Vol 18 No 9, September 2009
1674-1056/2009/18(09)/3772-05

Chinese Physics B

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Bifurcation and stability of an improved time-delayed fluid flow model in internet congestion control*

Liu Yu-Liang(刘玉良)[a)b)†], Zhu Jie(朱 杰)[a)], and Luo Xiao-Shu(罗晓曙)[c)]

[a)] *Department of Electronic Engineering, Shanghai Jiao Tong University, Shanghai* 200240, *China*
[b)] *Department of Electronic Information, Zhejiang Ocean University, Zhoushan* 316004, *China*
[c)] *Department of Physics and Electronic Science, Guangxi Normal University, Guilin* 541004, *China*

(Received 15 January 2009; revised manuscript received 7 March 2009)

Based on the fluid flow time-delayed model proposed by Misra *et al* in internet congestion control, one modified time-delayed model is presented, where the influence of the communication delay on the router queue length is investigated in detail. The main advantage of the new model is that its stability domain is larger even without an extra controller. By linear stability analysis and numerical simulation, the effectiveness and feasibility of the novel model in internet congestion control are verified.

Keywords: fluid flow model, congestion control, time delay, Hopf bifurcation
PACC: 0545, 0547

1. Introduction

At present, fluid flow models[1−6] represent a subject of recurring interest in the field of internet congestion control. One classic fluid flow model[1] with time delay was first proposed by Misra *et al*. Such a model is composed of coupled nonlinear differential equations as follows:

$$\begin{aligned}\frac{\mathrm{d}W(t)}{\mathrm{d}t} &= \frac{1}{R+q(t)/C} - \frac{1}{2}\frac{W(t)W(t-R-q(t)/C)}{R+q(t-R-q(t)/C)/C} \\ &\quad \times p(t-R-q(t)/C)\ , \\ \frac{\mathrm{d}q(t)}{\mathrm{d}t} &= \begin{cases} \dfrac{N(t)W(t)}{R+q(t)/C} - C,\ q>0\ , \\ \max(\dfrac{N(t)W(t)}{R+q(t)/C} - C, 0),\ \ q=0\ , \end{cases}\end{aligned} \tag{1}$$

where $W(t)$ denotes the average value of TCP window size (packets), $q(t)$ represents the average value of the router queue lengths (packets), $R(t)$ is the round-trip time (in s), C is the capacity (packets/s) of the bottleneck link, $N(t)$ is the number of TCP sessions, and $p(\cdot)$ is the probability function of a packet mark. Both the queue length $q(t)$ and the window size $W(t)$ are positive. In the expression for the time delay, $R(t) = R + q(t)/C$, R is the propagation delay (in s) and $q(t)/C$ is the queue delay.

The existence of the time-variant delay $R(t)$ in Eq.(1) makes the stability analysis very complicated, thus some simplified versions are proposed. For example, in Ref.[2], the TCP load $N(t)$ and the round-trip time $R(t)$ are assumed to be time-invariant, i.e. $N(t) = N$ and $R(t) = R$, where both N and R are constant. Then model (1) is reduced to

$$\begin{aligned}\frac{\mathrm{d}W(t)}{\mathrm{d}t} &= \frac{1}{R} - \frac{W(t)\,W(t-R)}{2R}p(t-R)\ , \\ \frac{\mathrm{d}q\ (t)}{\mathrm{d}t} &= \begin{cases} -C + \dfrac{NW(t)}{R},\ q>0\ , \\ \max\left(0, -C + \dfrac{NW(t)}{R}\right),\ q=0. \end{cases}\end{aligned} \tag{2}$$

In Ref.[3] the probability expressed as a function $p(t)$ was assumed to be proportional to the queue length and the first equation of Eq.(2) was sequentially abbreviated as

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = \frac{1}{R} - \frac{W(t)W(t-R)}{2R}Kq(t-R). \tag{3}$$

In Ref.[4] the time delay R in the window size in $W(t-R)$ was neglected and model (3) is further simplified to

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = \frac{1}{R} - \frac{W(t)^2}{2R}Kq(t-R). \tag{4}$$

Based on Eqs.(2)–(4), the stability analysis of the model equilibrium point is widely studied. For instance, Hollot and Chait[3] have proved that when the

*Project supported by the National Natural Science Foundation of China (Grant No 70571017) and the Research Foundation from Provincial Education Department of Zhejiang of China (Grant No 21186000507).
†E-mail: yl_liu@sjtu.edu.cn

delay is close to zero the equilibrium points of Eqs.(2)–(4) are asymptotically stable for $K > 0$. When the delay is not equal to zero a Lyapunov–Razumikhin approach is used to show the asymptotic stability of the equilibrium point of Eq.(4) for sufficiently small K/N. Then, in order to show the stability strictly, the fluid flow model is further simplified and the Hopf bifurcation is also discussed by using the normal form theory and centre manifold theorem.[7]

However, a common defect of these models is that the stable range, i.e. the time delay range guaranteeing that the model is stable, is too narrow, and it is not very suitable for the actual need in internet congestion control. In the present paper we propose a modified version of model (1). Compared with traditional models, the new model has a greater stability domain and its effectiveness for controlling network congestion is proved by theoretical analyses and numerical simulations.

The remainder of the present paper is organized as follows. In Section 2 the rationality of simplifying model (1) is discussed, then a modified model is presented in Section 3. In Section 4, the linear stability is studied and summarized as a theorem by Hopf bifurcation theory, and the numerical simulation result in point is described in Section 5. Finally, some conclusions are drawn from the present study in Section 6.

2. Rationality of the simplification of existing fluid flow models

Models (2)–(4), also consisting of the model studied in Ref.[7], are all obtained from Eq.(1) and the main modification strategy is attributed to the processing method of the time-variant delay $R(t)$ including R and $q(t)/C$. Generally speaking, the propagation delay $R(t)$ is far longer than the queue delay $q(t)/C$.[8] So $q(t)/C$ can be neglected in the differential equation about the change of the window size, and the simplification about the first equation of Eq.(1) is rational. However, in the second equation of Eq.(1), since the change of the queue length is closely related to the queue delay and a trifling variance of the queue will directly affect the probability of the packet mark and even the whole congestion state, the queue delay may not be neglected completely by virtue of the tenet of congestion control.

3. Modified fluid flow model

Based on the discussion of Section 2, we propose a novel fluid flow model as below,

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = \frac{1}{R} - \frac{W(t)^2}{2R} Kq(t-R) \ ,$$
$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = \begin{cases} -C + \dfrac{NW(t-R)}{R+q(t)/C}, & q > 0 \ , \\ \max\left(0, -C + \dfrac{NW(t-R)}{R+q(t)/C}\right), & q = 0. \end{cases} \quad (5)$$

The first differential equation of Eq.(5) is the same as Eq.(4) where the delay R in $R(t-W)$ is neglected, but in the other equations the time-variant delay$R(t)$ is sufficiently considered. Its main difference from traditional models above is that the delay R of the term $W(t-R)$ in the second equation of Eq.(5) is introduced to guarantee the accuracy of the packet mark probability, that is to say, the change of the queue length $q(t)$ is determined by the former window value $W(t-R)$ rather than the current value of $W(t)$. The reason for such a modification is described as follows: when sporadic data flows or noises occur, the window size $W(t)$ increases quickly and transitorily, but generally this case is not considered as a congestion danger (or state), then we can avoid or bring down the incorrect packet mark probability in that the queue variance $q(t)$ is determined by the former window size $W(t-R)$ rather than $W(t)$.

4. Linear stability analysis of the new model

Let the equilibrium point of Eq.(5) be (W^*, q^*) which is determined by $NW^* = RC + q^*$ and $W^{*2}Kq^* = 2$. Usually the queue length $q(t)$ is positive, so Eq.(5) can be linearized around the equilibrium point as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = a_1 x(t) + a_2 y(t-R),$$
$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = b_1 y(t) + b_2 x(t-R), \quad (6)$$

where $x(t) = W(t) - W^*$, $y(t) = q(t) - q^*$, $a_1 = -KW^*q^*/R$, $b_1 = -NW^*/(C(R + q^*/R)^2)$, $a_2 = -K(W^*)^2/(2R)$ and $b_2 = N/(R + q^*/C)$. The stability of Eq.(6) is governed by the real parts (abbr. Re

in this paper) of the roots of the associated characteristic equation given by

$$H(\lambda) = \lambda^2 - (a_1 + b_1)\lambda + a_1 b_1 - a_2 b_2 e^{-2\lambda R} = 0. \quad (7)$$

From Eq.(7), it is obvious that the improved model (5) is stable when $R \to 0$, which is consistent with Hollot and Chait's conclusion about another prior model.[3] Here our conclusion lies in the following analysis: when R is close to zero, the equation set (7) degenerate into $\lambda^2 - (a_1 + b_1)\lambda + a_1 b_1 - a_2 b_2 = 0$. Since a_1, a_2 and b_1 are all negative but b_2 is positive, Eq.(7) has no root with nonnegative real part, which means that the equilibrium point of Eq.(6) is locally asymptotically stable.

From the continuous dependence[13,14] of the roots of Eq.(7) on model parameters it follows that the model (5) is still stable even for a positive R which is small enough; say, there exists an R_0 such that all the roots of Eq.(10) satisfy

$$H(\lambda) = 0, \quad \mathrm{Re}(\lambda) < 0, \text{ for } R \in (0, R_0) \quad (8)$$

and $\mathrm{Re}(\lambda) < 0$ when $R = R_0$. In order to determine this R_0 and the associated purely imaginary root of Eq.(7), inserting $\lambda = \mathrm{i}\,\omega$, $\omega > 0$ into Eq.(7) yields the following equation set:

$$\begin{aligned} -\omega^2 + a_1 b_1 &= a_2 b_2 \cos(2\omega R), \\ -(a_1 + b_1)\omega &= -a_2 b_2 \sin(2\omega R). \end{aligned} \quad (9)$$

Eliminating R from Eq.(9), we obtain

$$\omega = \sqrt{\frac{-(a_1^2 + a_2^2) + \sqrt{(a_1^2 - b_1^2)^2 + 4a_2^2 b_2^2}}{2}} \quad (10)$$

which together with Eq.(9) leads to the following expression:

$$\begin{aligned} R = &\frac{1}{\sqrt{2\sqrt{(a_1^2 - b_1^2)^2 + 4a_2^2 b_2^2} - 2(a_1^2 + a_2^2)}} \\ &\times \left(2k\pi + \cos^{-1}\left(\frac{2a_1 b_1 - (a_1^2 + a_2^2) + \sqrt{(a_1^2 - b_1^2)^2 + 4a_2^2 b_2^2}}{2a_2 b_2}\right)\right), \quad k = 0, 1, 2, \ldots. \end{aligned} \quad (11)$$

We summarize the analysis as below,

Lemma 1 If $R \to 0$, then model (5) is linearly stable around the equilibrium point (W^*, q^*). If $R = R_0$

$$\begin{aligned} = &\frac{1}{\sqrt{2\sqrt{(a_1^2 - b_1^2)^2 + 4a_2^2 b_2^2} - 2(a_1^2 + a_2^2)}} \\ &\times \left(\cos^{-1}\left(\frac{2a_1 b_1 - (a_1^2 + a_2^2) + \sqrt{(a_1^2 - b_1^2)^2 + 4a_2^2 b_2^2}}{2a_2 b_2}\right)\right), \end{aligned}$$

then model (5) becomes unstable and it has periodic solutions with angular frequency

$$\omega_0 = \sqrt{\frac{-(a_1^2 + a_2^2) + \sqrt{(a_1^2 - b_1^2)^2 + 4a_2^2 b_2^2}}{2}}.$$

Next we present another lemma about the last condition of Hopf bifurcation[9−12] of Eq.(5).

Lemma 2 Let $\lambda(R) = \alpha(R) + \mathrm{i}\,\omega(R)$ be the root of Eq.(7) satisfying $\alpha(R_0) = 0$ and $\omega(R_0) = \omega_0$, then $\mathrm{d}\lambda(R)/Dr > 0$.

Proof: Differentiating Eq.(7) with respect to R and using the implicit function theorem, we obtain

$$\begin{aligned} \frac{\mathrm{d}\lambda(R)}{\mathrm{d}R}\,|_{R=R_0} &= \frac{-2a_2 b_2 \lambda \mathrm{e}^{-2\lambda R}}{-a_1 - b_1 + 2\lambda + 2a_2 b_2 R \mathrm{e}^{-2\lambda R}}\,|_{R=R_0} \\ &= \frac{2(a_1 + b_1)^2 \omega_0^2 + 2(a_1 + b_1)(\omega_0^3 + \omega_0 a_1 b_1)\,\mathrm{i}}{((a_1 + b_1) - 2a_2 b_2 R_0 \cos(2\omega_0 R_0))^2 + (2a_2 b_2 R_0 \sin(2\omega_0 R_0))^2}. \end{aligned}$$

Therefore,

$$\mathrm{Re}\left[\frac{\mathrm{d}\lambda(R)}{\mathrm{d}R}\,|_{R=R_0}\right] = \frac{2\,(a_1 + b_1)^2 \omega_0^2}{((a_1 + b_1) - 2a_2 b_2 R_0 \cos(2\omega_0 R_0))^2 + (2a_2 b_2 R_0 \sin(2\omega_0 R_0))^2} > 0,$$

and the proof is completed.

Applying the Hopf bifurcation theorem to delayed differential equations[14−17] and the two lemmas, we obtain

Theorem For a positive delay R_0, the following conclusions about the stability of model Eq.(5) hold:

1) when $R < R_0$, model (5) is locally asymptotically stable;

2) when $R = R_0$, model (5) exhibits a Hopf bifurcation;

3) when $R > R_0$, model (5) is unstable.

5. Simulation results

In this section, we use numerical simulation to verify the analytic results about the stability of Eq.(5). For comparing our model with traditional models, we set the same system parameters as those in Ref.[7], i.e. $N = 50$, $K = 0.001$ and $C = 1000$ packets/s. From Eq.(6) and expression (11), we plot R_0 against R in Fig.1. We find that if $R < 0.257$ s then $R < R_0$, which indicates that mode (5) is stable. In other words, Hopf bifurcation occurs when R reaches the critical value $R_0 = 0.257$ s.

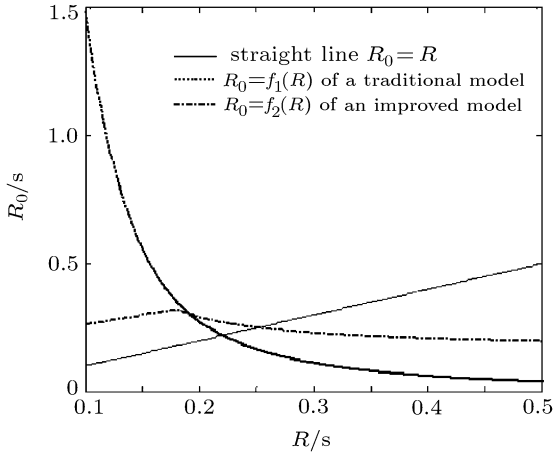


Fig.1. Relationship between R_0 and R.

If we choose $R = 0.256$ s, which is a little less than R_0, from the corresponding formulas in Section 4, we have $W^* = 6.184$, $q^* = 52.299$, $a_1 = -1.2589$, $a_2 = -0.0744$, $b_1 = -3.2342$, $b_2 = 161.7081$, $\omega_0 = 2.6087$ and $R_0 = 0.2571$. Since $R < R_0$, the point (W^*, q^*) of (5) is asymptotically stable as illustrated by waveform and phase plots in Fig.2.

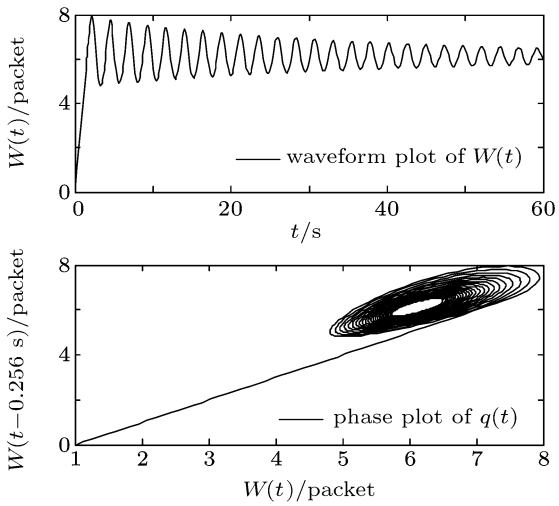


Fig.2. Waveform and phase plots of $W(t)$ with $R = 0.256$ s.

If the delay R increases to the critical value R_0, a Hopf bifurcation occurs, i.e. there are periodic solutions bifurcating out from the equilibrium point (W^*, q^*). For example, if we choose $R = 0.258$ s, then a Hopf bifurcation occurs as shown in Fig.3, which is obviously consistent with the theorem in Section 4. In contrast, if the delay R decreases through the critical value R_0, such as $R = 0.23$ s, Hopf bifurcation cannot occur and the congestion control model (5) is stable. But when R= 0.23 s and with the original fluid flow model in Ref.[7], Hopf bifurcation occurs as shown in Fig.4. These illustrations mean that the modified fluid flow model has a larger stability range for congestion control.

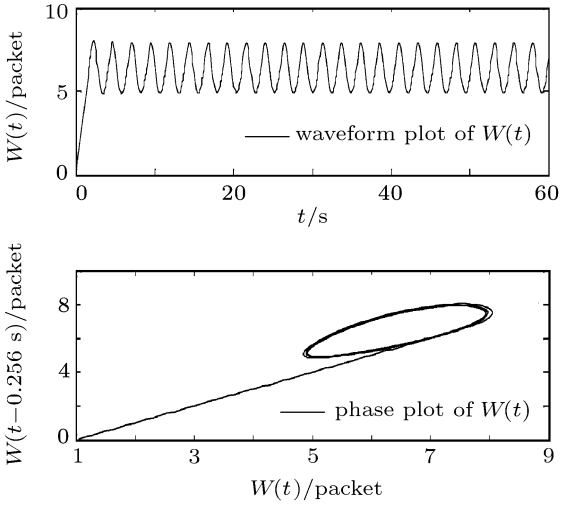


Fig.3. Waveform and phase plot of $W(t)$ with $R = 0.258$ s.

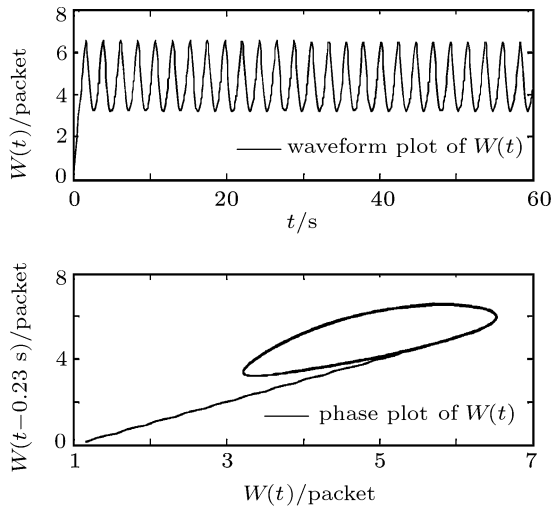


Fig.4. W(t) of the prior model in Ref.[7] with $R = 0.23$ s.

6. Conclusion

In this paper a modified fluid flow model is proposed and the local stability of its linearized system is studied. By combining analytical and numerical tools it is verified that the novel model is stable for a greater time delay than that of previous models, i.e. it yields better congestion control results without using any extra controller.[5] In the meantime, when studying the stability of the new model, we focus only on the linear stability analysis in the limited pages. In order to further obtain the stability characteristics in detail, it is necessary to investigate model (5) by nonlinear methods such as the perturbation method and homotopy analysis method, which is our work for the future.

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