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Geometrical description of denormalized thermodynamic manifold*

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In view of differential geometry, the state space of thermodynamic parameters is investigated. Here the geometrical structures of the denormalized thermodynamic manifold are considered. The relation of their geometrical metrics is obtained. Moreover an example is used to illustrate our conclusions.

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1. Introduction

Thermodynamics and statistical mechanics reflect the basic structure of the observed macroscopic physical world. The geometrical structure of the the state space of thermodynamic parameters was studied by Gibbs, dated back to the 1870's.[1] Ruppeiner[2] obtained the metric tensor via second moments of the fluctuations of the thermodynamic magnitudes.

Recently, some authors[3−11] have investigated the state space of thermodynamic parameters. Especially, Ingarden,[6] Janyszek[7] and other authors have obtained a Riemannian metric for a thermodynamic model based on the approach of differential geometry to parametric statistics developed by Chentsov,[12] Efron[13] and Amari.[14] For a given equilibrium density operator

$$\boldsymbol{\rho} = Z^{-1} \exp\left\{-\sum_{i=1}^{n} \beta_i \boldsymbol{F}_i\right\}, \tag{1}$$

where $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ are linearly independent and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ is the classical real parameter (statistical temperature, press, magnetic field) describing the environment of a physical system, and $Z = \mathrm{Tr}\exp\{-\sum_{i=1}^{n} \beta_i \boldsymbol{F}_i\}$ is a partition function. The set $S = \{\boldsymbol{\rho} | \boldsymbol{\rho} = Z^{-1}\exp(-\sum_{i=1}^{n} \beta_i \boldsymbol{F}_i), \mathrm{Tr}\boldsymbol{\rho} = 1\}$ of the thermodynamic parameters can be regarded as a differential manifold equipped with a Riemannian metric

$$g_{ij} = \mathrm{Tr}[\boldsymbol{\rho}(\partial_i \ln \boldsymbol{\rho})(\partial_j \ln \boldsymbol{\rho})], \tag{2}$$

for the case of commuting operators $\boldsymbol{F}_i$, where Tr denotes the trace of the matrix, ∂_i represents the partial derivative with respect to parameter β_i.

In Ref.[14], Amari investigated the denormalization of the statistical model. Here we consider the denormalization of the thermodynamic parametric model, a more general manifold

$$\widetilde{S} = \{\widetilde{\boldsymbol{\rho}} | \widetilde{\boldsymbol{\rho}} = f(\tau)\boldsymbol{\rho}, f(\tau) > 0, \mathrm{Tr}\boldsymbol{\rho} = 1\}, \tag{3}$$

where $f(\tau) > 0$ is an arbitrary-order differentiable function. Then $\widetilde{S}$ is a manifold which contains S as a submanifold with $\dim\widetilde{S} = \dim S + 1$. $\widetilde{S}$ is called the denormalization of S. By calculating the covariant derivative and the curvature tensor, we obtain the relation of the geometrical structures between S and $\widetilde{S}$. Finally, we use an example to illustrate our conclusions.

2. Geometric structure of the manifold $\widetilde{S}$

Let $\widetilde{S}$ be an $(n+1)$-dimensional manifold with the coordinate system $[\beta, \tau] = [\beta_1, \ldots, \beta_n, \tau]$ and $\widetilde{\partial}_i = \dfrac{\partial}{\partial \beta_i}$ and $\widetilde{\partial}_\tau = \dfrac{\partial}{\partial \tau}$ being its natural basis. Let $\boldsymbol{G} = (g_{ij})$ denote the Riemannian metric, V the volume, Γ_{ijk} the Riemanian connection coefficient, R_{ijkl} the curvature tensor, K_{ijij} the sectional curvature and R the scalar curvature of S, where $i, j, k, l \in \{1, \ldots, n\}$. Let $\widetilde{(\cdot)}$ denote the geometrical

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metric of $\widetilde{S}$ corresponding to that of S and let $\mu, \nu, \lambda, \gamma$ represent the subscripts of the geometrical metric of $\widetilde{S}$, where $\mu, \nu, \lambda, \gamma \in \{1, \ldots, n, n+1\}$. For the case of commuting operators $\boldsymbol{F}_i$, we consider the Riemannian metric of $\widetilde{S}$

$$\widetilde{g}_{\mu\nu} = \mathrm{Tr}(\widetilde{\boldsymbol{\rho}}\widetilde{\partial}_\mu \ln \widetilde{\boldsymbol{\rho}}\widetilde{\partial}_\nu \ln \widetilde{\boldsymbol{\rho}}). \tag{4}$$

Through calculation, we obtain

$$\begin{aligned} &\widetilde{g}_{ij} = f(\tau)g_{ij}, \quad \widetilde{g}_{i\tau} = 0, \\ &\text{and} \quad \widetilde{g}_{\tau\tau} = f(\tau)^{-1}(\widetilde{\partial}_\tau f(\tau))^2. \end{aligned} \tag{5}$$

The Riemannian metric of $\widetilde{S}$ and its inverse are given by

$$\begin{aligned} \widetilde{\boldsymbol{G}} &= \begin{pmatrix} f(\tau)\boldsymbol{G} & 0 \\ 0 & f(\tau)^{-1}(\widetilde{\partial}_\tau f(\tau))^2 \end{pmatrix} \\ \text{and} \quad \widetilde{\boldsymbol{G}}^{-1} &= \begin{pmatrix} f(\tau)^{-1}\boldsymbol{G}^{-1} & 0 \\ 0 & f(\tau)(\widetilde{\partial}_\tau f(\tau))^{-2} \end{pmatrix}. \end{aligned} \tag{6}$$

Since the square of the arc length of S is defined as

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}\beta_i \mathrm{d}\beta_j, \tag{7}$$

the square of the arc length of $\widetilde{S}$ is

$$\begin{aligned} \mathrm{d}\widetilde{s}^2 &= \widetilde{g}_{ij}\mathrm{d}\beta_i\mathrm{d}\beta_j + \widetilde{g}_{\tau\tau}\mathrm{d}\tau\mathrm{d}\tau \\ &= f(\tau)\mathrm{d}s^2 + f(\tau)^{-1}(\widetilde{\partial}_\tau f(\tau))^2(\mathrm{d}\tau)^2. \end{aligned} \tag{8}$$

Note that the volume element of S is given by

$$\mathrm{d}V = \sqrt{\det(\boldsymbol{G})}\mathrm{d}\beta_1 \Lambda \ldots \Lambda \mathrm{d}\beta_n, \tag{9}$$

we have the volume element of $\widetilde{S}$

$$\begin{aligned} &\mathrm{d}\widetilde{V} \\ &= \sqrt{\det(\widetilde{\boldsymbol{G}})}\mathrm{d}\beta_1 \Lambda \ldots \Lambda \mathrm{d}\beta_n \Lambda \mathrm{d}\tau \\ &= (f(\tau))^{\frac{n-1}{2}}\widetilde{\partial}_\tau f(\tau)\sqrt{\det(\boldsymbol{G})}\mathrm{d}\beta_1 \Lambda \ldots \Lambda \mathrm{d}\beta_n \Lambda \mathrm{d}\tau. \end{aligned} \tag{10}$$

Consequently, from the definition of the Riemannian connection coefficient

$$\widetilde{\Gamma}_{\mu\nu,\lambda} = \frac{1}{2}(\widetilde{\partial}_\mu \widetilde{g}_{\nu\lambda} + \widetilde{\partial}_\nu \widetilde{g}_{\mu\lambda} - \widetilde{\partial}_\lambda \widetilde{g}_{\mu\nu}), \tag{11}$$

we obtain

$$\begin{aligned} &\widetilde{\Gamma}_{ij,k} = f(\tau)\Gamma_{ij,k}, \\ &\widetilde{\Gamma}_{ij,\tau} = -\frac{1}{2}\widetilde{\partial}_\tau f(\tau) g_{ij}, \\ &\widetilde{\Gamma}_{i\tau,k} = \widetilde{\Gamma}_{\tau i,k} = \frac{1}{2}\widetilde{\partial}_\tau f(\tau) g_{ik}, \\ &\widetilde{\Gamma}_{i\tau,\tau} = \widetilde{\Gamma}_{\tau i,\tau} = 0, \quad \widetilde{\Gamma}_{\tau\tau,k} = 0, \\ &\widetilde{\Gamma}_{\tau\tau,\tau} = -\frac{1}{2}f(\tau)^{-2}(\widetilde{\partial}_\tau f(\tau))^3 \\ &\qquad + f(\tau)^{-1}\widetilde{\partial}_\tau f(\tau)\widetilde{\partial}_\tau \widetilde{\partial}_\tau f(\tau). \end{aligned} \tag{12}$$

These expressions enable us to verify that the following relations hold for the covariant derivatives of the Riemannian metric on $\widetilde{S}$ and S, which we denote by $\widetilde{\nabla}$ and ∇, respectively:

$$\begin{aligned} \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} &= (\nabla_X Y)^{\sim} - \frac{1}{2}(\widetilde{\partial}_\tau f(\tau))^{-1} < \widetilde{X}, \widetilde{Y} > \widetilde{\partial}_\tau, \\ \widetilde{\nabla}_{\widetilde{X}}\widetilde{\partial}_\tau &= \widetilde{\nabla}_{\widetilde{\partial}_\tau}\widetilde{X} = \frac{1}{2}f(\tau)^{-1}\widetilde{\partial}_\tau f(\tau)\widetilde{X}, \\ \widetilde{\nabla}_{\widetilde{\partial}_\tau}\widetilde{\partial}_\tau &= \Big[-\frac{1}{2}f(\tau)^{-1}\widetilde{\partial}_\tau f(\tau) \\ &\qquad + (\widetilde{\partial}_\tau f(\tau))^{-1}\widetilde{\partial}_\tau \widetilde{\partial}_\tau f(\tau) \Big]\widetilde{\partial}_\tau, \end{aligned} \tag{13}$$

where X and Y are arbitrary vector fields on S, and $(\cdot)^{\sim}$ denotes $\widetilde{(\cdot)}$.

In the following, we consider the curvature tensor of $\widetilde{S}$. Let X, Y, Z, W be vector fields on S. By using expression (13), we have

$$\begin{aligned} \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} &= \widetilde{\nabla}_{\widetilde{X}}[(\nabla_Y Z)^{\sim} - \frac{1}{2}(\widetilde{\partial}_\tau f(\tau))^{-1} < \widetilde{Y}, \widetilde{Z} > \widetilde{\partial}_\tau] \\ &= (\nabla_X \nabla_Y Z)^{\sim} - \frac{1}{2}(\widetilde{\partial}_\tau f(\tau))^{-1} < \widetilde{X}, (\nabla_Y Z)^{\sim} > \widetilde{\partial}_\tau \\ &\quad -\frac{1}{4}f(\tau)^{-1} < \widetilde{Y}, \widetilde{Z} > \widetilde{X} - \frac{1}{2}(\widetilde{\partial}_\tau f(\tau))^{-1}\widetilde{X} < \widetilde{Y}, \widetilde{Z} > \widetilde{\partial}_\tau, \\ \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z} &= (\nabla_{[X,Y]}Z)^{\sim} - \frac{1}{2}(\widetilde{\partial}_\tau f(\tau))^{-1} < [X,Y]^{\sim}, \widetilde{Z} > \widetilde{\partial}_\tau. \end{aligned} \tag{14}$$

Since the definition of the curvature tensor is

$$\begin{aligned} < \widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z}, \widetilde{W} >: = &< \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} \\ &-\widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}, \widetilde{W} >, \end{aligned} \tag{15}$$

substituting expression (14) into expression (15), we obtain

$$\begin{aligned}
&< \widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z},\widetilde{W} > \\
=&< (R(X,Y)Z)^{\sim},\widetilde{W} > \\
&-\frac{1}{4}f(\tau)^{-1} < \widetilde{Y},\widetilde{Z} >< \widetilde{X},\widetilde{W} > \\
&+\frac{1}{4}f(\tau)^{-1} < \widetilde{X},\widetilde{Z} >< \widetilde{Y},\widetilde{W} > \\
=& f(\tau) < R(X,Y)Z,W > \\
&-\frac{1}{4}f(\tau) < Y,Z >< X,W > \\
&+\frac{1}{4}f(\tau) < X,Z >< Y,W > . \qquad (16)
\end{aligned}$$

Similarly, for other cases we have

i)

$$\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{\partial}_\tau = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\partial}_\tau - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{\partial}_\tau - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{\partial}_\tau = 0. \qquad (17)$$

In fact, notice that

$$\begin{aligned}
\widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\partial}_\tau &= \widetilde{\nabla}_{\widetilde{X}}\left[\frac{1}{2}f(\tau)^{-1}\widetilde{\partial}_\tau f(\tau)\widetilde{Y}\right] \\
&= \frac{1}{2}f(\tau)^{-1}\widetilde{\partial}_\tau f(\tau)(\nabla_X Y)^{\sim} \\
&\quad -\frac{1}{4}f(\tau)^{-1} < \widetilde{X},\widetilde{Y} > \widetilde{\partial}_\tau, \\
\widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{\partial}_\tau &= \frac{1}{2}f(\tau)^{-1}\widetilde{\partial}_\tau f(\tau)[\widetilde{X},\widetilde{Y}], \qquad (18)
\end{aligned}$$

we can obtain expression (17).

ii)

$$\widetilde{R}(\widetilde{\partial}_\tau,\widetilde{Y})\widetilde{Z} = 0. \qquad (19)$$

Since

$$\begin{aligned}
\widetilde{\nabla}_{\widetilde{\partial}_\tau}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} &= \widetilde{\nabla}_{\widetilde{\partial}_\tau}\left\{(\nabla_Y Z)^{\sim} - \frac{1}{2}(\widetilde{\partial}_\tau f(\tau))^{-1} < \widetilde{Y},\widetilde{Z} > \widetilde{\partial}_\tau\right\} \\
&= \frac{1}{2}f(\tau)^{-1}(\widetilde{\partial}_\tau f(\tau))(\nabla_Y Z)^{\sim} - \frac{1}{2} < \widetilde{Y},\widetilde{Z} > \widetilde{\nabla}_{\widetilde{\partial}_\tau}(\widetilde{\partial}_\tau f(\tau))^{-1}\widetilde{\partial}_\tau - \frac{1}{2}\widetilde{\partial}_\tau < \widetilde{Y},\widetilde{Z} > (\widetilde{\partial}_\tau f(\tau))^{-1}\widetilde{\partial}_\tau \\
&= \frac{1}{2}f(\tau)^{-1}(\widetilde{\partial}_\tau f(\tau))(\nabla_Y Z)^{\sim} - \frac{1}{4}f(\tau)^{-1} < \widetilde{Y},\widetilde{Z} > \widetilde{\partial}_\tau, \\
\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{\partial}_\tau}\widetilde{Z} &= \widetilde{\nabla}_{\widetilde{Y}}(-\frac{1}{2}f(\tau)^{-1}(\widetilde{\partial}_\tau f(\tau))\widetilde{Z}) = \frac{1}{2}f(\tau)^{-1}(\widetilde{\partial}_\tau f(\tau))(\nabla_Y Z)^{\sim} - \frac{1}{4}f(\tau)^{-1} < \widetilde{Y},\widetilde{Z} > \widetilde{\partial}_\tau, \\
\widetilde{\nabla}_{[\widetilde{\partial}_\tau,\widetilde{Y}]}\widetilde{Z} &= 0 \qquad (20)
\end{aligned}$$

hold, we acquire expression (19).

iii)

$$\widetilde{R}(\widetilde{\partial}_\tau,\widetilde{Y})\widetilde{\partial}_\tau = 0. \qquad (21)$$

It is easy to check that

$$\begin{aligned}
\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{\partial}_\tau}\widetilde{\partial}_\tau &= \widetilde{\nabla}_{\widetilde{Y}}\left[-\frac{1}{2}f(\tau)^{-1}(\widetilde{\partial}_\tau f(\tau)) + (\widetilde{\partial}_\tau f(\tau))^{-1}\widetilde{\partial}_\tau\widetilde{\partial}_\tau f(\tau)\right]\widetilde{\partial}_\tau \\
&= -\frac{1}{4}f(\tau)^{-2}(\widetilde{\partial}_\tau f(\tau))^2\widetilde{Y} + \frac{1}{2}f(\tau)^{-1}\widetilde{\partial}_\tau\widetilde{\partial}_\tau f(\tau)\widetilde{Y}, \\
\widetilde{\nabla}_{\widetilde{\partial}_\tau}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\partial}_\tau &= -\frac{1}{4}f(\tau)^{-2}(\widetilde{\partial}_\tau f(\tau))^2\widetilde{Y} + \frac{1}{2}f(\tau)^{-1}\widetilde{\partial}_\tau\widetilde{\partial}_\tau f(\tau)\widetilde{Y}, \\
\widetilde{\nabla}_{[\widetilde{\partial}_\tau,\widetilde{Y}]}\widetilde{\partial}_\tau &= 0. \qquad (22)
\end{aligned}$$

So we have expression (21).

Taking $\widetilde{X} = \widetilde{\partial}_k, \widetilde{Y} = \widetilde{\partial}_l, \widetilde{Z} = \widetilde{\partial}_i$, and $\widetilde{W} = \widetilde{\partial}_j$, from expressions (16), (17), (19) and (21), we obtain the components of the curvature tensor

$$\begin{aligned}
&\widetilde{R}_{ijkl} = f(\tau)R_{ijkl} - \frac{1}{4}f(\tau)(g_{il}g_{jk} - g_{jl}g_{ik}), \\
&\widetilde{R}_{ijk\tau} = \widetilde{R}_{i\tau k\tau} = 0. \qquad (23)
\end{aligned}$$

Now, we consider the sectional curvature and the scalar curvature of $\widetilde{S}$. Through a direct calculation, we have

$$\begin{aligned}
\widetilde{K}_{ijij} :&= -\frac{\widetilde{R}_{ijij}}{\widetilde{g}_{ii}\widetilde{g}_{jj} - \widetilde{g}_{ij}\widetilde{g}_{ji}} \\
&= \frac{-f(\tau)R_{ijij} - \frac{1}{4}f(\tau)(g_{jj}g_{ii} - g_{ij}g_{ji})}{(f(\tau))^2(g_{ii}g_{jj} - g_{ij}g_{ji})}
\end{aligned}$$

$$= (f(\tau))^{-1}K_{ijij} - \frac{1}{4}(f(\tau))^{-1}, \tag{24}$$

and

$$\begin{aligned}
&\widetilde{R}\\
&= \widetilde{R}_{\mu\nu\lambda\gamma}\widetilde{g}^{\mu\gamma}\widetilde{g}^{\nu\lambda}\\
&= \left[f(\tau)R_{ijij} - \frac{1}{4}f(\tau)(g_{jj}g_{ii} - g_{ij}g_{ji})\right] g^{il}g^{kj}(f(\tau))^{-2}\\
&= f(\tau)^{-1}R + \frac{1}{4}f(\tau)^{-1}(n-n^2).
\end{aligned} \tag{25}$$

In the case of $n = 2$, from expressions (3) and (25) it follows that

$$\widetilde{R} = \frac{f(\tau)^{-1}}{2\det(\boldsymbol{G})^2}\begin{pmatrix} \partial_1^2\ln Z & \partial_1^3\ln Z & \partial_1^2\partial_2\ln Z \\ \partial_1\partial_2\ln Z & \partial_1^2\partial_2\ln Z & \partial_1\partial_2^2\ln Z \\ \partial_2^2\ln Z & \partial_1\partial_2^2\ln Z & \partial_2^3\ln Z \end{pmatrix} - \frac{f(\tau)^{-1}}{2}. \tag{26}$$

From expression (26), we can see that $\widetilde{R}$ is given in the terms of the second and third derivatives of $\ln Z$ and the function $f(\tau)$. In physics, it means that $\widetilde{R}$ depends on the second and third moments of the variables $\boldsymbol{F}_i$ and function $f(\tau)$.

3. Example

Here we use an example to verify our conclusions above. Consider the set $S = \{\boldsymbol{\rho}|\boldsymbol{\rho} = Z^{-1}\exp\big[-\sum_{i=1}^{2}\beta_i\boldsymbol{F}_i\big]\}$ with two thermodynamical degrees of freedom, and the matrices F_i given by

$$\boldsymbol{F}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{F}_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}. \tag{27}$$

Taking $f(\tau) = \tau > 0$ yields

$$\widetilde{S} = \left\{\widetilde{\boldsymbol{\rho}}|\widetilde{\boldsymbol{\rho}} = \tau Z^{-1}\exp\left[-\sum_{i=1}^{2}\beta_i\boldsymbol{F}_i\right]\right\}. \tag{28}$$

An equilibrium density $\boldsymbol{\rho} \in S$ can be represented as

$$\boldsymbol{\rho} = \frac{1}{Z}\exp\left[\begin{pmatrix} -\beta_2 & -\beta_2-\beta_1 & 0 \\ 0 & -\beta_1 & 0 \\ \beta_2 & -\beta_2 & 0 \end{pmatrix}\right], \tag{29}$$

where the partition function of $\boldsymbol{\rho}$ is $Z = \exp(-\beta_1) + \exp(-\beta_2) + 1$. Under the coordinate system (β_1, β_2), the Riemannian metric of S is given by

$$\begin{aligned}
\boldsymbol{G} &= (g_{ij}) = (\partial_i\partial_j\ln Z),\\
\det(\boldsymbol{G}) &= \partial_1^2\ln Z\partial_2^2\ln Z - (\partial_1\partial_2\ln Z)^2\\
&= \frac{\exp(2(\beta_1+\beta_2))}{(\exp(\beta_1)+\exp(\beta_2)+\exp(\beta_1+\beta_2))^3}.
\end{aligned} \tag{30}$$

Through calculation, we acquire the independent component of the curvature tensor

$$\begin{aligned}
R_{1212} &= -\frac{1}{4\det(\boldsymbol{G})}\begin{vmatrix} \partial_1^2\ln Z & \partial_1^3\ln Z & \partial_1^2\partial_2\ln Z \\ \partial_1\partial_2\ln Z & \partial_1^2\partial_2\ln Z & \partial_1\partial_2^2\ln Z \\ \partial_2^2\ln Z & \partial_1\partial_2^2\ln Z & \partial_2^3\ln Z \end{vmatrix}\\
&= \frac{\exp(2(\beta_1+\beta_2))}{4(\exp(\beta_1)+\exp(\beta_2)+\exp(\beta_1+\beta_2))^3}.
\end{aligned} \tag{31}$$

As a consequence, the sectional curvature and the scalar curvature of S are given respectively by

$$K_{1212} = \frac{R_{1212}}{\det(\boldsymbol{G})} = \frac{1}{4}$$

and

$$R = \frac{2}{\det(\boldsymbol{G})}R_{1212} = \frac{1}{2}. \tag{32}$$

Under the coordinate system (β_1, β_2, τ), we obtain the Riemannian metrics of $\widetilde{S}$

$$\begin{aligned}
\widetilde{\boldsymbol{G}} &= \begin{pmatrix} -\tau\partial_1^2\ln Z & -\tau\partial_1\partial_2\ln Z & 0 \\ -\tau\partial_1\partial_2\ln Z & -\tau\partial_2^2\ln Z & 0 \\ 0 & 0 & \tau^{-1} \end{pmatrix}\\
&= \begin{pmatrix} \tau\boldsymbol{G} & 0 \\ 0 & \tau^{-1} \end{pmatrix}.
\end{aligned} \tag{33}$$

From

$$\begin{aligned}
\widetilde{R}_{\mu\nu\lambda\gamma} &= (\widetilde{\partial}_\nu\widetilde{\Gamma}^{\iota}_{\mu\lambda} - \widetilde{\partial}_\mu\widetilde{\Gamma}^{\iota}_{\nu\lambda})\widetilde{g}_{\iota\lambda}\\
&\quad +(\widetilde{\Gamma}_{\nu\kappa\gamma}\widetilde{\Gamma}^{\kappa}_{\mu\lambda} - \widetilde{\Gamma}_{\mu\kappa\gamma}\widetilde{\Gamma}^{\kappa}_{\nu\lambda}),
\end{aligned} \tag{34}$$

we obtain the geometrical metrics of $\widetilde{S}$ corresponding to those of S:

$$\begin{aligned}
&\widetilde{R}_{1212} = \tau R_{1212} - \frac{1}{4}\tau\det(\boldsymbol{G}) = 0,\\
&\widetilde{R}_{1\tau1\tau} = \widetilde{R}_{121\tau} = \widetilde{R}_{212\tau} = \widetilde{R}_{1\tau2\tau} = \widetilde{R}_{2\tau2\tau} = 0,\\
&\widetilde{K}_{1212} = \tau^{-1}K_{1212} - \frac{1}{4}\tau^{-1} = 0.
\end{aligned} \tag{35}$$

The scalar curvature of $\widetilde{S}$ is given by

$$\widetilde{R} = \frac{\tau^{-1}}{2\det(\boldsymbol{G})^2} \begin{pmatrix} \partial_1^2 \ln Z & \partial_1^3 \ln Z & \partial_1^2\partial_2 \ln Z \\ \partial_1\partial_2 \ln Z & \partial_1^2\partial_2 \ln Z & \partial_1\partial_2^2 \ln Z \\ \partial_2^2 \ln Z & \partial_1\partial_2^2 \ln Z & \partial_2^3 \ln Z \end{pmatrix} - \frac{\tau^{-1}}{2} = 0. \quad (36)$$

4. Conclusions

In the present paper, we consider the denormalization of the space of thermodynamic parameters in view of differential geometry. By calculating the covariant derivative and the curvature tensor, we obtain the relation of their geometrical structures. Finally, we use an example to illustrate our conclusions.

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