{19}$$

Let $B^{-1}\boldsymbol{N}=\psi$, $\boldsymbol{b}^*=B^{-1}\boldsymbol{b}$, the equation becomes

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau}=\begin{bmatrix} -\sigma_{k+1} & & & & \\ & \ddots & & & \\ & & -\sigma_{k+i} & & \\ & & & \ddots & \\ & & & & -\sigma_{k+m} \end{bmatrix}\psi+\sigma_k N_k\boldsymbol{b}^*,\quad \psi=\begin{bmatrix}\psi_1\\ \vdots\\ \psi_i\\ \vdots\\ \psi_m\end{bmatrix}. \tag{20}$$

Because of $\boldsymbol{b}^*=B^{-1}\boldsymbol{b}$, $\boldsymbol{b}=[\,1\ 0\ \dots\ 0\,]^{\mathrm{T}}$, the result of $B^{-1}\boldsymbol{b}$ is just the first column of matrix B^{-1}. Let $\sigma_k N_k=f_k$ and we can have

$$\sigma_k N_k\boldsymbol{b}=[f_k,a_2f_k,a_3f_k,...,a_mf_k]\,. \tag{21}$$

Then we can obtain the differential equation for ψ_i

$$\frac{\mathrm{d}\psi_i}{\mathrm{d}\tau}=-\sigma_{k+i}\psi_i+a_if_k(\tau). \tag{22}$$

With the variation of the constants method, we can acquire

$$\psi_i=\mathrm{e}^{-\sigma_{k+i}\tau}\cdot\int_0^{\tau}\mathrm{e}^{\sigma_{k+i}\tau'}\cdot a_i\cdot\sigma_k N_k(\tau')\mathrm{d}\tau'. \tag{23}$$

In the same way, we have

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}\tau}=-\sigma_{k+1}\psi_1+f_k(\tau),$$

$$\text{thus}\quad \psi_1=\mathrm{e}^{-\sigma_{k+1}\tau}\cdot\int_0^{\tau}\mathrm{e}^{\sigma_{k+i}\tau'}\cdot\sigma_k N_k(\tau')\mathrm{d}\tau'. \tag{24}$$

Because of $\boldsymbol{N}=B\psi$, it is easy to obtain

$$\begin{bmatrix}N_{k+1}\\N_{k+2}\\N_{k+3}\\\vdots\\\vdots\\N_{k+m}\end{bmatrix}=\begin{bmatrix} 1 & \dots & \dots & \dots & 0\\ \dfrac{\sigma_{k+1}}{\sigma_{k+2}-\sigma_{k+1}} & \ddots & \vdots & & \\ \prod\limits_{j=2}^{3}\dfrac{\sigma_{k+j-1}}{\sigma_{k+j}-\sigma_{k+1}} & \ddots & 1 & & \vdots\\ & & \dfrac{\sigma_{k+i}}{\sigma_{k+i+1}-\sigma_{k+i}} & & \vdots\\ & & \vdots & \ddots & \\ \prod\limits_{i=2}^{m}\dfrac{\sigma_{k+i-1}}{\sigma_{k+i}-\sigma_{k+1}} & & \prod\limits_{j=i+1}^{m}\dfrac{\sigma_{k+j-1}}{\sigma_{k+j}-\sigma_{k+i}} & \dots & 1 \end{bmatrix}\cdot\begin{bmatrix}\psi_1\\\psi_2\\\psi_3\\\vdots\\\vdots\\\psi_m\end{bmatrix} \tag{25}$$

$$\text{thus } N_{k+1}=\psi_1, \tag{26}$$

$$N_{k+i}=\psi_i+\sum_{\upsilon=2}^{i}\left(\prod_{j=\upsilon}^{i}\frac{\sigma_{k+j-1}}{\sigma_{k+j}-\sigma_{k+\upsilon-1}}\right)\psi_{\upsilon-1}. \tag{27}$$

3.2. Branching chain

Let the branching chain start from the unstable nuclide A_1, which means that there are other physical processes (they include electron-capture, positron-emission, beta decay *et al* which were taken into account by Ward *et al*[6]) in competition with neutron capture. As an example where the branching chain at ^{84}Kr is in accordance with the result given by Ward *et al*,[6] these processes are shown in Fig.1.

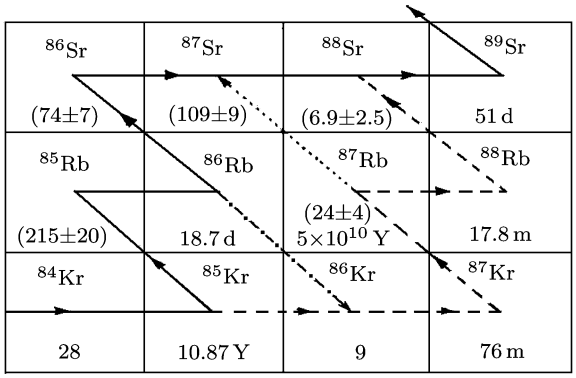


Fig.1. Branching s-process path through the isotopes of Kr, Rb and Sr. The arrows denote the possible multifarious flows leading from ^{85}Kr to ^{88}Sr. The dash dot arrow at ^{86}Rb indicates the positron-emission. The neutron capture cross section (in mb) and half-lives of nuclides are written below the nuclide.

Because the positron-emission of ^{86}Rb was considered by Ward *et al*, the branching path becomes complicated. A direct effect is the complex main matrix A.

According to the up-to-date nuclear data,[11] the half-life of the unstable nuclide ^{86}Rb is 18.63 days and there are two decay models, i.e. beta decay and positron-emission. Because the branching ratio of positron-emission is 0.0052%,[11] it can be ignored and the branching chain will be simplified. In fact, if a branching point nuclide brings on branching chain intricacy, it must be unstable and decaying with multiplicate models. Checking the latest nuclear data,[11] we find only 10 candidates in the s-process path (see Table 1).

Table 1. Intricate branching point of the s-process.

nuclide	decay model and branching ratio,	half-life
^{64}Cu	$\varepsilon+\beta^+(61\%)$, $\beta^-(39\%)$,	12.7 h
^{70}Ga	$\varepsilon(0.41\%)$, $\beta^-(99.59\%)$,	21.14 m
^{104}Rh	$\varepsilon+\beta^+(0.45\%)$, $\beta^-(99.54\%)$,	42.3 s
^{110}Ag	$\varepsilon(0.3\%)$, $\beta^-(99.7\%)$,	24.6 s
^{122}Sb	$\varepsilon+\beta^+(2.4\%)$, $\beta^-(97.6\%)$,	2.724 d
^{128}I	$\varepsilon+\beta^+(6.9\%)$, $\beta^-(93.1\%)$,	24.99 m
^{152}Eu	$\varepsilon+\beta^+(72.08\%)$, $\beta^-(27.92\%)$,	13.537 y
^{186}Re	$\varepsilon(7.47\%)$, $\beta^-(92.53\%)$,	3.7182 d
^{192}Ir	$\varepsilon+\beta^+(4.87\%)$, $\beta^-(95.13\%)$,	73.827 d
^{204}Tl	$\varepsilon+\beta^+(2.9\%)$, $\beta^-(97.1\%)$,	3.78 y

Here, s,m,h,d,y denote second, minute, hour, day and year respectively.

As the table shows, only two nuclides, ^{152}Eu and ^{204}Tl can induce the complexity of the s-process branching chain. The half-lives of the others, however, are too short or there are too many obvious differences between the branching ratios of the different decay models. So we think over the following standard branching chain which includes the most cases of s-process branching.

Let nuclide A_1 be unstable, $\tau_n \sim O(\tau_\beta)$, so the neutron capture and decay take place synchronously. We suppose that A_1 captures the neutron to create A_2 and decays into B_2, and then A_2 and B_2 keep on capturing neutrons separately. Because A_2 departs from the beta stable line, it will be unstable after a series of neutron captures and will enter into the stable line by means of beta decay. See the following sketch:

$$A_0+n\to A_1\begin{array}{c}\overset{\beta\text{ decay}}{\nearrow}\\ \underset{+n}{\searrow}\end{array}\begin{array}{l}B_2+n\to B_3+n\to\ldots\to B_n+n\searrow\\ A_2+n\to A_3+n\to\ldots\to A_n+n\nearrow\end{array}A_{n+1}.$$

Here, A_1 is the branching point and A_{n+1} is the convergence point. The network equation at branching point A_1 is

$$\frac{\mathrm{d}X_1}{\mathrm{d}\tau}=-(\sigma_1+\sigma_1^*)X_1+\sigma_0X_0, \tag{28}$$

where, σ_1 and σ_1^* denote the neutron capture cross section and the beta decay rate of A_1 respectively. σ_0 is the neutron capture cross section of A_0. It is easy to obtain

$$X_1 = \mathrm{e}^{-(\sigma_1+\sigma_1^*)\tau} \cdot \int_0^\tau \sigma_0 \mathrm{e}^{(\sigma_1+\sigma_1^*)\tau'} X_0(\tau')\mathrm{d}\tau'. \quad (29)$$

Let σ_i (σ_i^*) and $X_i(Y_i$) be the neutron capture cross section and the abundance of nucleus A_i (B_i) respectively. The branching chain can be considered as two no branching chains from B_2 and A_2 separately and the solutions can be written as the following formulae with the solutions of no branching chains (26) and (27).

$$\psi_1 = \mathrm{e}^{-\sigma_2\tau} \cdot \int_0^\tau \mathrm{e}^{\sigma_2\tau'} \cdot \sigma_1 X_1(\tau')\mathrm{d}\tau'. \quad (30)$$

$$\psi_i = \mathrm{e}^{-\sigma_{i+1}\tau} \cdot \int_0^\tau \mathrm{e}^{\sigma_{i+1}\tau'} \cdot \left(-\prod_{j=2}^{i} \frac{\sigma_j}{\sigma_{j+1}-\sigma_2} \right) \times \sigma_1 X_1(\tau')\mathrm{d}\tau'. \quad (31)$$

$$X_2 = \psi_1, \quad (32)$$

$$X_{i+1} = \psi_i + \sum_{\upsilon=2}^{i} \left(\prod_{j=\upsilon}^{i} \frac{\sigma_j}{\sigma_{j+1}-\sigma_\upsilon} \right) \psi_{\upsilon-1} \quad (i = 2, 3, \dots), \quad (33)$$

$$\Psi_1 = \mathrm{e}^{-\sigma_2^*\tau} \cdot \int_0^\tau \mathrm{e}^{\sigma_2^*\tau'} \cdot \sigma_1^* X_1(\tau')\mathrm{d}\tau', \quad (34)$$

$$\Psi_i = \mathrm{e}^{-\sigma_{i+1}^*\tau} \cdot \int_0^\tau \mathrm{e}^{\sigma_{i+1}^*\tau'} \cdot \left(-\prod_{j=2}^{i} \frac{\sigma_j^*}{\sigma_{j+1}^*-\sigma_2^*} \right) \times \sigma_1^* X_1(\tau')\mathrm{d}\tau'. \quad (35)$$

$$X_2 = \Psi_1, \quad (36)$$

$$X_{i+1} = \Psi_i + \sum_{\upsilon=2}^{i} \left(\prod_{j=\upsilon}^{i} \frac{\sigma_j^*}{\sigma_{j+1}^*-\sigma_\upsilon^*} \right) \Psi_{\upsilon-1} \quad (i = 2, 3, \dots). \quad (37)$$

The abundance of the last element A_{n+1} satisfies

$$\frac{\mathrm{d}X_{n+1}}{\mathrm{d}\tau} = -\sigma_{n+1}X_{n+1} + \sigma_n X_n + \sigma_n^* Y_n,$$

so we can obtain the expression of abundance X_{n+1} as

$$X_{n+1} = \mathrm{e}^{-\sigma_{n+1}\tau} \cdot \int_0^\tau \mathrm{e}^{\sigma_{n+1}\tau'} \cdot \sigma_n X_n(\tau')\mathrm{d}\tau' + \mathrm{e}^{-\sigma_{n+1}\tau} \times \int_0^\tau \mathrm{e}^{\sigma_{n+1}\tau'} \cdot \sigma_n^* Y_n(\tau')\mathrm{d}\tau'.$$

4. Heavy element abundance in the solar system

As an application, we compute the heavy element abundance of the no branching s-process of the solar system with the analytic solutions. It is known that the distribution of the s-process abundances observed in the solar system is a superposition of different numbers of seed nuclei ρ (τ) exposed to irradiation for τ per interval $\mathrm{d}\tau$ since Clayton *et al* published their results in 1961[5]

$$(\sigma N)_i = \int_0^\infty \rho(\tau) N_i(\tau)\mathrm{d}\tau. \quad (38)$$

In their paper, $\rho(\tau)$ proved to be a decreasing function of τ. Subsequently, Seeger, Fowler and Clayton[6] pointed out that ρ (τ) might be approximated well by an exponential distribution

$$\rho(\tau) = G\exp(-\tau/\tau_0). \quad (39)$$

Now we suppose that the reaction chain starts from ^{56}Fe and ends at ^{209}Bi . Let the initial value of ^{56}Fe be N_0, then the abundance of ^{56}Fe will be

$$N_{56} = N_0\mathrm{e}^{-\sigma_{56}\tau}. \quad (40)$$

Using formula (23) and (24), we have

$$\begin{aligned}(\sigma\psi)_1 &= \int_0^\infty \rho(\tau)\psi_1(\tau)\mathrm{d}\tau \\ &= \int_0^\infty \rho(\tau)\mathrm{e}^{-\sigma_{57}\tau} \int_0^\tau \sigma_{56}N_0\mathrm{e}^{(\sigma_{57}-\sigma_{56})\tau'}\mathrm{d}\tau'\mathrm{d}\tau \\ &= \frac{\sigma_{56}N_0\tau_0}{(\tau_0\sigma_{57}+1)(\tau_0\sigma_{56}+1)}, \end{aligned} \quad (41)$$

$$\begin{aligned}(\sigma\psi)_i &= \int_0^\infty \rho(\tau)\psi_i(\tau)\mathrm{d}\tau \\ &= \int_0^\infty \rho(\tau)\mathrm{e}^{-\sigma_{56+i}\tau} \\ &\quad \times \int_0^\tau \sigma_{56}N_0 \cdot a_i \cdot \mathrm{e}^{(\sigma_{56+i}-\sigma_{56})\tau'}\mathrm{d}\tau'\mathrm{d}\tau \\ &= \frac{\sigma_{56}N_0\tau_0}{(\tau_0\sigma_{56+i}+1)(\tau_0\sigma_{56}+1)} \cdot a_i, \end{aligned} \quad (42)$$

$$(\sigma N)_{57} = (\sigma\psi)_1, \quad (43)$$

$$\begin{aligned}&(\sigma N)_{56+i} \\ &= (\sigma\psi)_i + \sum_{\upsilon=2}^{i} \left(\prod_{j=\upsilon}^{i} \frac{\sigma_{k+j-1}}{\sigma_{k+j}-\sigma_{k+\upsilon-1}} \right) (\sigma\psi)_{\upsilon-1}. \end{aligned} \quad (44)$$

During calculation, following the way by Almeida *et al*[15] and Käppeler *et al*,[16] we take the neutron irradiation distribution $\rho(\tau)$ as the superposition of two different irradiations

$$\begin{aligned}\rho(\tau) &= \rho_1(\tau) + \rho_2(\tau) \\ &= f_1 \cdot G_1 \cdot \mathrm{e}^{-\tau/\tau_{10}} + f_2 \cdot G_2 \cdot \mathrm{e}^{-\tau/\tau_{20}},\end{aligned} \tag{45}$$

where these parameters are

$$\begin{aligned}&f_1 = 2.7, \quad f_2 = 0.092, \\ &\tau_{10} = 0.056(\mathrm{mb}^{-1}), \quad \tau_{20} = 0.24(\mathrm{mb}^{-1}), \\ &G_i = N_0/\tau_{i0} \quad (i = 1, 2).\end{aligned}$$

The cross sections have been cited from Refs.[14, 17]. The initial abundance N_0 of ^{56}Fe is the observed solar abundance, $8.25\times10^5/10^6$Si (See Ref.[18]). The calculation results are shown in Fig.2.

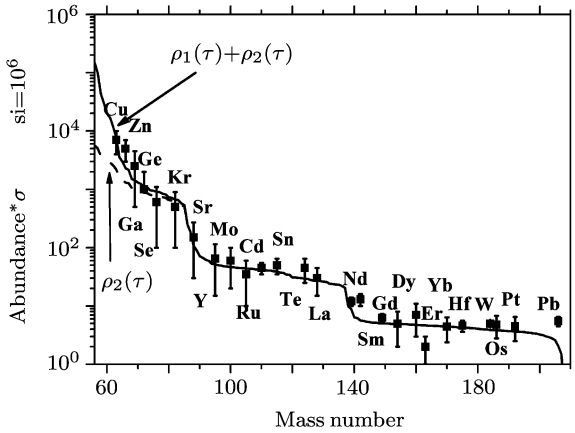


Fig.2. Product of neutron capture cross section (30 keV in mb) times solar system abundance (relative to Si=10^6) versus atomic mass for nucleus produced in the s-process. The dash line and the solid line represent theoretical calculations from a single exponential distribution ρ_2 (τ) (where τ is the neutron exposure time) and from the sum of two such distributions ρ_1 (τ)+ρ_2 (τ), respectively. The square with error bar denotes the astronomical measure.

5. Summary

As basic tools, network equations play an important role in the s-process. Generally, there are several hundreds of nuclei that take part in the s-process network reaction. When using the numerical method, several hundreds of differential equations have to be solved. While the network equations of the s-process are solved, it is necessary to superpose the differential equations several hundred times. Because of the accumulation of errors in the superposition, it is possible to increase the errors of numerical solutions. On the contrary, the analytic solutions are accurate. Moreover, just as mentioned above, the analytic solution of the s-process is important to understanding the s-process nucleosynthesis and the effects of different physical parameters on nucleosynthesis. In the preceding investigations, with the Laplace transform, Clayton *et al* gave the analytic solutions for the no branching s-process in 1961.[5] But at the branching point, the Laplace transform is too complicated to compute. In their research of the branching s-process in 1976, Ward *et al*[7] gave the analytic solutions for the s-process network equations with the matrix diagonalization method. In their analytic solutions of the branching s-process, however, they just solved the network from ^{85}Kr to ^{88}Sr as an exemplification of the branching points, and they could give the analytic expression of the transformation matrix of the diagonalization matrix. In the present paper, we divide the s-process into two standard forms, the no branching chain and the branching chain which include the majority of the cases of s-process branchings. Calculating the eigenvalue and the eigenvector of the matrix of the network differential equation system, we obtain the analytic representation of the transformation matrix and give the analytic solutions of the network equations. With the analytic solutions, we compute the heavy element abundance in a solar system and the results are in good agreement with astrophysical measurements.

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