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Robust lag synchronization between two different chaotic systems via dual-stage impulsive control*

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In this paper, an improved impulsive lag synchronization scheme for different chaotic systems with parametric uncertainties is proposed. Based on the new definition of synchronization with error bound and a novel impulsive control scheme (the so-called dual-stage impulsive control), some new and less conservative sufficient conditions are established to guarantee that the error dynamics can converge to a predetermined level, which is more reasonable and rigorous than the existing results. In particular, some simpler and more convenient conditions are derived by taking the same impulsive distances and control gains. Finally, some numerical simulations for the Lorenz system and the Chen system are given to demonstrate the effectiveness and feasibility of the proposed method.

Keywords: chaotic systems, lag synchronization, dual-stage impulsive control, parametric uncertainty

PACC: 0545

1. Introduction

Chaos synchronization has gained extensive attention among scientists from different research fields over the past decade. A variety of control methods have been developed to achieve chaos synchronization between two identical chaotic systems with different initial conditions, which include linear and nonlinear feedback controls,[1,2] backstepping control,[3] sliding mode control,[4] adaptive control,[5−7] active control,[8−10] nonlinear observer approach,[11,12] impulsive control,[13−15] etc. Among them, impulsive synchronization provides an efficient method to deal with the dynamical systems which cannot be controlled by continuous control. In the synchronization process, the response system receives the information from the drive system only at discrete time instants. This drastically reduces the amount of synchronization information transmitted from the drive system to the response system and increases the efficiency of bandwidth usage, which makes this method more efficient and thus useful in a great number of real-life applications.[15,16]

In practice, it is hardly the case that every component of drive and response systems can be assumed to be identical, especially when chaos synchronization is applied to secure communication. Thus, some studies on the synchronization between two different chaotic systems have been performed.[16−23] In Refs.[16] and [17], impulsive synchronization schemes for different chaotic systems with parametric uncertainties were proposed. Based on the new definition of synchronization, the derived criteria can guarantee that the error dynamics converges to a point below a given bound $\xi > 0$. It is worth pointing out that once the error magnitude $\|\boldsymbol{e}\|$ is less than or equal to ξ, the criteria in Refs.[16] and [17] will become unsupported and during the time interval between any two successive impulsive instants, the error may be likely to exceed ξ. Motivated by this, we propose a new definition of synchronization and a novel impulsive control scheme (so-called dual-stage impulsive control) to overcome this possible instability and keep the error below ξ all along in the present paper.

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The contributions of this paper can be summarized as follows: (1) a kind of dual-stage impulsive control scheme on robust lag synchronization of different chaotic systems with different time-varying parametric uncertainties is designed, which is first investigated to the best of our knowledge; (2) compared with the existing results, the conditions of this paper are more reasonable and rigorous, which can provide a more practical framework for the synchronization of multi-perturbation chaotic systems.

Notations: Throughout the paper, R, R^n and $R^{n\times m}$ denote the real number, the n-dimensional Euclidean space and the set of all $n\times m$ real matrices, respectively. $N:=\{1,2,\ldots\}$. $\boldsymbol{I}$ represents the identity matrix with an appropriate dimension. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix 2-norm.

2. Problem statement

Consider the drive system as follows:

$$\dot{\boldsymbol{x}}(t)=(\boldsymbol{A}+\Delta\boldsymbol{A}(t))\boldsymbol{x}(t)+f(\boldsymbol{x}(t)), \tag{1}$$

where $\boldsymbol{x}(t)\in R^n$ is the state vector, $\boldsymbol{A}\in R^{n\times n}$ is a known constant matrix, and continuous nonlinear function $f(\boldsymbol{x}(t))\in R^n$ satisfies the Lipschitz condition with a Lipschitz constant $\sigma>0$, i.e.

$$||f(\boldsymbol{x}(t))-f(\tilde{\boldsymbol{x}}(t))||\le\sigma||\boldsymbol{x}(t)-\tilde{\boldsymbol{x}}(t)||, \tag{2}$$

where $\boldsymbol{x}(t)\in R^n$ and $\tilde{\boldsymbol{x}}(t)\in R^n$.

Due to modelling errors and parametric perturbations, the system parameter may contain uncertainty, which is often time-varying and unknown. Now, we assume that the uncertainty satisfies the following assumptions:

Assumption 1 The parametric uncertainty $\Delta\boldsymbol{A}(t)$ is in the form

$$\Delta\boldsymbol{A}(t)=\boldsymbol{E}\boldsymbol{F}(t)\boldsymbol{H}, \tag{3}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are known real constant matrices with appropriate dimensions, and the uncertain matrix $\boldsymbol{F}(t)\in R^{n\times n}$ satisfies

$$\|\boldsymbol{F}(t)\|\le 1. \tag{4}$$

Assumption 2[16] The parametric uncertainty $\Delta\boldsymbol{A}(t)$ does not destroy the chaotic behaviour of the chaotic system (1).

Considering channel time-delay η, we replace t with $t-\eta$ in drive system (1) and obtain

$$\dot{\boldsymbol{x}}(t-\eta)=(\boldsymbol{A}+\Delta\boldsymbol{A}(t-\eta))\boldsymbol{x}(t-\eta)+f(\boldsymbol{x}(t-\eta)). \tag{5}$$

We consider the response system as follows:

$$\dot{\tilde{\boldsymbol{x}}}(t)=(\tilde{\boldsymbol{A}}+\Delta\tilde{\boldsymbol{A}}(t))\tilde{\boldsymbol{x}}(t)+g(\tilde{\boldsymbol{x}}(t)), \tag{6}$$

where $\tilde{\boldsymbol{x}}(t)\in R^n$ is the state vector, $\tilde{\boldsymbol{A}}\in R^{n\times n}$ is a known constant matrix, and $g(\tilde{\boldsymbol{x}}(t))\in R^n$ is a continuous nonlinear function. The parametric uncertainty $\Delta\tilde{\boldsymbol{A}}(t)$ satisfies the same assumptions as those of $\Delta\boldsymbol{A}(t)$, i.e. $\Delta\tilde{\boldsymbol{A}}(t)=\tilde{\boldsymbol{E}}\tilde{\boldsymbol{F}}(t)\tilde{\boldsymbol{H}}$ and $||\tilde{\boldsymbol{F}}(t)||\le 1$.

Due to the boundedness of the chaotic signals, there exists a constant $\varsigma>0$ such that $\|\tilde{\boldsymbol{x}}\|\le\varsigma$.

Assumption 3[16] For the nonlinear functions $f(\cdot)$ and $g(\cdot)$, there exists a positive constant μ such that the following condition holds:

$$||f(\tilde{\boldsymbol{x}}(t))-g(\tilde{\boldsymbol{x}}(t))||\le\mu||\tilde{\boldsymbol{x}}(t)||. \tag{7}$$

Suppose that a discrete instant set $\{t_k\}$ satisfies $0<t_1<t_2<\ldots<t_k<t_{k+1}<\ldots$, $\lim\limits_{k\to\infty}t_k=\infty$, $k\in N, 0\le t_0<t_1$. At discrete time t_k, the state variables of the drive system are transmitted to the response system as the control input, so that the state variables of the response system are suddenly changed at these time instants. Therefore, the response system can be written in the following form:

$$\begin{aligned}
\dot{\tilde{\boldsymbol{x}}}(t) &= (\tilde{\boldsymbol{A}}+\Delta\tilde{\boldsymbol{A}}(t))\tilde{\boldsymbol{x}}(t)+g(\tilde{\boldsymbol{x}}(t)),\ t\ne t_k,\\
\Delta\tilde{\boldsymbol{x}}(t_k) &= \tilde{\boldsymbol{x}}(t_k^+)-\tilde{\boldsymbol{x}}(t_k^-)\\
&= \tilde{\boldsymbol{x}}(t_k^+)-\tilde{\boldsymbol{x}}(t_k)\\
&= -\boldsymbol{B}_k\boldsymbol{e}(t_k),\ t=t_k,\ k\in N,\\
\tilde{\boldsymbol{x}}(t_0^+) &= \tilde{\boldsymbol{x}}_0,
\end{aligned} \tag{8}$$

where $\boldsymbol{e}(t)=\boldsymbol{x}(t-\eta)-\tilde{\boldsymbol{x}}(t)=[x_1(t-\eta)-\tilde{x}_1(t),x_2(t-\eta)-\tilde{x}_2(t),\ldots,x_n(t-\eta)-\tilde{x}_n(t)]^{\mathrm{T}}$ is the synchronization error, $\lim\limits_{t\to t_k^+}\tilde{\boldsymbol{x}}(t)=\tilde{\boldsymbol{x}}(t_k^+)$, $\lim\limits_{t\to t_k^-}\tilde{\boldsymbol{x}}(t)=\tilde{\boldsymbol{x}}(t_k^-)$, $\tilde{\boldsymbol{x}}(t_k)=\tilde{\boldsymbol{x}}(t_k^-)$ implies that $\tilde{\boldsymbol{x}}(t)$ is left-continuous at $t=t_k$, $\boldsymbol{B}_k\in R^{n\times n}$ is the impulsive control gain, and $\Delta\boldsymbol{x}(t_k)=\boldsymbol{x}(t_k^+)-\boldsymbol{x}(t_k^-)=0$. Then from Eqs.(5) and (8), the following error system is obtained:

$$\begin{aligned}
\dot{\boldsymbol{e}}(t) &= (\boldsymbol{A}+\Delta\boldsymbol{A}(t-\eta))\boldsymbol{e}(t)+h(t)\\
&\quad +(\boldsymbol{A}-\tilde{\boldsymbol{A}}+\Delta\boldsymbol{A}(t-\eta)-\Delta\tilde{\boldsymbol{A}}(t))\tilde{\boldsymbol{x}}(t)\\
&\quad +f(\tilde{\boldsymbol{x}}(t))-g(\tilde{\boldsymbol{x}}(t)),\ t\ne t_k,\ k\in N,\\
\Delta\boldsymbol{e}|_{t=t_k} &= \boldsymbol{e}(t_k^+)-\boldsymbol{e}(t_k)=\boldsymbol{B}_k\boldsymbol{e}(t_k),\ t=t_k,\\
\boldsymbol{e}(t_0^+) &= \boldsymbol{e}_0,
\end{aligned} \tag{9}$$

where $h(t):=f(\boldsymbol{x}(t-\eta))-f(\tilde{\boldsymbol{x}}(t))$.

Note that the origin $\boldsymbol{e}(t)=0$ is not the equilibrium point of the error system (9), which implies that

it is impossible to achieve complete synchronization[24] between systems (5) and (8). Therefore, the following definition of synchronization with error bound ξ_1 is employed.

Definition 1 The synchronization between systems (5) and (8) is said to be achieved with error bound ξ_1 if, for arbitrary initial conditions $\boldsymbol{x}_0$ and $\tilde{\boldsymbol{x}}_0$, the trivial solution of the error system (9) converges to a predetermined neighbourhood of the origin for any admissible parametric uncertainty that satisfies assumptions 1 and 2, i.e. there exist $\xi_1 > 0$ and $T_1 > 0$ such that synchronization error $\boldsymbol{e}(t)$ converges to the region $\Upsilon_1 := \{\boldsymbol{e} \in R^n : \|\boldsymbol{e}\| \leq \xi_1\}$ for all $t \geq T_1$.

Here our goal is to set the control gain $\boldsymbol{B}_k$ and the impulsive distance $\delta_k = t_k - t_{k-1}$ $(k \in N)$ such that the error magnitude $\|\boldsymbol{e}\|$ converges to a neighbourhood of the origin with a given radius ξ_1 (i.e. the region Υ_1) if the error starts with $\|\boldsymbol{e}\| > \xi_1$, which implies the impulsive controlled response system (8) is synchronized with the drive system (5) for arbitrary initial conditions.

In Refs.[16] and [17], based on the theory of Lyapunov stability and impulsive differential equations, one concludes that the synchronization error $\boldsymbol{e}(t)$ can enter the region Υ_1 if the error starts with $\|\boldsymbol{e}\| > \xi_1$ by using the impulsive controller $(\boldsymbol{B}_k, \delta_k)$ (the corresponding results can also be seen in the first half part of the proof of Theorem 1 subsequently). However, theoretically speaking, after synchronization error $\boldsymbol{e}(t)$ enters the region Υ_1, the designed impulsive controller $(\boldsymbol{B}_k, \delta_k)$ becomes unsupported. During the time interval δ_k between any two successive impulsive instants as shown in Fig.1, $\boldsymbol{e}(t)$ may be likely to go away from Υ_1. To overcome this possible instability and keep $\boldsymbol{e}(t)$ within Υ_1 all along, in this paper, we introduce another region $\Upsilon_2 := \{\boldsymbol{e} \in R^n : \|\boldsymbol{e}\| \leq \xi_2\}$, where $0 < \xi_2 < \xi_1$. As shown in Fig.2, under the control gain $\boldsymbol{B}_k$, if error $\boldsymbol{e}(t)$ starts with $\|\boldsymbol{e}\| > \xi_1$, we choose impulsive distance δ_{1k} such that $\boldsymbol{e}(t)$ enters Υ_1. If error $\boldsymbol{e}(t)$ enters Υ_1, we choose impulsive distance δ_{2k} such that $\boldsymbol{e}(t)$ enters Υ_2. After that, during the time interval δ_{2k} between any two successive impulsive instants, $\boldsymbol{e}(t)$ may be likely to go away from Υ_2, if we can ensure that the maximum value of $\|\boldsymbol{e}\|$ is not larger than ξ_1 by choosing appropriate values of $\boldsymbol{B}_k$, δ_{1k} and δ_{2k}, then our goal is achieved. This control scheme is so-called dual-stage impulsive control and the corresponding controller $(\boldsymbol{B}_k, \delta_{1k}, \delta_{2k})$ can be described as follows:

$$\boldsymbol{B}_k \text{ and } \delta_k = \begin{cases} \delta_{1k}, & \text{if } \|\boldsymbol{e}\| > \xi_1, \\ \delta_{2k}, & \text{if } \|\boldsymbol{e}\| \leq \xi_1. \end{cases} \tag{10}$$

For simplicity, in the following sections, $\boldsymbol{x}(t-\eta)$, $\tilde{\boldsymbol{x}}(t)$, $h(t)$, $\Delta\boldsymbol{A}(t-\eta)$, $\Delta\tilde{\boldsymbol{A}}(t)$ are denoted by $\boldsymbol{x}$, $\tilde{\boldsymbol{x}}$, h, $\Delta\boldsymbol{A}$, $\Delta\tilde{\boldsymbol{A}}$, respectively.

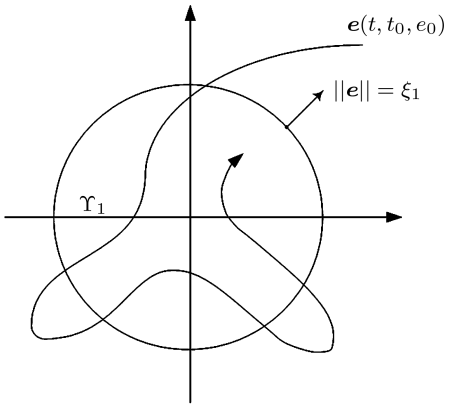


Fig.1. Geometric representation of the portrait of synchronization error via impulsive controller $(\boldsymbol{B}_k, \delta_k)$.

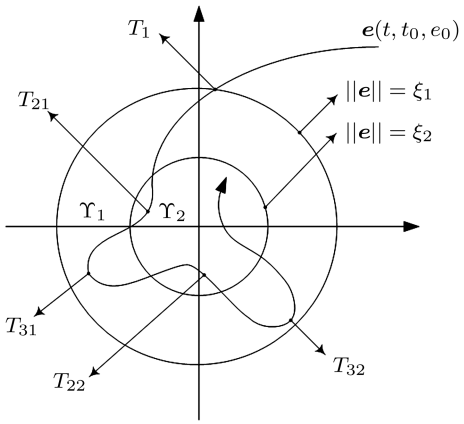


Fig.2. Geometric representation of the portrait of synchronization error via dual-stage impulsive controller $(\boldsymbol{B}_k, \delta_{1k}, \delta_{2k})$.

3. Main results

The following theorem gives the sufficient conditions for robust stability of the error system (9), which implies that the impulsive controlled response system (8) is synchronized with the drive system (5) with error bound ξ_1.

Theorem 1 If the control gain and the impulsive distance are chosen as those in expression (10) and there exist constants $\rho > 1$ and $\xi_2 > 0$ such that

$$\ln(\rho\beta_k) + \theta_m\delta_{mk} \leq 0,\ k \in N,\ m = 1, 2, \tag{11}$$

$$\alpha_1 \exp(\alpha_2\bar{\delta}_2) \leq \xi_1, \tag{12}$$

then the response system (8) is synchronized with the drive system (5) with error bound ξ_1, where $\theta_m = \lambda_A + 2\sigma + 2\|\boldsymbol{E}\|\|\boldsymbol{H}\| + 2\xi_m^{-1}\vartheta$, $\alpha_1 = \xi_2 + \vartheta\bar{\delta}_2$, $\alpha_2 = \|\boldsymbol{A}\| + \|\boldsymbol{E}\|\|\boldsymbol{H}\| + \sigma$, $\vartheta = \varsigma(\left\|\boldsymbol{A} - \tilde{\boldsymbol{A}}\right\| + \|\boldsymbol{E}\|\|\boldsymbol{H}\| + \left\|\tilde{\boldsymbol{E}}\right\|\left\|\tilde{\boldsymbol{H}}\right\| + \mu)$, $\bar{\delta}_2 = \sup_{k\in N}\{\delta_{2k}\}$, β_k and λ_A are the largest eigenvalues of $(\boldsymbol{I}+\boldsymbol{B}_k)^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_k)$ and $\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}}$, respectively.

Proof Choose the following Lyapunov function:

$$V(\boldsymbol{e}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}. \tag{13}$$

For $t \in (t_{k-1}, t_k]$, $k \in N$, the time derivative of $V(\boldsymbol{e})$ along the solution of (9) is

$$\begin{aligned}\dot{V}(\boldsymbol{e}(t)) &= [(\boldsymbol{A}+\Delta\boldsymbol{A})\boldsymbol{e} + h + (\boldsymbol{A} - \tilde{\boldsymbol{A}} + \Delta\boldsymbol{A} - \Delta\tilde{\boldsymbol{A}})\tilde{\boldsymbol{x}} + f(\tilde{\boldsymbol{x}}) - g(\tilde{\boldsymbol{x}})]^{\mathrm{T}}\boldsymbol{e} \\ &\quad + \boldsymbol{e}^{\mathrm{T}}[(\boldsymbol{A}+\Delta\boldsymbol{A})\boldsymbol{e} + h + (\boldsymbol{A} - \tilde{\boldsymbol{A}} + \Delta\boldsymbol{A} - \Delta\tilde{\boldsymbol{A}})\tilde{\boldsymbol{x}} + f(\tilde{\boldsymbol{x}}) - g(\tilde{\boldsymbol{x}})] \\ &= \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}})\boldsymbol{e} + 2\boldsymbol{e}^{\mathrm{T}}h + \boldsymbol{e}^{\mathrm{T}}(\Delta\boldsymbol{A} + \Delta\boldsymbol{A}^{\mathrm{T}})\boldsymbol{e} + 2\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{A} - \tilde{\boldsymbol{A}} + \Delta\boldsymbol{A} - \Delta\tilde{\boldsymbol{A}})\tilde{\boldsymbol{x}} + 2\boldsymbol{e}^{\mathrm{T}}(f(\tilde{\boldsymbol{x}}) - g(\tilde{\boldsymbol{x}})) \\ &\le (\lambda_A + 2\sigma + 2\|\boldsymbol{E}\|\|\boldsymbol{H}\|)\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} + 2\vartheta\|\boldsymbol{e}\|. \end{aligned} \tag{14}$$

To make the error enter the region Υ_1, i.e. $\|\boldsymbol{e}\| \le \xi_1$, here we are concerned only with the case of $\|\boldsymbol{e}\| > \xi_1$, which implies that $\|\boldsymbol{e}\| < \xi_1^{-1}\|\boldsymbol{e}\|^2 = \xi_1^{-1}V(\boldsymbol{e}(t))$, and therefore expression (14) further becomes

$$\begin{aligned}\dot{V}(\boldsymbol{e}(t)) &\le \left(\lambda_A + 2\sigma + 2\|\boldsymbol{E}\|\|\boldsymbol{H}\| + 2\xi_1^{-1}\vartheta\right)V(\boldsymbol{e}(t)) \\ &= \theta_1 V(\boldsymbol{e}(t)), \end{aligned} \tag{15}$$

which implies that

$$V(\boldsymbol{e}(t)) \le V(\boldsymbol{e}(t_{k-1}^+))\exp[\theta_1(t - t_{k-1})], \quad t \in (t_{k-1}, t_k],\ k \in N. \tag{16}$$

On the other hand, from the second equation of Eq.(9), we obtain

$$\begin{aligned}V(\boldsymbol{e}(t_k^+)) &= [(\boldsymbol{I}+\boldsymbol{B}_k)\boldsymbol{e}(t_k)]^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_k)\boldsymbol{e}(t_k) \\ &= \boldsymbol{e}^{\mathrm{T}}(t_k)(\boldsymbol{I}+\boldsymbol{B}_k)^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_k)\boldsymbol{e}(t_k) \\ &\le \beta_k V(\boldsymbol{e}(t_k)). \end{aligned} \tag{17}$$

Thus, letting $k = 1$ in the inequality (16), we have

$$V(\boldsymbol{e}(t)) \le V(\boldsymbol{e}(t_0^+))\exp[\theta_1(t - t_0)],\ t \in (t_0, t_1],$$

which leads to

$$V(\boldsymbol{e}(t_1)) \le V(\boldsymbol{e}(t_0^+))\exp[\theta_1(t_1 - t_0)],$$

and from expression (17), we further have

$$\begin{aligned}V(\boldsymbol{e}(t_1^+)) &\le \beta_1 V(e(t_1)) \\ &\le V(\boldsymbol{e}(t_0^+))\beta_1\exp[\theta_1(t_1 - t_0)].\end{aligned}$$

Similarly, for $t \in (t_1, t_2]$,

$$\begin{aligned}V(\boldsymbol{e}(t)) &\le V(\boldsymbol{e}(t_1^+))\exp[\theta_1(t - t_1)] \\ &\le V(\boldsymbol{e}(t_0^+))\beta_1\exp[\theta_1(t - t_0)].\end{aligned}$$

In general, for any $t \in (t_k, t_{k+1}]$, from inequality (11) with $m = 1$, we have

$$\begin{aligned}V(\boldsymbol{e}(t)) &\le V(\boldsymbol{e}(t_0^+))\beta_1\beta_2\ldots\beta_k\exp[\theta_1(t - t_0)] \\ &\le V(\boldsymbol{e}_0)\beta_k\exp[\theta_1(t_k - t_{k-1})]\beta_{k-1}\exp[\theta_1(t_{k-1} - t_{k-2})]\ldots \\ &\quad \times\beta_1\exp[\theta_1(t_1 - t_0)]\exp[\theta_1(t - t_k)] \\ &= V(\boldsymbol{e}_0)\beta_k\exp(\theta_1\delta_{1k})\beta_{k-1}\exp(\theta_1\delta_{1(k-1)})\ldots\times\beta_1\exp(\theta_1\delta_{11})\exp[\theta_1(t - t_k)] \\ &\le \frac{V(e_0)}{\rho^k}\exp[\theta_1(t - t_k)] \\ &\le \begin{cases}\dfrac{V(\boldsymbol{e}_0)}{\rho^k}\exp(\theta_1\delta_{1(k+1)}), & \text{for } \theta_1 > 0, \\ \dfrac{V(\boldsymbol{e}_0)}{\rho^k}, & \text{for } \theta_1 \le 0.\end{cases}\end{aligned} \tag{18}$$

It is easy to see that if the error starts with $\|\boldsymbol{e}\| > \xi_1$, the synchronization error can asymptotically enter Υ_1. By a similar analytical procedure (let $m = 2$ in the above analysis), it is easy to conclude that under the impulsive distance δ_{2k} and control gain $\boldsymbol{B}_k$, the error can asymptotically enter Υ_2 if the error magnitude satisfies $\xi_2 < \|\boldsymbol{e}\| \leq \xi_1$, and the relevant proof is omitted for avoiding unnecessary redundancies.

To keep the synchronization error $\boldsymbol{e}(t)$ within Υ_1 in the case of error going away from Υ_2, the appropriate control gain $\boldsymbol{B}_k$ and impulsive distance δ_{2k} have to be set. In the following, we give the sufficient condition to ensure that the maximum value of $\|\boldsymbol{e}\|$ does not exceed ξ_1 even if the synchronization error goes away from Υ_2.

Firstly, in the process of the synchronization error entering Υ_1, we define three special time instants (cf. Fig.2):

(1) T_1, which satisfies $\|\boldsymbol{e}(t)\| \leq \xi_1$ for $t \geq T_1$, especially, $\|\boldsymbol{e}(T_1)\| = \xi_1$.

(2) T_{2j}, the previous impulsive instant before $\boldsymbol{e}(t)$ goes away from Υ_2 for the jth time, which satisfies $\left\|\boldsymbol{e}(T_{2j}^+)\right\| \leq \xi_2$, $j \in N$.

(3) T_{3j}, the next impulsive instant after $\boldsymbol{e}(t)$ goes away from Υ_2 for the jth time, which satisfies $\xi_2 < \|\boldsymbol{e}(T_{3j})\| \leq \xi_1$, $j \in N$.

Obviously, we can obtain $T_{3j} - T_{2j} \leq \bar{\delta}_2$.

Considering the first equation of system (9) as follows:

$$\begin{aligned}\dot{\boldsymbol{e}}(t) =\ & (\boldsymbol{A} + \Delta\boldsymbol{A}(t-\eta))\boldsymbol{e}(t) + h(t) \\ & +(\boldsymbol{A} - \tilde{\boldsymbol{A}} + \Delta\boldsymbol{A}(t-\eta) - \Delta\tilde{\boldsymbol{A}}(t))\tilde{x}(t) \\ & +f(\tilde{\boldsymbol{x}}(t)) - g(\tilde{\boldsymbol{x}}(t)). \end{aligned} \tag{19}$$

Integrating Eq.(19) from T_{2j} to t, where $t \in (T_{2j}, T_{3j}]$, yields

$$\begin{aligned}\boldsymbol{e}(t) =\ & \boldsymbol{e}(T_{2j}) - \int_{T_{2j}}^{t} [(\boldsymbol{A} + \Delta\boldsymbol{A}(s-\eta))\boldsymbol{e}(s) + h(s) \\ & +(\boldsymbol{A} - \tilde{\boldsymbol{A}} + \Delta\boldsymbol{A}(s-\eta) - \Delta\tilde{\boldsymbol{A}}(s))\tilde{\boldsymbol{x}}(s) \\ & +f(\tilde{\boldsymbol{x}}(s)) - g(\tilde{\boldsymbol{x}}(s))]\mathrm{d}s. \end{aligned} \tag{20}$$

Taking the norm on both sides of expression (20) gives

$$\begin{aligned} & \|\boldsymbol{e}(t)\| \\ \leq\ & \xi_2 + \int_{T_{2j}}^{t} [(\|\boldsymbol{A}\| + \|\boldsymbol{E}\|\,\|\boldsymbol{H}\| + \sigma)\,\|\boldsymbol{e}(s)\| \\ & +\varsigma\left(\left\|\boldsymbol{A} - \tilde{\boldsymbol{A}}\right\| + \|\boldsymbol{E}\|\,\|\boldsymbol{H}\| + \left\|\tilde{\boldsymbol{E}}\right\|\left\|\tilde{\boldsymbol{H}}\right\| + \mu\right)\Big]\,\mathrm{d}s \\ \leq\ & \alpha_1 + \alpha_2 \int_{T_{2j}}^{t} \|\boldsymbol{e}(s)\|\,\mathrm{d}s. \end{aligned} \tag{21}$$

By using the Grownwell–Bellman inequality,[25] it follows from inequalities (12) and (21) that

$$\begin{aligned}\|\boldsymbol{e}(t)\| &\leq \alpha_1 \exp[\alpha_2(t - T_{2j})] \\ &\leq \alpha_1 \exp[\alpha_2(T_{3j} - T_{2j})] \\ &= \alpha_1 \exp(\alpha_2 \bar{\delta}_2) \leq \xi_1,\ j \in N. \end{aligned} \tag{22}$$

For $t \in (T_{3j}, T_{2(j+1)}]$, $j \in N$, it is shown by the first half part of this proof that the synchronization error enters Υ_2 again. Thus, it is clear that $\|\boldsymbol{e}(t)\| \leq \xi_1$, $\forall t \geq T_1$. Therefore, we can conclude that the synchronization error $\boldsymbol{e}(t)$ can converge to the region Υ_1 if $\boldsymbol{e}(t)$ starts with $\|\boldsymbol{e}(t)\| > \xi_1$, which implies that the response system (8) is synchronized with the drive system (5) with error bound ξ_1. The proof is completed here.

In practice, for the sake of convenience, the gain matrix $\boldsymbol{B}_k$ is always selected as a constant matrix and the impulsive distances δ_{1k} and δ_{2k} are set to be positive constants, then we have the following corollary straightforwardly.

Corollary 1 Assume that $\delta_{1k} = \delta_1$, $\delta_{2k} = \delta_2$ and $\boldsymbol{B}_k = \boldsymbol{B}$ $(k \in N)$. If there exist constants $\rho > 1$ and $\xi_2 > 0$ such that the following conditions hold:

$$\ln(\rho\beta) + \theta_m \delta_m \leq 0,\ m = 1, 2, \tag{23}$$

$$\alpha_1 \exp(\alpha_2 \delta_2) \leq \xi_1, \tag{24}$$

then the response system (8) is synchronized with the drive system (5) with error bound ξ_1, where β is the largest eigenvalue of $(\boldsymbol{I} + \boldsymbol{B})^{\mathrm{T}}(\boldsymbol{I} + \boldsymbol{B})$. ϑ, α_1, α_2 and θ_m are defined as those in Theorem 1.

Remark 1 Note that the dual-stage impulsive control scheme in Corollary 1 can be described as

$$\boldsymbol{B}_k = \boldsymbol{B},\ \delta_k = \begin{cases} \delta_1, & \text{if } \|\boldsymbol{e}\| > \xi_1, \\ \delta_2, & \text{if } \|\boldsymbol{e}\| \leq \xi_1, \end{cases} \tag{25}$$

which has fixed control gain and switched impulsive distance. Alternatively, we can also use another version of the dual-stage impulsive control scheme with fixed impulsive distance and switched control gain, which can be described as follows:

$$\delta_k = \delta,\ \boldsymbol{B}_k = \begin{cases} \boldsymbol{B}_{1k} = \boldsymbol{B}_1, & \text{if} \quad \|\boldsymbol{e}\| > \xi_1, \\ \boldsymbol{B}_{2k} = \boldsymbol{B}_2, & \text{if} \quad \|\boldsymbol{e}\| \leq \xi_1, \end{cases} \tag{26}$$

where $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are the constant control gain matrices. The analytical procedure is similar to those of Theorem 1 and Corollary 1. In order to avoid unnecessary redundancies, the relevant details are omitted.

Remark 2 To achieve synchronization between different chaotic systems, complicated continuous controllers are often required to compensate the structure mismatch.[18−23] Compared with continuous control, impulsive control studied in the present paper is realized just by linear feedback of the error signal and only the state information at discrete time instants is needed, which is more efficient, secure and thus useful in a great number of applications, especially for secure communication.

4. Simulation results

In the simulation, the impulsive distances δ_{1k} and δ_{2k} are set to be positive constants δ_1 and δ_2, respectively, and the gain matrix $\boldsymbol{B}_k$ is selected as a constant matrix, i.e. $\boldsymbol{B}_k = \boldsymbol{B} = \mathrm{diag}\{d, d, d\}$, and then $\beta = (d+1)^2$.

In order to observe the synchronization behaviour in different chaotic systems, we assume that the Lorenz system[26] drives the Chen system.[27] Therefore, the drive and the response systems are as follows:

$$\begin{aligned} \dot{x}_1 &= a_1(x_2 - x_1), \\ \dot{x}_2 &= a_2 x_1 - x_1 x_3 - x_2, \\ \dot{x}_3 &= x_1 x_2 - a_3 x_3 \end{aligned} \tag{27}$$

and

$$\begin{aligned} \dot{\tilde{x}}_1 &= b_1(\tilde{x}_2 - \tilde{x}_1), \\ \dot{\tilde{x}}_2 &= (b_2 - b_1)\tilde{x}_1 - \tilde{x}_1\tilde{x}_3 + b_2\tilde{x}_2, \\ \dot{\tilde{x}}_3 &= \tilde{x}_1\tilde{x}_2 - b_3\tilde{x}_3. \end{aligned} \tag{28}$$

From systems (9), (27) and (28), the following compact form of the error system is obtained:

$$\begin{aligned} &\dot{\boldsymbol{e}} = (\boldsymbol{A} + \Delta\boldsymbol{A})\boldsymbol{e} + h + (\boldsymbol{A} - \tilde{\boldsymbol{A}} + \Delta\boldsymbol{A} - \Delta\tilde{\boldsymbol{A}})\tilde{\boldsymbol{x}} \\ &\qquad + f(\tilde{\boldsymbol{x}}) - g(\tilde{\boldsymbol{x}}),\ t \neq t_k,\ k \in N, \\ &\Delta\boldsymbol{e}|_{t=t_k} = \boldsymbol{e}(t_k^+) - \boldsymbol{e}(t_k) = \boldsymbol{B}_k\boldsymbol{e}(t_k),\ t = t_k, \\ &\boldsymbol{e}(t_0^+) = \boldsymbol{e}_0, \end{aligned} \tag{29}$$

Let $a_1 = 10$, $a_2 = 28$, $a_3 = 8/3$, $b_1 = 35$, $b_2 = 28$ and $b_3 = 3$. Systems (27) and (28) are chaotic for the above parameters. Then we have

$$\boldsymbol{A} = \begin{pmatrix} -a_1 & a_1 & 0 \\ a_2 & -1 & 0 \\ 0 & 0 & -a_3 \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix},$$

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} -b_1 & b_1 & 0 \\ b_2 - b_1 & b_2 & 0 \\ 0 & 0 & -b_3 \end{pmatrix} = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix},$$

$$\Delta\boldsymbol{A} = -\Delta\tilde{\boldsymbol{A}} = 0.04\begin{pmatrix} \sin t & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & \sin t \end{pmatrix},$$

where

$$\boldsymbol{E} = \boldsymbol{H} = \tilde{\boldsymbol{E}} = \tilde{\boldsymbol{H}} = 0.2\boldsymbol{I},$$

$$\boldsymbol{F}(t) = -\tilde{\boldsymbol{F}}(t) = \begin{pmatrix} \sin t & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & \sin t \end{pmatrix}.$$

From $f(\tilde{\boldsymbol{x}}) = g(\tilde{\boldsymbol{x}}) = (0, -\tilde{x}_1\tilde{x}_3, \tilde{x}_1\tilde{x}_2)^{\mathrm{T}}$, we have $\mu = 0$.

In this simulation, we choose $\sigma = 27.5$ and $\varsigma = 60$. By simple computation, we can obtain $\lambda_A = 28.0512$, $\|\boldsymbol{A} - \tilde{\boldsymbol{A}}\| = 57.5257$, and $\|\boldsymbol{E}\|\|\boldsymbol{H}\| = \|\tilde{\boldsymbol{E}}\|\|\tilde{\boldsymbol{H}}\| = 0.04$. Let $\rho = 1.1$, $\xi_1 = 0.5$, $\xi_2 = 0.1022$, and $d = -0.98$, the 'unknown' channel time-delay $\eta = 3$ s, the initial conditions of the drive and response systems are taken as $\boldsymbol{x}_0 = (4, 1, 3)^{\mathrm{T}}$ and $\tilde{x}_0 = (3, -1, 5)^{\mathrm{T}}$, respectively. Hence, from inequality (23) in Corollary 1, the estimate of the bound of the stable region is given by

$$0 < \delta_1 \leq -\frac{\ln(\rho\beta)}{\theta_1} = 5.5568 \times 10^{-4}.$$

From inequality (24), we further obtain $\delta_2 = 1.1412 \times 10^{-4}$. Robust impulsive lag synchronization between the Lorenz system and the Chen system with impulsive distances $\delta_1 = 5.5\times10^{-4}$ and $\delta_2 = 1.1\times10^{-4}$ is shown in Fig.3. As can be seen, the synchronization has been achieved practically and $\|\boldsymbol{e}\|$ is smaller than $\xi_1 = 0.5$.

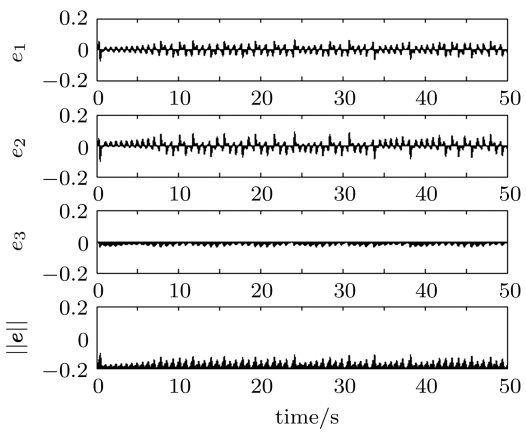


Fig.3. Synchronization errors and their magnitude with $\delta_1 = 5.5 \times 10^{-4}$ and $\delta_2 = 1.1 \times 10^{-4}$.

5. Conclusions

In this paper, we have investigated the issue of robust lag synchronization for different chaotic systems with different parametric uncertainties by using the dual-stage impulsive control method. Based on the new definition of synchronization with error bound and the theory of impulsive differential equations, some new and less conservative sufficient conditions are established. Finally, numerical simulations are given to demonstrate the effectiveness of the method. It should be pointed out that the method can also be applied to most typical chaotic systems that can be described by Eq.(1), and inequalities (2) and (7), such as the Rössler system, the Lü system, the unified system, the Liu system, several variants of Chua's circuits, *etc.* Since parameter mismatch caused by different structures, different parametric uncertainties and other internal (or external) perturbations are ubiquitous in both natural and man-made systems, complete synchronization is always impossible and unnecessary. Thus, for the synchronization of multi-perturbation chaotic systems, the idea and the approach developed in this paper can provide a more practical framework by keeping the synchronization error within a reasonable region, which is significant for applications of creating secure communication systems.

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