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Collective coordination of multi-agent systems guided by multiple leaders

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A neighbour-based coordination scheme is proposed for a multi-agent system with multiple leaders. Under assumptions of the connectivity of the interconnection topology and a simple first-order dynamics model for each mobile agent, the results show that all the agents will flock to the polytope region formed by the leaders.

Keywords: multiple leaders, neighbour-based rule, ploytope
PACC: 0565, 0540

1. Introduction

Recently, widespread interest in multi-agent systems has been aroused by the breadth of natural and technological systems that exhibit collective coordinated activity of a group of autonomous agents, partly due to a broad application of multi-agent systems including flocking/swarming (e.g., Refs.[1] and [2]), consensus,[3,4] formation control,[5,6] and sensor networks.[7] In multi-agent coordination, leader-follower is an important architecture. Even in some leaderless cases, concepts like "virtual leader" are proposed to study cooperative behaviours.[8]

In most leader-following problems, there is only one leader involving in the collective motion (e.g., Refs.[9] and [10]), though the leader, sometimes, may be active with unmeasured states (e.g., Ref.[11]). However, in some practical situations of formation or foraging, the formulation of multiple leaders may be needed. In Ref.[9], Lin *et al* discussed an interesting model for a group of agents with straight-line formation containing two "edge leaders", where all the agents converge to a uniform distribution on the line segment specified by the two edge leaders. In Ref.[12], a simple model was given to simulate foraging and demonstrate that the larger the group is, the smaller the proportion of "leaders" needed to guide the group is.

Inspired by Refs.[9] and [12], in this paper we consider the roles of multiple leaders in the guidance of the collective behaviour of mobile agents. By using a neighbourhood coordination rule, we will show that a group of agents will converge to the polytope formed by the leaders (that is, the leaders form a vertex set of the polytope) on the assumption that the interconnection topology associated with the agents and leaders is fixed and connected, and demonstrate that the collective behaviour changes as the connectivity of the interconnection between agents and a leader increases through some simulations.

2. Problem formulation

Before formulating our problem, we first introduce some basic concepts and notations in graph theory that will be used.[13] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a weighted undirected graph of order n with a set of nodes $\mathcal{V} = \{1, 2, \ldots, n\}$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $A = [a_{ij}] \in \boldsymbol{R}^{n\times n}$ with non-negative elements. The element a_{ij} associated with the edge of the graph is positive, i.e. $a_{ij} > 0 \Leftrightarrow (i, j) \in \mathcal{E}$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{I}$. The set of neighbours of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. A diagonal matrix $D = \mathrm{diag}\{d_1, \ldots, d_n\} \in \boldsymbol{R}^{n\times n}$ is a degree matrix of $\mathcal{G}$, with diagonal elements $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$ for $i = 1, \ldots, n$. Then the Laplacian of the weighted graph $\mathcal{G}$ is defined as $L = D - A \in \boldsymbol{R}^{n\times n}$. From a well-known result in algebraic graph theory (e.g. Ref.[13]), we know that all eigenvalues of L are non-negative and that there is only one zero eigenvalue if the graph $\mathcal{G}$

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is connected.

Next, we introduce some notations in linear algebra [14] and convex analysis[15]. Let $S \subset \boldsymbol{R}^m$. S is said to be convex if $(1-\gamma)x+\gamma y \in S$ whenever $x \in S, y \in S$ and $0 < \gamma < 1$. A vector sum $\gamma_1 x_1 + \ldots + \gamma_n x_n$ is called a convex combination of $x_1, \ldots, x_n$ if the coefficients γ_i are all non-negative and $\gamma_1 + \ldots + \gamma_n = 1$. Here, γ_i can be interpreted as proportions. The intersection of all convex sets containing S is the convex hull of S, denoted by $\mathrm{co}(S)$. The convex hull of a finite set of points $x_1, \ldots, x_n \in \boldsymbol{R}^m$ is a polytope, denoted by $\mathrm{co}\{x_1, \ldots, x_n\}$. If $x_2 - x_1, \ldots, x_n - x_1$ are linearly independent, the set of points $x_1, \ldots, x_n$ is said to be affinely independent. Then the polytope is called an $(n-1)$-dimensional simplex and $x_1, \ldots, x_n$ are called the vertices of the simplex.

In the present paper, we consider a system consisting of n agents and k leaders, and the interconnection topology among them can easily be described by a simple graph. The purpose of the leaders is to guide the multi-agent behaviour. Denote the positions of these static leaders by $x_0^j \in \boldsymbol{R}^m, j = 1, \ldots, k$. The interconnection topology $\mathcal{G}$ associated with n agents can be fixed or varied but may not be connected. If we regard the polytope formed by leaders as a virtual node and if one agent can see a vertex (i.e. some leader) of the polytope, we say that the agent is connected to the virtual node. By "the graph, $\bar{\mathcal{G}}$, of this system is connected", we mean that at least one agent in each component of $\mathcal{G}$ is connected to the virtual node. Then we define a diagonal matrix B to be a leader adjacency matrix associated with $\bar{\mathcal{G}}$ with diagonal elements b_i $(i \in \mathcal{I})$ such that each b_i is some positive number if agent i is connected to the virtual node and 0 otherwise.

A continuous-time dynamics of n agents is described as follows:

$$\dot{x}_i = u_i, \tag{1}$$

where $x_i \in \boldsymbol{R}^m$ can be the position of agent i and $u_i \in \boldsymbol{R}^m$ its interconnection control inputs for $i = 1, \ldots, n$. As usual, we propose a neighbour-based feedback control as follows:

$$u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) + \sum_{q=1}^{k} b_i^q (x_0^q - x_i);\ i = 1, \ldots, n. \tag{2}$$

where non-negative functions $b_i^q > 0$ if and only if agent i is connected to leader q $(q = 1, \ldots, k)$ when $t \in [t_l, t_{l+1})$ for $l = 0, 1, \ldots$.

Denote x, x_0 as the concatenations of x_i, x_0^j $(i = 1, \ldots, n; j = 1, \ldots, k)$. Let $B^q \in \boldsymbol{R}^{n \times n}$ be a diagonal matrix with non-negative diagonal entry b_i^q for $q = 1, \ldots, k$ and $B = [B^1, \ldots, B^k] \in \boldsymbol{R}^{n \times nk}$. Let $\Xi = \{\xi = (\xi_1, \ldots, \xi_n)^{\mathrm{T}} | \xi_i \in \mathrm{co}\{x_0^1, \ldots, x_0^k\}\}$.

Throughout the paper, $\boldsymbol{R}^+ = [0, \infty), ||\cdot||$ denotes the Euclidean norm and $\langle \cdot, \cdot \rangle$ denotes the inner product on $\boldsymbol{R}^{mn}$. For non-empty $\Xi \subset \boldsymbol{R}^{mn}$, $d_\Xi : \boldsymbol{R}^{mn} \to \boldsymbol{R}^+$ denotes its Euclidean distance function given by $d_\Xi(x) = \frac{1}{2} \inf\limits_{\xi \in \Xi} ||x - \xi||^2$.

Then a closed-loop system is followed by exerting the coordination scheme (2) on the agent dynamics (1):

$$\dot{x} = -(H \otimes \boldsymbol{I}_m)x + [B(\boldsymbol{I}_k \otimes \boldsymbol{1}_n)] \otimes \boldsymbol{I}_m x_0, \tag{3}$$

where

$$H = L + B(\mathbf{1}_k \otimes I_n),$$

with $\otimes$ being the Kronecker product.

The objective of this paper is to lead all the agents to enter the region formed by the leaders; namely, x_i $(i = 1, \ldots, n)$, for each agent i, will be contained in a convex hull of x_0^j $(j = 1, \ldots, k)$ as $t \to \infty$. In other words,

$$\lim_{t \to \infty} d_\Xi(x(t)) = 0. \tag{4}$$

In the following, the convergence of the system (3) will be studied based on $d_\Xi(x)$ with fixed interconnection topology.

3. Convergence analysis

At first, we consider the coordination behaviour of multi-agent system (3) with fixed interconnection topology. The next lemma shows the relationship between the positive definiteness of a matrix $H = L + B(\mathbf{1}_k \otimes I_n)$ and the connectivity of $\bar{\mathcal{G}}$, which was proved in Ref.[11].

Lemma 1 If graph $\bar{\mathcal{G}}$ is connected, then the symmetric matrix H associated with $\bar{\mathcal{G}}$ is positive definite.

Now the main result can be drawn as follows:

Theorem 1 For the multi-agent system (1), the agents can enter the region formed by the leaders if and only if $\bar{\mathcal{G}}$ is connected,

Proof: (Sufficiency) Let $\bar{x} = x - [H^{-1}B(I_k \otimes \mathbf{1}_n)] \otimes I_m x_0$. Then we have

$$\dot{\bar{x}} = -(\boldsymbol{H} \otimes I_m)\bar{x}. \tag{5}$$

For any initial value $\bar{x}(t_0)$ at initial time t_0, the solution of the system (5) is

$$\bar{x}(t) = \mathrm{e}^{-(H \otimes I_m)(t - t_0)} \bar{x}(t_0),\ t \geq t_0.$$

From Lemma 1, the eigenvalues of $H \otimes \boldsymbol{I}_m$ are positive, and therefore, $\bar{x}(t) \to 0$; namely, $x \to x^* = [H^{-1}B(\boldsymbol{I}_k \otimes \boldsymbol{1}_n)] \otimes \boldsymbol{I}_m x_0$, as $t \to \infty$.

Then we only need to prove that each vector $x_i^* \in \boldsymbol{R}^m$ $(i = 1, \dots, n)$ can be represented by a convex combination of $x_0^p \in \boldsymbol{R}^m$ for $p = 1, \dots, k$. This is equivalent to proving that $[H^{-1}B(\boldsymbol{I}_k \otimes \boldsymbol{1}_n)] \otimes \boldsymbol{I}_m$ is a row stochastic matrix which is a non-negative matrix and the sum of the entries in every row equals 1.

For H, there exists a positive number α such that $H = \alpha I - M$, where M is a non-negative matrix and $\alpha > \rho(M)$ with $\rho(M) = \max\{\sqrt{\lambda}, \lambda \text{ is an eigenvalue of } M^{\mathrm{T}}M\}$. In fact, the eigenvalue $\lambda_i(H) = \alpha - \lambda_i(M) > 0, i = 1, \dots, n$. Thus,

$$(\alpha I - M)^{-1} = \frac{1}{\alpha}\left(I + \frac{1}{\alpha}M + \frac{1}{\alpha^2}M^2 + \dots\right) \geq 0_{n\times n}$$

or equivalently, $(H^{-1} \otimes \boldsymbol{I}_m)$ is a non-negative matrix, and so is $[H^{-1}B(\boldsymbol{I}_k \otimes \boldsymbol{1}_n)] \otimes \boldsymbol{I}_m$.

Additionally, since

$$(H \otimes \boldsymbol{I}_m)(\boldsymbol{1}_n \otimes \boldsymbol{1}_m) = \left(\sum_{q=1}^{k} B^q \boldsymbol{1}_n\right) \otimes \boldsymbol{1}_m,$$

we have

$$\begin{aligned}&(H \otimes \boldsymbol{I}_m)^{-1}\left(\sum_{q=1}^{k} B^q \boldsymbol{1}_n\right) \otimes \boldsymbol{1}_m\\ =&((H \otimes \boldsymbol{I}_m)^{-1}[B(\boldsymbol{I}_k \otimes \boldsymbol{1}_n)] \otimes \boldsymbol{I}_m)(\boldsymbol{1}_k \otimes \boldsymbol{1}_m)\\ =&([H^{-1}B(\boldsymbol{I}_k \otimes \boldsymbol{1}_n)] \otimes \boldsymbol{I}_m)(\boldsymbol{1}_k \otimes \boldsymbol{1}_m) = \boldsymbol{1}_n \otimes \boldsymbol{1}_m.\end{aligned}$$

Note that $[H^{-1}B(\boldsymbol{I}_k \otimes \boldsymbol{1}_n)] \otimes \boldsymbol{I}_m$ is a matrix with every row sum equal to 1, which leads to the conclusion.

(Necessity) Suppose that $\bar{\mathcal{G}}$ is not connected. Without loss of generality, we also suppose that there are κ components in $\mathcal{G}$, where some components are connected with the virtual node. We can renumber the nodes of $\mathcal{V}$ such that the Laplacian matrix associated with $\mathcal{G}$ takes the following form

$$L = \begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix}$$

and, correspondingly,

$$B^q = \begin{pmatrix} B_{11}^q & 0 \\ 0 & 0 \end{pmatrix}$$

where $L_{11} \in \boldsymbol{R}^{s\times s}(0 \leq s < n)$ is the Laplacian associated with those components connected with the virtual node, $B_{11}^q \in \boldsymbol{R}^{s\times s}$ is a nonzero diagonal matrix and 0's denote some appropriate zero matrices.

Let $x = (y_1^{\mathrm{T}}, y_2^{\mathrm{T}})^{\mathrm{T}}$ with $y_1 \in \boldsymbol{R}^{sm}; y_2 \in \boldsymbol{R}^{(n-s)m}$. Then (3) become:

$$\begin{aligned}&\dot{y}_1 = -(H_1 \otimes \boldsymbol{I}_m)y_1 + [B_1(\boldsymbol{I}_k \otimes \boldsymbol{1}_s)] \otimes \boldsymbol{I}_m x_0,\\ &\dot{y}_2 = -(L_{22} \otimes \boldsymbol{I}_m)y_2,\end{aligned}$$

where $B_1 = [B_{11}^1, \dots, B_{11}^k], H_1 = L_{11} + B_1(\boldsymbol{I}_k \otimes \boldsymbol{1}_s)$. From the proof for the sufficient condition, it follows that s agents corresponding to y_1 will approach the leaders for an arbitrary initial condition. However, the other agents corresponding to y_2 will stay static or diverge to some distinct locations which can arbitrarily exist and may not belong to the polytope $\mathrm{co}\{x_0^1, \dots, x_0^k\}$. This leads to a contradiction.

Secondly, we consider multi-agent coordination with multiple leaders in a noisy environment, that is, the information obtained by each agent from its neighbours or some leader contains noise. Here, a modified neighbour-based feedback control is proposed for each agent as follows:

$$\begin{aligned}&u_i = \sum_{j\in N_i} a_{ij}(x_j - x_i) + \sum_{q=1}^{k} b_i^q(x_0^q - x_i) + \omega_i,\\ &i = 1, \dots, n,\end{aligned} \tag{6}$$

where $\omega_i(t)$ $(i = 1, \dots, n)$ are uncorrelated zero-mean Gaussian white noises with covariance matrices $E[\omega_i(t_1)\omega_i^{\mathrm{T}}(t_2)] = \Omega\delta(t_1 - t_2)$, $\delta(t)$ is a Dirac function, $E[\cdot]$ is the expectation operator.

For a general Ito stochastic equation as follows:

$$\mathrm{d}e(t) = f(e(t))\mathrm{d}t + g(e(t))\mathrm{d}\tilde{\omega}(t),\ e(t_0) = e(0), \tag{7}$$

where $e(t) \in \boldsymbol{R}^n$, $\tilde{\omega}(t) \in \boldsymbol{R}^m$ is a standard Brown motion, nonlinear functions $f(e(t))$ and $g(e(t))$ are continuous in $\boldsymbol{R}^n$ and satisfy a global Lipschitz condition, namely, for all solutions $e_1(t), e_2(t) \in \boldsymbol{R}^n$,

$$\begin{aligned}&\|f(e_1(t)) - f(e_2(t))\|_2 + \|g(e_1(t)) - g(e_2(t))\|_2\\ &\leq k_L\|e_1(t) - e_2(t)\|_2, k_L \geq 0.\end{aligned}$$

The differential generator associated with $e(t)$ is defined by:

$$\mathcal{L}(\cdot) = \frac{\partial(\cdot)}{\partial e}f(e(t)) + \frac{1}{2}\mathrm{tr}(g(e(t))g^{\mathrm{T}}(e(t))\mathrm{Hess}(\cdot)) \tag{8}$$

where $\mathrm{Hess}(\cdot)$ denotes the Hessian matrix.

Now a key lemma to show the stochastic boundedness or stability is given as follows:[7]

Lemma 2 For stochastic differential equation (7), if there is a stochastic process $V(e(t))$ and positive numbers λ_1, λ_2, λ_3, ϵ such that

$$\lambda_1\|e(t)\|^2 \leq V(e(t)) \leq \lambda_2\|e(t)\|^2 \tag{9}$$

and

$$\mathcal{L}V(e(t)) \leq -\lambda_3 V(e(t)) + \epsilon \tag{10}$$

are fulfilled. Then the stochastic process $e(t)$ is exponentially bounded in mean square.

Applying the controller (6) to system (1), a close-loop system is obtained as:

$$\dot{x} = -(H \otimes \boldsymbol{I}_m)x + [B(\boldsymbol{I}_k \otimes \mathbf{1}_n)] \otimes \boldsymbol{I}_m x_0 + \omega, \tag{11}$$

where ω is the concatenation of ω_i.Then, for the multi-agent system (11), we say that the agents can enter the region formed by the leaders in the noisy environment if $[H^{-1}B(\boldsymbol{I}_k \otimes \mathbf{1}_n)] \otimes \boldsymbol{I}_m$ is row stochastic and the error $\bar{x} = x - [H^{-1}B(\boldsymbol{I}_k \otimes \mathbf{1}_n)] \otimes \boldsymbol{I}_m x_0$ is exponentially bounded in mean square. Then another main result is given as follows:

Theorem 2 For the system (11), if $\bar{\mathcal{G}}$ is connected, then the agents can enter the region formed by the leaders in a noisy environment.

Proof: Let $\bar{x} = x - [H^{-1}B(\boldsymbol{I}_k \otimes \mathbf{1}_n)] \otimes \boldsymbol{I}_m x_0$, then we have

$$\dot{\bar{x}} = -(H \otimes \boldsymbol{I}_m)\bar{x} + \omega, \tag{12}$$

Define $V(\bar{x}) = \bar{x}^{\mathrm{T}}\bar{x}$, then the differential generator is given by

$$\begin{aligned} \mathcal{L}V(\bar{x}) &= -2\bar{x}^{\mathrm{T}}(H \otimes \boldsymbol{I}_m)\bar{x} + mn \\ &\leq -2\lambda_{\min} V(\bar{x}) + mn, \end{aligned}$$

where $\lambda_{\min} > 0$ is the minimal eigenvalue of $H \otimes \boldsymbol{I}_m$, for $\bar{\mathcal{G}}$ is connected and H is positive definite. By Lemma 2, $\bar{x}$ is exponential bounded in mean square. On the basis of theorem 1, the conclusion follows.

4. Numerical examples

In this section, we give numerical simulations to illustrate the coordination of multi-agent systems with multiple leaders. The following two interconnection topologies associated with multi-agent systems with multiple leaders are given (see Figs.1 and 2).

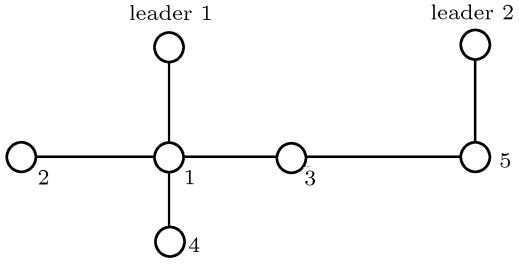


Fig.1. $\bar{\mathcal{G}}_1$ and $\mathcal{G}_1$.

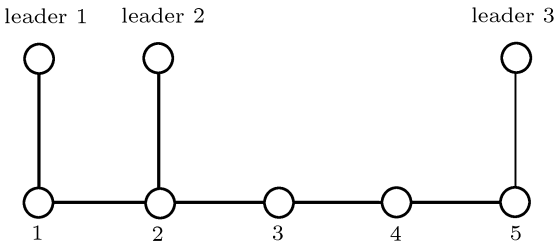


Fig.2. $\bar{\mathcal{G}}_2$ and $\mathcal{G}_2$.

Example 1 In this example, we consider five agents guided by two leaders, which are moving in a horizontal line. In addition, the initial positions of agents and leaders are given as follows:

$$x_1(0) = 5,\ x_2(0) = 5.5,\ x_3(0) = 6,\ x_4(0) = 7,$$

$$x_5(0) = 6.5,\ x_0^1 = 1,\ x_0^2 = 2.$$

Then the simulation results show that the group of agents will approach the segment $[0, 2]$, which connects the two leaders in Figs.3–6.

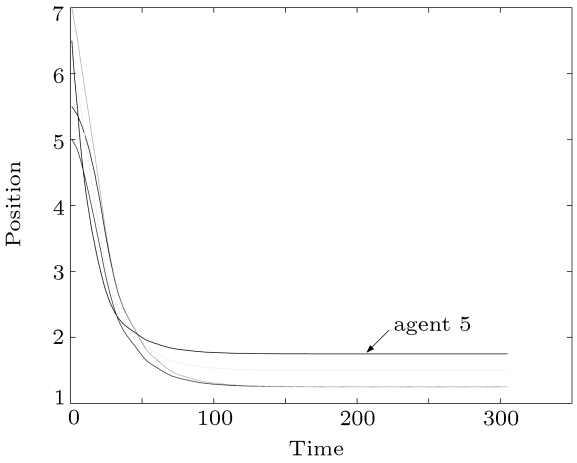


Fig.3. Coordination behaviour with the the topology shown in Fig.1.

In Fig.4, when the number of agents linked to leader 1 increases, the agent group will move to leader 1 more closely. Moreover, in Fig.6, when agent 5 is sensed by other agents, then it also moves to leader 1. In Fig.5, though agent 2 is not connected with other

agents but linked to leader 1, then it will reach the locality of leader 1 finally. This example shows that if information on a leader can be obtained by more agents, the guidance role of the leader will play more weightily in the group decision.

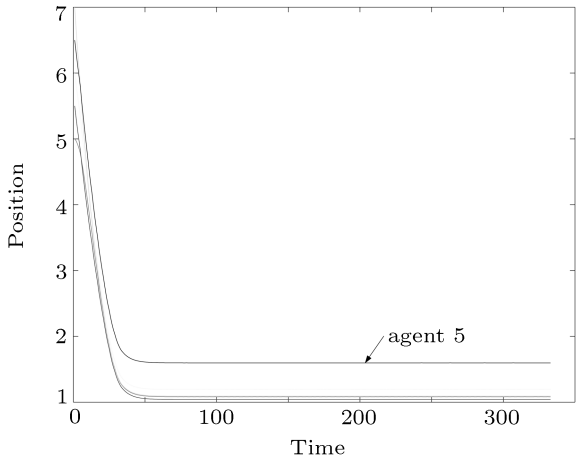


Fig.4. Coordination behaviour of the revised Fig.1, where agents 2, 3 and 4 are connected with leader 1.

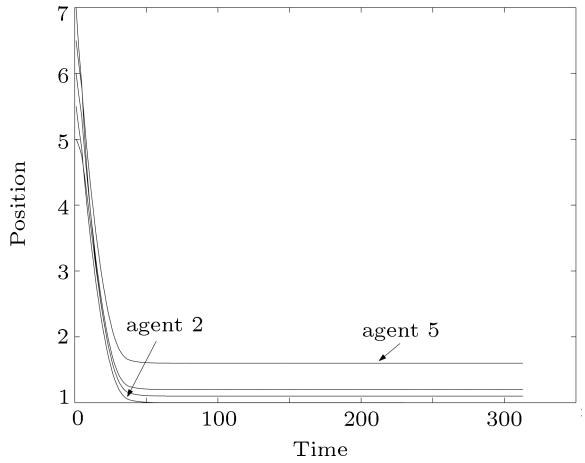


Fig.5. Coordination behaviour of Fig.3 with agent 2 disconnected from agents 3 and 4.

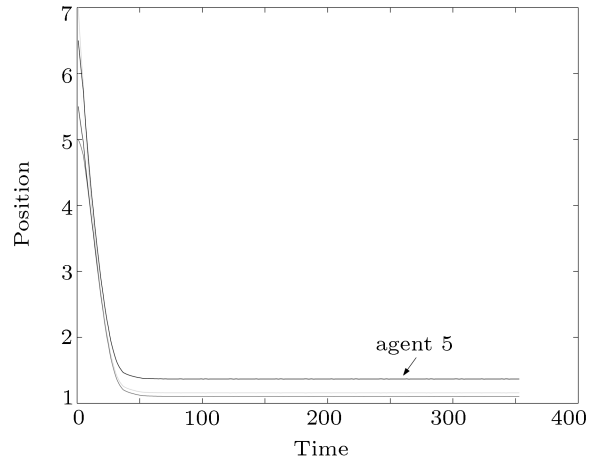


Fig.6. Coordination behaviour of Fig.3 with agents 2 and 4 connected with agent 5.

Example 2 In this example, we consider five agents are guided by three leaders moving in a plane. The initial positions of agents and leaders are given as follows:

$$x_1(0) = (0,0)^{\mathrm{T}},\ x_2(0) = (1,0)^{\mathrm{T}},\ x_3(0) = (2,0)^{\mathrm{T}},$$

$$x_4(0) = (3,0)^{\mathrm{T}},\ x_5(0) = (4,0)^{\mathrm{T}},\ x_0^1 = (1,1)^{\mathrm{T}},$$

$$x_0^2 = (2,2)^{\mathrm{T}},\ x_0^3 = (1,2)^{\mathrm{T}}.$$

Here, the agents take a straight-line formation at the beginning.

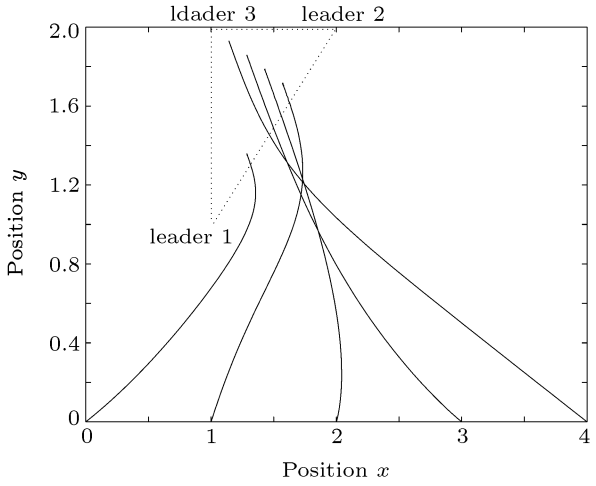


Fig.7. Coordination behaviour related to Fig.2.

From Fig.7, we observe that this group of agents will enter the triangle formed by three leaders. Moreover, during the evolution, agents 2, 3, 4, and 5 remain in a straight-line formation. This example illustrates that the initial interconnection topology of multi-agent systems may exert an influence on the final formation while guided by multiple leaders.

5. Conclusion

In this paper, a coordination problem of a multi-agent system with multiple leaders is studied. By applying the neighbour-based coordination scheme, the group of agents approaches the region "contained" by the leaders if the interconnection graph is connected even though the information obtained by the agents is noisy. Moreover, numerical simulations are given to illustrate the theoretical analysis.

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