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A viscous continuum traffic flow model with consideration of the coupling effect for two-lane freeways*

Sun Di-Hua (孙棣华) and Peng Guang-Han(彭光含)†

College of Automation, Chongqing University, Chongqing 400030, *China*

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In this paper, the viscous continuum traffic flow model for a single lane is extended to the traffic flow for two-lane freeways. The proposed model is a higher-order continuum model considering the coupling and lane changing effects of the vehicles on two adjacent lanes. It results from integrating the Taylor series expansion of the viscous continuum traffic flow model proposed by Ge (2006 *Physica* A **371** 667) into the multi-lane model presented by Daganzo (1997 *Transpn. Res.* B **31** 83). Our proposed model may be used to describe non-anisotropic behaviour because of lane changing in multi-lane traffic. A linear stability analysis is given and the neutral stability condition is obtained. Also, issues related to lane changing, shock waves and rarefaction waves, local clustering and phase transition are investigated through a simulation experiment. The simulation results show that the proposed model is capable of explaining some particular traffic phenomena commonly observable in real world traffic flow.

Keywords: two-lane traffic, viscous continuum traffic flow model, numerical simulation
PACC: 0520, 5110, 0547

1. Introduction

LWR theory is the earliest and most fundamental first-order continuum traffic model proposed by Lighthill and Whitham[1] and Richards[2] independently. According to this theory, traffic flow on a long homogeneous freeway is governed by the following conservation equation:

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = s(x,t), \tag{1}$$

where k is the traffic density, q is the traffic flow rate, $s(x,t)$ is the flow generation rate, while x and t are the space and the time, respectively. If the freeway has no on-off ramp, then $s(x,t) = 0$; otherwise, $s(x,t) \neq 0$. The equivalent flow–density relationship between flow rate q and density k is

$$q = ku. \tag{2}$$

For the speed u, as in the LWR, the existence of an equilibrium speed–density relationship is assumed to be

$$u = u_{\rm e}(k). \tag{3}$$

According to the above three equations, analytical solutions can be obtained. Some traffic flow problems such as shock waves and traffic jams can be addressed from these solutions.

However, it is impossible for the LWR model to describe traffic flow in nonequilibrium states correctly, for the speed is assumed to be always determined by the equilibrium speed-density Eq.(3). Therefore, high-order continuum traffic flow models, which incorporate a dynamic equation or momentum equation to represent general car-following behaviours, have been developed by many scholars.[3−11] In this momentum equation the acceleration and the inertia of driving are taken into account. Hence, the high-order models make up for some deficiencies in the simple continuum models and improve their ability to reproduce complex traffic behaviours.

Nevertheless, the existing high-order models have a fundamental flaw that one of the characteristic speeds resulting from the momentum and conservation equations is larger than the macroscopic flow speed, which implies that vehicles would affect the vehicles in front of them.[12] This evidently violates the anisotropy of traffic flow and does not accord with real traffic.

On the basis of their full velocity difference

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†E-mail: pengguanghan@yahoo.com.cn, d3sun@163.com

(FVD) car-following model,[13] which is one of the car-following models presented with the consideration of velocity difference,[13−22] Jiang *et al*[12] proposed a new dynamic equation in which the density gradient term is replaced by a speed gradient term. Recently, Jiang and Wu[23] further analysed the structural properties of the solutions to the speed gradient traffic flow model, which showed that the characteristic speed of the model was not larger than the macroscopic flow speed. Hence, vehicles respond only to their frontal stimulus. But the propagation speed of disturbance, c_0, in this model is a constant, independent of density. In fact, the propagation speed of disturbance in a Payne–Whitham-like model should depend on density.[24,25] Subsequently, Xue and Dai[26] improved the model by treating c_0 as a function of density, which indicates that the undesirable 'wrong-way travel' phenomenon and gas-like behaviour have been eliminated and the formation and diffusion of traffic shock can be better simulated in their model. However, Zhang[27] pointed out that the anisotropic property could not be expected to hold in multi-lane traffic. After this, Ge and Han[28] introduced a density viscous continuum traffic flow model, which shows that the model is consistent with the situation of multi-lane traffic flow. In addition, other different kinds of models were developed to describe traffic.[29−39]

In the above models, the following car is not allowed to overtake the leading car. Hence, these models are applicable only to traffic flow on freeways with a single lane. Daganzo[40] developed a continuum theory of traffic dynamics for freeways with two lanes allocated for two sorts of vehicles (lane 1 for faster vehicles and lane 2 for slower vehicles). The theory defines the conservation equation as

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = S, \tag{4}$$

where $P = (k_1, k_2)^{\mathrm{T}}$, $Q = [q_1(k_1, k_2), q_2(k_1, k_2)]^{\mathrm{T}}$, $S = (s_{12} - s_{21},\ s_{21} - s_{12})^{\mathrm{T}}$, k_m and q_m represent the traffic density and the flow rate on lane m ($m = 1, 2$), respectively, s_{mn} represents the lane changing rate from lane m to lane n. Daganzo first studied the possible relationships between the speed and density based on the density ratio of the two lanes, then formulated the traffic flow model for each of a total of four $k_1 - k_2$ regions. To carry out analysis graphically, Daganzo adopted the simple continuum model only; hence his approach cannot be used to describe non-equilibrium traffic flow dynamics. Another multi-lane traffic model, considering the specific traffic conditions in Chinese cities (e.g. untidy flows and lower speeds on roads), was also proposed by Wu.[41] But Wu's model tends to equalize all lane densities through averaging them, which conflicts with real traffic since the density on the lane allocated for faster vehicles is generally less than that on the lane allocated for slower vehicles. Tang and Huang[42,43] recently developed a continuum model for freeways with two lanes, in which the momentum equation proposed by Jiang *et al*[12] was integrated into Daganzo's modeling framework for multi-lane traffic flow.[40] However, this model allows only the faster vehicles to change lanes. Recently, Huang, Tang and Gao[44] further extended the work of Tang and Huang[42,43] by allowing vehicles on both lanes to change their lanes according to the traffic condition then and there, which contains speed gradient-based momentum equations derived from a car-following theory suited to two-lane traffic flow. Tang, Jiang and Wu[45] also extended the speed gradient model for traffic flow on two-lane freeways. Other different two-lane models were presented to reproduce the traffic phenomenon.[46−51] But their models have anisotropy, which is not reasonable for describing multiple-lane traffic flow in the case where vehicles are allowed to change their lanes.[27] The coupling effect of the interactions between vehicles on two lanes has not been taken into account in these models either.

In the present paper, we extend the density viscous continuum traffic flow model proposed by Ge and Han[28] to asymmetric two-lane traffic with consideration of the coupling effect, allowing vehicles on both lanes to change their lanes according to the traffic condition then and there. We also investigate the lane changing rate resulting in shock waves and rarefaction waves, and the local cluster wave effect in two-lane freeways as well.

The remainder of the present paper is organized as follows. In Section 2, we establish a density viscous continuum traffic flow model with consideration of the coupling effect on both lanes. In Section 3, we analyse the linear stability of the new model. Numerical results are presented in Section 4, which validates the correctness of the theoretical analysis. Some nonequilibrium phenomena are investigated, such as small disturbance instability, shock waves and rarefaction waves, local cluster waves and phase transition. Finally, some conditions drawn from the present study are presented in Section 5.

2. Modelling and analysis

The two-delay timescale model proposed by Xue and Dai is based on the assumption of forward anticipation,[26] in which the stream speed u in the relaxation time T just reaches the anticipated value u_e corresponding to the density in time $t + t_r$ at location $x + ut_r$, where t_r is the driver reaction time. The speed–density relationship can be expressed as follows:

$$u(x + uT, t + T) = u_e(k(x + ut_r, t + t_r)). \quad (5)$$

The relaxation time T includes the driver reaction time t_r as well as the mechanical delay of the vehicle. But it must be noticed that the speed at the location (x, t) is determined by the traffic density at the location $(x + \Delta x,\ t)$, which accords with the idea of Payne.[3] So Ge and Han[28] gave the relation as

$$u(x + uT, t + T) = u_e(k(x + \Delta x + ut_r, t + t_r)). \quad (6)$$

In the car following model, it is reasonable to include the effects of relative speed because in real traffic drivers' reactions are affected not only by the distance of two successive vehicles, but also by the relative speed between them.[13] Such car following behaviour, when it is considered in continuum traffic flow, gives rise to the viscosity model. From the viewpoint of the macroscopic model, Ge and Han[28] thought that the relative density was more reasonable and deduced the following relation:

$$\begin{aligned} u(x + uT, t + T) = {} & u_e(k(x + \Delta x + ut_r, t + t_r)) \\ & + \mu(k(x + \Delta x) - k(x)), \end{aligned} \quad (7)$$

where $\mu = b_0 u_e'(k) < 0$, b_0 is a positive constant. Letting $\Delta x = 1/k(x, t)$, using the Taylor series expansion for Eq.(7), and applying Eq.(1) yield[28]

$$\begin{aligned} & \frac{\partial u}{\partial t} + (u - 2c_0)\frac{\partial u}{\partial x} \\ = {} & \frac{u_e(k) - u}{T} \\ & + \frac{u_e'(k) + u}{kT}\frac{\mathrm{d}k}{\mathrm{d}x} + \frac{\mu}{Tk^2}\frac{\mathrm{d}^2 k}{\mathrm{d}x^2}, \\ c_0 = {} & -\frac{1}{2}k\frac{t_r}{T}\frac{\mathrm{d}u_e}{\mathrm{d}k} \geq 0. \end{aligned} \quad (8)$$

On two-lane freeways, as is well known, lane changing behaviour will occur. To model this effect, the interactions between vehicles on two lanes must be taken into account to describe the flow rate changing from the other lane, where we take the coupling effect into account for the interactions between two lanes. Accordingly, the coupling effect term is added to Eq.(7) to describe the lateral lane effect and flow rate changing from the other lane.

$$\begin{aligned} & u_m(x + u_m T_m, t + T_m) \\ = {} & u_{em}[k_m(x + \Delta x_m + u_m t_{rm}, t + t_{rm}), \beta_{nm} k_n] \\ & + \mu_m[k_m(x + \Delta x_m) - k_m(x)], \\ & m, n = 1, 2, \quad \text{and} \quad m \neq n, \end{aligned} \quad (9)$$

where u_m and u_{em} are the stream speed and the anticipated value (u_{em} is a function of k_1 and k_2) on lane m, respectively. T_m and t_{rm}, respectively, are the relaxation time and driver reaction time on lane m. Here β_{nm} is the coupling coefficient from lane n to lane m. On the basis of the principle of coupling interaction, we consider $\beta_{nm} = \beta_{mn} = \beta$. Using the Taylor series expansion for Eq.(9) and neglecting higher-order terms yields

$$\begin{aligned} u_m(x, T_m) + \frac{\partial u_m}{\partial t}T_m + u_m T_m \frac{\partial u_m}{\partial x} = {} & u_{em}(k_m, \beta k_n) + \frac{\partial u_{em}}{\partial k_m} \times \left[t_{rm}\frac{\partial k_m}{\partial t} + (\Delta x_m + u_m t_{rm})\frac{\partial k_m}{\partial x}\right] \\ & + \mu_m \Delta x_m \frac{\mathrm{d}k_m}{\mathrm{d}x_m} + \mu_m \Delta x_m^2 \frac{\mathrm{d}^2 k_m}{\mathrm{d}x^2}. \end{aligned} \quad (10)$$

Equation (10) can be rearranged as

$$\begin{aligned} \frac{\partial u_m}{\partial t} + u_m \frac{\partial u_m}{\partial x} = {} & \frac{u_{em}(k_m, \beta k_n) - u_m}{T_m} + \frac{t_{rm}}{T_m}\frac{\partial u_{em}}{\partial k_m} \times \left(\frac{\partial k_m}{\partial t} + u_m \frac{\partial k_m}{\partial x}\right) \\ & + \frac{\Delta x_m}{T_m}\frac{\partial u_{em}}{\partial k_m}\frac{\partial k_m}{\partial x} + \frac{\mu_m \Delta x_m}{T_m}\frac{\mathrm{d}k_m}{\mathrm{d}x_m} + \frac{\mu_m \Delta x_m^2}{T_m}\frac{\mathrm{d}^2 k_m}{\mathrm{d}x^2}, \end{aligned} \quad (11)$$

where $\mu_m = b_0 u'_{\rm e}(k_m, \beta k_n) < 0$, b_0 is a positive constant. Let $\Delta x_m = 1/k_m(x,t)$. Applying Eq.(4), we have

$$\frac{\partial u_m}{\partial t} + (u_m - 2c_{m0})\frac{\partial u_m}{\partial x} = \frac{u_{em}(k_m, \beta k_n) - u_m}{T_m} + \frac{u'_{em}(k_m, \beta k_n) + \mu_m}{k_m T_m}\frac{{\rm d}k_m}{{\rm d}x} + \frac{\mu_m}{T_m k_m^2}\frac{{\rm d}^2 k_m}{{\rm d}x^2} + \gamma_m s_{mn} - \gamma_m s_{nm}, \tag{12}$$

where $c_{m0} = -\frac{1}{2}k_m \frac{t_{{\rm r}m}}{T_m}\frac{\partial u_{\rm e}}{\partial k_m} \geq 0$, and $\gamma_m = \frac{t_{{\rm r}m}}{T_m}\frac{\partial u_{em}(k_m, \beta k_n)}{\partial k_m}$.

The dynamic model for two-lane traffic consists of the following equations:

$$\frac{\partial k_m}{\partial t} + \frac{\partial q_m}{\partial x} = s_{mn}(x,t) - s_{nm}(x,t), \tag{13}$$

$$\frac{\partial u_m}{\partial t} + (u_m - 2c_{m0})\frac{\partial u_m}{\partial x} = \frac{u_{em}(k_m, \beta k_n) - u_m}{T_m} + \frac{u'_{em}(k_m, \beta k_n) + \mu_m}{k_m T_m}\frac{{\rm d}k_m}{{\rm d}x_m} + \frac{\mu_m}{T_m k_m^2}\frac{{\rm d}^2 k_m}{{\rm d}x^2} + \gamma_m s_{mn} - \gamma_m s_{nm}, \quad m, n = 1, 2, \ \text{and} \ m \neq n. \tag{14}$$

Note that the model given by Eqs.(13) and (14) for a specific lane is formally the same as the viscous continuum traffic flow model proposed by Ge and Han[28] (when $\beta = 0$ and $S = 0$ on a single lane). However, attention should be paid to such a difference that all speeds in Eqs.(13) and (14) are both functions of k_1 and k_2. According to Liu's ideal[52] and Tang's method,[45] we suppose the analogy terms of lane changing rate have the form as follows:

$$s_{12} = aQ_{\rm e1}k_1(1 - (k_2/k_{2\rm jam})^{\sigma}), \tag{15}$$

$$s_{21} = a(1 + b(Q_{\rm e1} - Q_{\rm e2})) \times Q_{\rm e2}k_2(1 - (k_1/k_{1\rm jam})^{\phi}), \tag{16}$$

where Q_{ei} is the equilibrium flow ($Q_{ei} = ku_{ei}(k)$) and σ and ϕ are each a parameter that reflect the intensity of the lane-change effect from one lane to another, $0 \leq \sigma, \ \phi \leq 1$, and depend on the condition of the road. The special case of $Q_{\rm e1} = Q_{\rm e2}$ corresponds to symmetric lane changing rules, while $Q_{\rm e1} \neq Q_{\rm e2}$ corresponds to asymmetric lane changing rules; a and b are constant parameters, $k_{i\rm jam}$ is the jam density of lane i.

In real traffic, the lane changing rate, say from lane 1 to lane 2, should be zero when either $k_2 = k_{2\rm jam}$ (lane 2 is in complete jam) or $k_1 = 0$ (there is no vehicle on lane 1). Furthermore, it obviously depends on the equilibrium speed–density relationship on the present lane and on coupling density of another lane. These properties are fulfilled in expressions (15) and (16).

We rewrite Eqs.(13) and (14) as follows:

$$\frac{\partial V}{\partial t} + [A]\frac{\partial V}{\partial x} = E, \tag{17}$$

where

$$V = \begin{bmatrix} k_1 \\ u_1 \\ k_2 \\ u_2 \end{bmatrix}, \quad E = \begin{bmatrix} s_{12} - s_{21} \\ \dfrac{(u_{1\rm e} - u_1)}{T_1} + \dfrac{\mu_1}{T_1 k_1^2}\dfrac{{\rm d}^2 k_1}{{\rm d}x^2} + \gamma_1 s_{12} - \gamma_1 s_{21} \\ s_{21} - s_{12} \\ \dfrac{(u_{2\rm e} - u_2)}{T_2} + \dfrac{\mu_2}{T_2 k_2^2}\dfrac{{\rm d}^2 k_2}{{\rm d}x^2} + \gamma_2 s_{21} - \gamma_2 s_{12} \end{bmatrix},$$

and

$$[A] = \begin{bmatrix} u_1 & k_1 & 0 & 0 \\ \dfrac{u'_{\rm e1} + \mu_1}{k_1 T_1} & u_1 - 2c_{10} & 0 & 0 \\ 0 & 0 & u_2 & k_2 \\ 0 & 0 & \dfrac{u'_{\rm e2} + \mu_2}{k_2 T_2} & u_2 - 2c_{20} \end{bmatrix}.$$

Comparing our new model with other continuum models presented by Daganzo,[40] we can see that the new model includes two new motion equations. Next we use the matrixes to analyse the characteristic speeds of Eq.(17). The eigenvalues, λ, of the $[A]$ matrix are found by setting

$$\det([A]-\lambda[I])=0, \tag{18}$$

where $[I]$ is the identity matrix. From Eq.(18), we have

$$\lambda_{11}=u_1-c_{10}-\sqrt{c_{10}^2-[u'_{\mathrm{e}1}(k_1,\beta k_2)+\mu_1]/k_1},$$

$$\lambda_{12}=u_1-c_{10}+\sqrt{c_{10}^2-[u'_{\mathrm{e}1}(k_1,\beta k_2)+\mu_1]/k_1},$$

$$\lambda_{21}=u_2-c_{20}-\sqrt{c_{20}^2-[u'_{\mathrm{e}2}(\beta k_1,k_2)+\mu_2]/k_2},$$

and

$$\lambda_{22}=u_2-c_{20}+\sqrt{c_{20}^2-[u'_{\mathrm{e}2}(\beta k_1,k_2)+\mu_2]/k_2}.$$

The homogeneous system of Eq.(17) is strictly hyperbolic, with real eigenvalues $\lambda_{m1}<u_m<\lambda_{m2}$ for $c_{m0}\geq 0$ on lane m. From the above-mentioned, it is found that one of the characteristic speeds of each subsystem is larger than their macroscopic flow speeds, which shows that traffic flow exhibits gas-like behaviour, which means that the rear vehicle speeds the vehicle ahead up. Zhang[27] pointed out that the anisotropic property of traffic flow could not be expected to hold in multi-lane traffic because of lane changing. Therefore, our proposed model may describe non-anisotropic behaviour in multi-lane traffic. It also guarantees that in a system consisting of two lanes the rear disturbance will affect the driving behaviour of the leading vehicle because lane changing is allowed in these two-lane traffic systems.

3. Linear stability condition

We analyse the qualitative properties of the viscous model with the linear stability method. According to the method introduced by Huang *et al*[44] and Tang *et al*,[45] steady state solutions of the extended density viscous model can be obtained by setting the derivative terms in Eqs.(13) and (14) to zero. It means that here $s_{mn}-s_{nm}=0$ (m, $n=12$). Suppose that traffic flow is initially in a state differing infinitesimally from homogeneous flow:

$$\begin{pmatrix}k_m(x,t)\\u_m(x,t)\end{pmatrix}=\begin{pmatrix}k_{m0}\\u_{m0}\end{pmatrix}+\sum\nolimits_g\begin{pmatrix}\hat{k}_{mg}\\\hat{u}_{mg}\end{pmatrix}\exp(\mathrm{i}\,gx+\sigma_g t). \tag{19}$$

Substituting Eq.(19) into Eqs. (13) and (14), and neglecting the higher-order terms of the small perturbations yields

$$(\sigma_{mg}+\mathrm{i}\,k_m u_{m0})\hat{k}_{mg}=0, \tag{20}$$

$$\begin{aligned}\sigma_{mg}\hat{u}_{mg}+\mathrm{i}\,g(u_{m0}-2c_{m0})\hat{u}_{mg}=&\frac{1}{T_m}[u'_{\mathrm{e}m}(k_{m0},\beta k_{n0})\hat{k}_{m0}-\hat{u}_{mg}]+\mathrm{i}\,g\frac{u'_{\mathrm{e}m}(k_{m0},\beta k_{n0})+\mu_m}{k_{m0}T_m}\hat{k}_{mg}\\&-\frac{g^2\mu_m}{T_m k_{m0}^2}\hat{k}_{mg}.\end{aligned} \tag{21}$$

For the linear algebraic equation in $\hat{k}_{mg}$ and $\hat{u}_{mg}$, the solvability condition reads

$$(\sigma_{mg}+\mathrm{i}\,k_m u_{m0})^2+\left(\frac{u_{m0}}{T_m}-2\,\mathrm{i}\,gc_{m0}\right)(\sigma_g+\mathrm{i}\,gu_{m0})+\frac{\mathrm{i}\,gk_{m0}}{T_m}\left[u'_{\mathrm{e}}(k_{m0},\beta k_{n0})+\mathrm{i}\,g\frac{u'_{\mathrm{e}m}(k_{m0},\beta k_{n0})+\mu_m}{k_{m0}}-\frac{g^2\mu_m}{k_{m0}^2}\right]=0. \tag{22}$$

The traffic flow will remain stable provided that both roots of σ_{mg} have negative real parts. We can obtain the neutral stability line as

$$T_{ms}=\frac{u'_{\mathrm{e}m}(k_{m0},\beta k_{n0})+\mu_m(k_{m0},\beta k_{n0})}{-2c_{m0}(k_{m0},\beta k_{n0})k_{m0}u'_{\mathrm{e}m}(k_{m0},\beta k_{n0})-k_{m0}^2u'_{\mathrm{e}m}(k_{m0},\beta k_{n0})^2}. \tag{23}$$

Substituting c_{m0} and μ_m into expression (23), and neglecting the higher order of g, we have

$$T_{ms} = t_{rm} - \frac{b_0 + 1}{k_{m0}^2 u'_{em}(k_{m0}, \beta k_{n0})}, \tag{24}$$

$$\mathrm{Im}(\sigma_{mg}) \approx -g[u_{m0} + k_{m0} u'_{em}(k_{m0}, \beta k_{n0})] + O(g^3). \tag{25}$$

From expression (25), we conclude that the critical disturbance travels at a speed of

$$c(k_{m0}) = u_{m0} + k_{m0} u'_{em}(k_{m0}, \beta k_{n0}), \tag{26}$$

which accords with the kinematic wave speed for a single lane (when $\beta = 0$ and $S = 0$) mentioned in Refs.[12] and [28].

4. Numerical simulation

In this section, using the finite difference method, we numerically solve the model to demonstrate the model's ability to capture complex traffic phenomena. Let index j and index i represent the time interval and the road section, respectively. Then the difference equations corresponding to Eqs.(13) and (14) are as follows:

$$\begin{aligned} k_{mi}^{j+1} = k_{mi}^j + \frac{\Delta t}{\Delta x} k_{mi}^j \left(u_{mi}^j - u_{m(i+1)}^j\right) \\ \times \frac{\Delta t}{\Delta x} u_{mi}^j \left(k_{m(i-1)}^j - k_{mi}^j\right) + \Delta t s_{mni}^j \\ - \Delta t s_{nmi}^j, \quad m, n = 1, 2. \end{aligned} \tag{27}$$

(I) When traffic is heavy, $u_{mi}^j < c_{m0}$,

$$\begin{aligned} u_{mi}^{j+1} = u_{mi}^j + \frac{\Delta t}{\Delta x}\left[2c_{m0} - u_{mi}^j\right]\left(u_{m(i+1)}^j - u_{mi}^j\right) + \frac{\Delta t}{T}\left(u_{em} - u_{mi}^j\right) + \frac{\Delta t}{\Delta x}\frac{u'_{em} + \mu_m}{k_{mi}^j T_m}\left(k_{mi}^j - k_{m(i-1)}^j\right) \\ + \frac{\Delta t}{\Delta^2 x}\frac{\mu_m}{(k_{mi}^j)^2 T_m}\left(k_{mi}^j - k_{m(i-1)}^j\right)^2 + \gamma_m \Delta t s_{mni}^j - \gamma_m \Delta t s_{nmi}^j, \quad m, n = 1, 2. \end{aligned} \tag{28}$$

(II) When traffic is light, $u_{mi}^j \geq c_{m0}$,

$$\begin{aligned} u_{mi}^{j+1} = u_{mi}^j + \frac{\Delta t}{\Delta x}\left[2c_{m0} - u_{mi}^j\right]\left(u_{mi}^j - u_{m(i-1)}^j\right) + \frac{\Delta t}{T}\left(u_{em} - u_{mi}^j\right) + \frac{\Delta t}{\Delta x}\frac{u'_{em} + \mu_m}{k_{mi}^j T_m}\left(k_{mi}^j - k_{m(i-1)}^j\right) \\ + \frac{\Delta t}{\Delta^2 x}\frac{\mu_m}{(k_{mi}^j)^2 T_m}\left(k_{mi}^j - k_{m(i-1)}^j\right)^2 + \gamma_m \Delta t s_{mni}^j - \gamma_m \Delta t s_{nmi}^j, \quad m, n = 1, 2. \end{aligned} \tag{29}$$

4.1. Shock waves and rarefaction waves

In order to investigate the congestion and the dissipation of the traffic flow, we use two Riemann initial conditions.[26,43]

$$\text{(i)} \begin{cases} k_{1\mathrm{u}}^1 = 0.03, \\ k_{2\mathrm{u}}^1 = 0.04, \end{cases} \quad \begin{cases} k_{1\mathrm{d}}^1 = 0.12, \\ k_{2\mathrm{d}}^1 = 0.18, \end{cases} \tag{30a}$$

$$\text{(ii)} \begin{cases} k_{1\mathrm{u}}^2 = 0.12, \\ k_{2\mathrm{u}}^2 = 0.18, \end{cases} \quad \begin{cases} k_{1\mathrm{d}}^2 = 0.03, \\ k_{2\mathrm{d}}^2 = 0.04, \end{cases} \tag{30b}$$

where $k_{1\mathrm{u}}^1$, $k_{2\mathrm{u}}^1$, $k_{1\mathrm{d}}^1$, $k_{2\mathrm{d}}^1$ and $k_{1\mathrm{u}}^2$, $k_{2\mathrm{u}}^2$, $k_{1\mathrm{d}}^2$, $k_{2\mathrm{d}}^2$ are, respectively, the upstream and downstream densities of two lanes. Equation (30a) corresponds to the appearance of shock waves when free-flow traffic meets stopped vehicles, while Eq.(30b) corresponds to the rarefaction wave as a queue dissolves. Two initial conditions are

$$u_{1\mathrm{u}}^{1,2} = u_{1\mathrm{e}}(k_{1\mathrm{u}}^{1,2}, k_{2\mathrm{u}}^{1,2}), \quad u_{1\mathrm{d}}^{1,2} = u_{1\mathrm{e}}(k_{1\mathrm{d}}^{1,2}, k_{2\mathrm{d}}^{1,2}), \tag{31a}$$

$$u_{2\mathrm{u}}^{1,2} = u_{2\mathrm{e}}(k_{1\mathrm{u}}^{1,2}, k_{2\mathrm{u}}^{1,2}), \quad u_{2\mathrm{d}}^{1,2} = u_{2\mathrm{e}}(k_{1\mathrm{d}}^{1,2}, k_{2\mathrm{d}}^{1,2}). \tag{31b}$$

The equilibrium speed–density relationships are as follows, in which the $k_m + \beta k_n$ has been substituted into the equilibrium speed–density relationship developed by Del Castillo and Benitez:[53]

$$u_{\mathrm{e}1}(k) = u_{1\mathrm{f}}\left[1 - \exp\left(1 - \exp\left(\frac{c_{1\mathrm{jam}}}{u_{1\mathrm{f}}}\left(\frac{k_{1\mathrm{jam}} + \beta k_{2\mathrm{jam}}}{k_1 + \beta k_2} - 1\right)\right)\right)\right], \tag{32a}$$

$$u_{\mathrm{e}2}(k) = u_{2\mathrm{f}}\left[1 - \exp\left(1 - \exp\left(\frac{c_{2\mathrm{jam}}}{u_{2\mathrm{f}}}\left(\frac{\beta k_{1\mathrm{jam}} + k_{2\mathrm{jam}}}{\beta k_1 + k_2} - 1\right)\right)\right)\right], \tag{32b}$$

where $k_{1\mathrm{jam}}$, $k_{2\mathrm{jam}}$ and $u_{1\mathrm{f}}$, $u_{2\mathrm{f}}$ are, respectively, the jam densities and free-flow speed of lane 1 and lane 2. $c_{1\mathrm{jam}}$ and $c_{2\mathrm{jam}}$ are the propagation speeds of a disturbance under the jam densities $k_{1\mathrm{jam}}$ and $k_{2\mathrm{jam}}$.

Free boundary conditions are applied here,[12,26,43] i.e. $\partial k_m/\partial x = 0$ and $\partial u_m/\partial x = 0$. Consider a section of freeway 20 km long, divide it into 100 cells, and take the time interval as 1 s. According to real observation and the parameter identification process, the values of the parameters are chosen as follows:[26,28,43] $u_{1\mathrm{f}} = 40$ m/s, $T_1 = 8$ s, $t_{\mathrm{r}1} = 1$ s, $k_{1\mathrm{jam}} = 0.15$ veh/m, $c_{1\mathrm{jam}} = 7$ m/s, $u_{2\mathrm{f}} = 30$ m/s, $T_2 = 7$ s, $t_{\mathrm{r}2} = 0.75$ s, $k_{2\mathrm{jam}} = 0.2$ veh/m, $c_{2\mathrm{jam}} = 6$ m/s, $a = 0.01$, $b = 5$ s, $b_0 = 1/6$, $\beta = 0.1$, $\sigma = \phi = 1$.

Note that the jam density of lane 2 is higher than that of lane 1. The computational results under the two Riemann initial conditions (30a) and (30b) are shown in Figs.1–6 separately.

Figures 1, 2, 4 and 5 show that different Riemann initial conditions lead to different fronts between the congested and free-flow traffic, which results in proper figures of traffic flow density change on both lanes with lane changing (see Figs.3 and 6). The wave shapes of two lanes are in accordance with those of the single lane by Jiang *et al*[12] (see Fig.2 in Ref.[12]) and by Xue and Dai[26] (see Figs.1 and 2 in Ref.[26]), which indicates that our new two-lane model is reasonable. Figures 1 and 2 show how the backward-moving shock wave front evolves under the condition (30a) on lanes 1 and 2, respectively. This means that traffic becomes more congested, which we often see in rush hours. Figures 4 and 5 show how the rarefaction wave front evolves under the condition (30b). This is a queue in the process of dissolution, which is consistent with our daily experiences in real traffic.

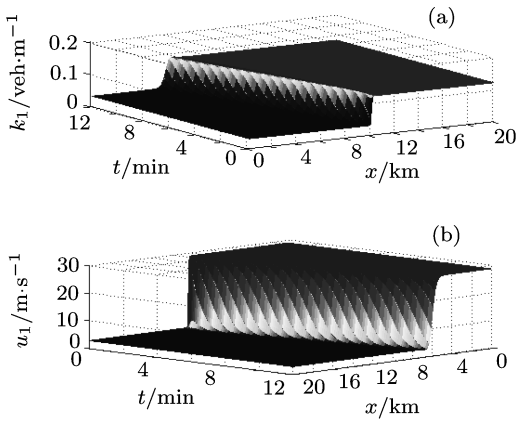


Fig.1. Shock waves under the Riemann initial conditions of expression (30a): (a) temporal evolution of density $k_1(x,t)$ and (b) speed $u_1(x,t)$ on lane 1.

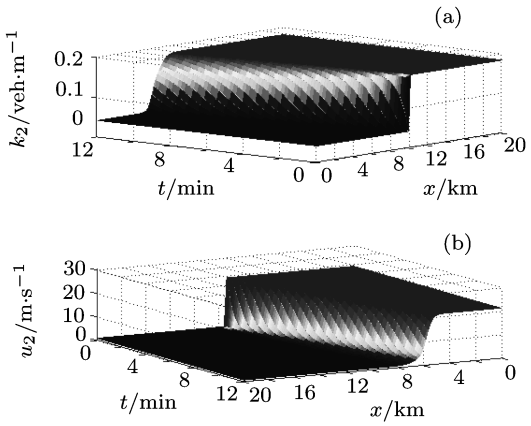


Fig.2. Shock waves under the Riemann initial conditions of expression (30a): (a) temporal evolution of density $k_2(x,t)$ and (b) speed $u_2(x,t)$ on lane 2.

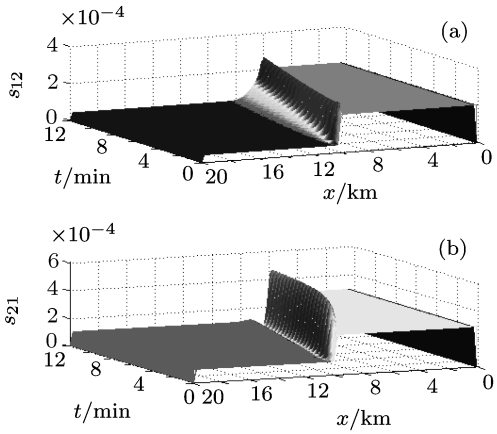


Fig.3. Lane change rate under the Riemann initial conditions of expression (30a): (a)s_{12} is the lane change rate from lane 1 to lane 2, and (b)s_{21} is the lane change rate from lane 2 to lane 1.

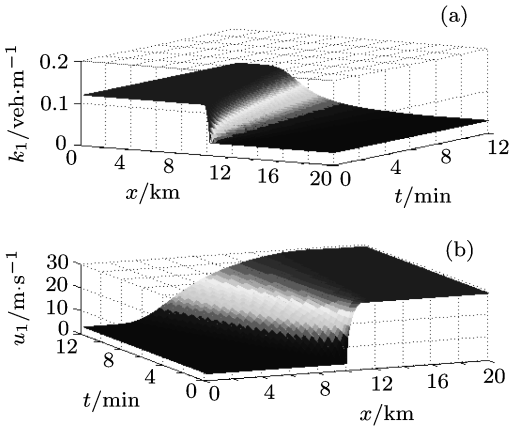


Fig.4. Shock waves under the Riemann initial conditions of expression (30b): (a) temporal evolution of density $k_1(x,t)$ and (b) speed $u_1(x,t)$ on lane 1.

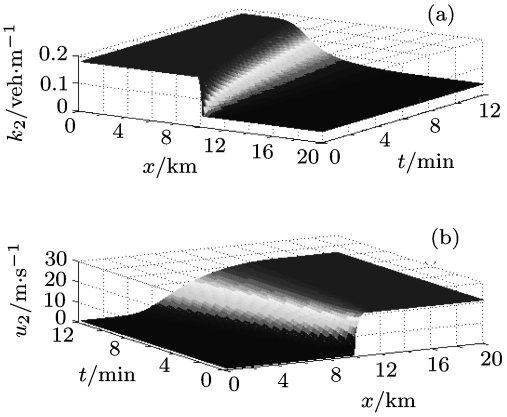


Fig.5. Shock waves under the Riemann initial conditions of expression (30b): (a) temporal evolution of density $k_2(x,t)$ and (b) speed $u_2(x,t)$ on lane 2.

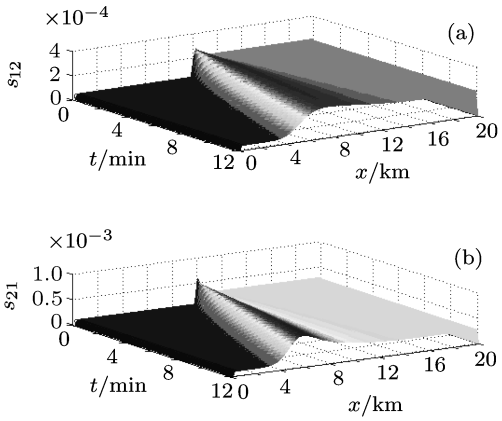


Fig.6. Lane change rate under the Riemann initial conditions of expression (30b): (a) s_{12} is the lane change rate from lane 1 to lane 2 and (b)s_{21} is the lane change rate from lane 2 to lane 1.

4.2. Local cluster effect

In order to investigate the consequences caused by a small localized disturbance under an initial homogeneous condition, we adopt the following initial variations of the average density $k_{m0}(m = 1, 2)$ which was used by Herrmann and Kerner:[54]

$$k_m(x,0) = k_{m0} + \Delta k_{m0}\Bigg\{ \cosh^{-2}\left[\frac{160}{L}\left(x - \frac{5L}{16}\right)\right] - \frac{1}{4}\cosh^{-2}\left[\frac{40}{L}\left(x - \frac{11L}{32}\right)\right]\Bigg\}, \quad m = 1, 2, \tag{33}$$

where $L = 32.2$ km is the length of the road section under consideration. Δk_{m0} represents a perturbation. The following periodic boundary conditions are adopted:

$$k_m(L,t) = k_m(0,t),$$

and

$$u_m(L,t) = u_m(0,t), \quad m = 1, 2. \tag{34}$$

The equilibrium speed–density relationships are as follows, in which the $k_m + \beta k_n$ has been substituted into the equilibrium speed–density relationship developed by Kerner and Konhäuser:[55]

$$u_{em} = u_{m0}\Bigg\{\left(1 + \exp\left[\frac{(k_m + \beta k_n)/k_{m\mathrm{jam}} - 0.25}{0.06}\right]\right)^{-1} - 3.72 \times 10^{-6}\Bigg\}, \quad m, n = 1, 2, \quad \text{and} \quad m \neq n, \tag{35}$$

where $k_{1\mathrm{jam}}$, $k_{2\mathrm{jam}}$ and u_{10}, u_{20} are, respectively, the jam densities and free-flow speed of lane 1 and lane 2. Assume the initial flows to be in local equilibrium everywhere, i.e. $u_1(x,0) = u_{1\mathrm{e}}(k_1(x,0) + \beta k_2(x,0))$, and $u_2(x,0) = u_{2\mathrm{e}}(\beta k_1(x,0) + k_2(x,0))$. Other parameters are as follows:[26,28,44] $\beta = 0.1$, $\Delta k_{10} = 0008$ veh/m, $\Delta k_{20} = 001$ veh/m, $\Delta x = 100$ m, $\Delta t = 1$ s, $u_{10} = 40$ m/s, $u_{20} = 30$ m/s, $t_{\mathrm{r}1} = 1$ s, $t_{\mathrm{r}2} = 1$ s, $T_1 = 8$ s, $T_2 = 7$ s, $k_{1\mathrm{jam}} = 0.15$ veh/m, $k_{2\mathrm{jam}} = 0.2$ veh/m, $a = 0.01$, $b = 5$ s, $\beta = 0.1$, $\sigma = \phi = 1$, and $b_0 = 1/6$.

Figures 7–18 show the temporal evolution of traffic with localized perturbations with different values of k_{m0} $(m = 1, 2)$. The scales of space x, time t and density k_m are 100 m, 1 s and 1 veh/m, respectively. Some new different results are shown as follows.

(i) Under uncongested traffic conditions as shown in Figs.7 and 8 ($k_{10} = 0.02$; $k_{20} = 0.035$), the perturbation dissipates quickly and the traffic flow in both lanes is in a stable condition (Fig.7). A few vehicles change their lanes on both lanes (Fig.8). The traffic flow density is very low. Hence the perturbation dies out with time. Therefore the front car can speed up freely, which results in the perturbation wave surface propagating ahead.

(ii) When the traffic density is increased to a certain level on lane 2 while k_{10} remains unchanged, traffic breakdown may occur if a perturbation is exerted on two lanes. As the density is just above the down-critical density, a single local cluster forms on both lanes as shown in Fig.9. A number of vehicles on both lanes change lanes (Fig.10). This implies that a higher frequency of lane changing occurs at large densities, which is consistent with our daily experiences in real traffic and the psychological factor of drivers.

(iii) When k_{20} increases even further to 0.075 while k_{10} still remains unchanged, a complex localized structure consisting of more clusters forms as the density becomes higher on both lanes, which corresponds to stop-and-go traffic (See Fig.11). It shows that lane 1 attracts a considerable number of vehicles from lane 2 (Fig.12) and more cars change lanes with the increment of large densities.

(iv) When k_{10} increases to 0.068 while k_{20} remains unchanged, we observe a dipole-like structure on both lanes as illustrated in Fig.13. But lane changing still occurs seriously from one lane to another as shown in Fig.14.

(v) When k_{10} decreases to 0.03 while k_{20} increases to 0.105, the stop-and-go traffic phenomenon occurs in lane 1 but the traffic on lane 2 is stable as seen from Fig.15. Figure 16 implies that the excessive lane changing from lane 2 to lane 1 (many vehicles on lane 2 shift to lane 1, then a few of them go back to lane 2) results in local jams on lane 1. It also shows that lane 1 attracts a considerable number of vehicles from lane 2 (Fig.16) and thus alleviates the jam on lane 2.

(vi) Finally, as the density becomes greater than the up-critical density, a stable regime is reached again on both lanes as can be seen in Fig.17. Lane changing hardly occurs according to Fig.18.

The above results are consistent with the properties of traffic flow in Refs.[12], [28], [44], [54] and [56]. These local cluster effects are also consistent with the diverse nonlinear dynamic phenomena observed in realistic traffic flow.

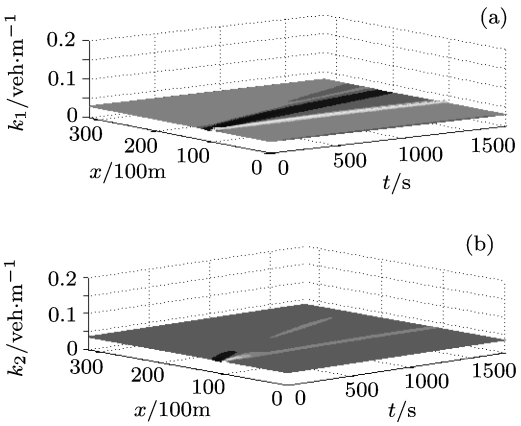


Fig.7. $k_{10} = 0.03$ and $k_{20} = 0.035$.

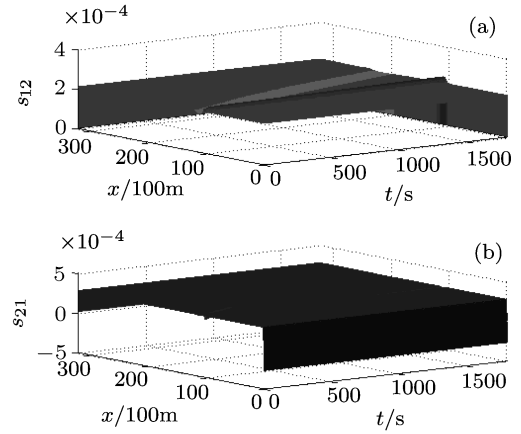


Fig.8. $k_{10} = 0.03$ and $k_{20} = 0.035$: (a) s_{12} is the lane changing rate from lane 1 to lane 2 and (b) s_{21} is the lane changing rate from lane 2 to lane 1.

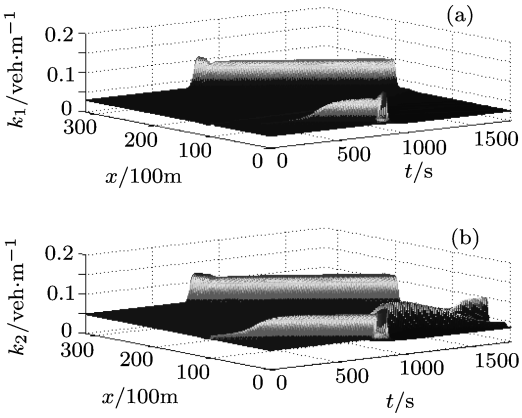


Fig.9. $k_{10} = 0.03$ and $k_{20} = 0.048$.

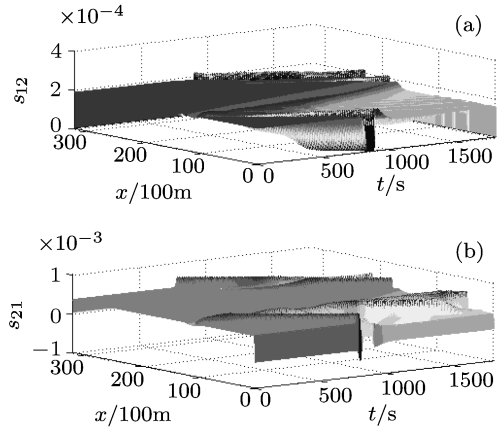


Fig.10. $k_{10} = 0.03$ and $k_{20} = 0.048$: (a) s_{12} is the lane changing rate from lane 1 to lane 2 and (b) s_{21} is the lane changing rate from lane 2 to lane 1.

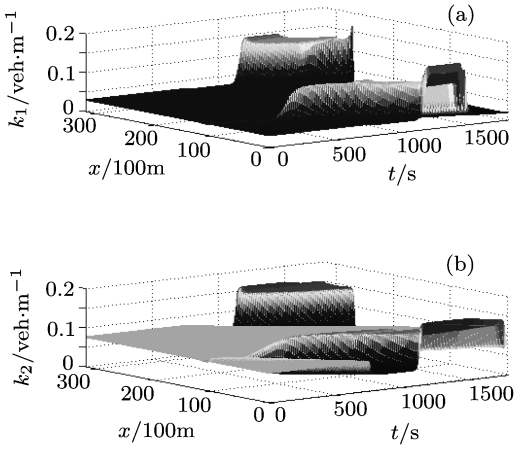


Fig.11. $k_{10} = 0.03$ and $k_{20} = 0.075$.

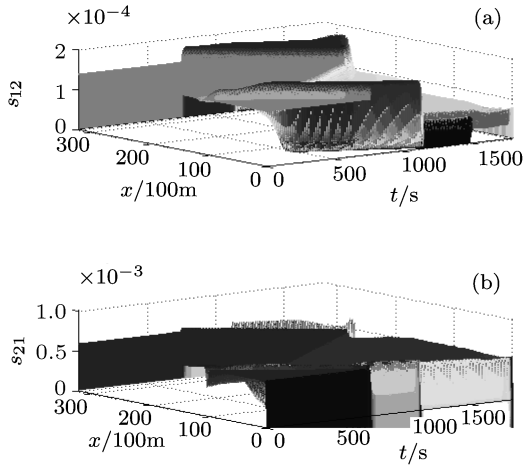


Fig.12. $k_{10} = 0.034$ and $k_{20} = 0.075$: (a) s_{12} is the lane changing rate from lane 1 to lane 2 and (b) s_{21} is the lane changing rate from lane 2 to lane 1.

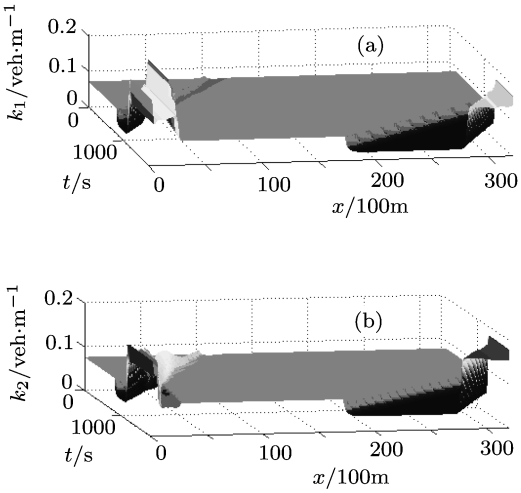


Fig.13. $k_{10} = 0.068$ and $k_{20} = 0.075$.

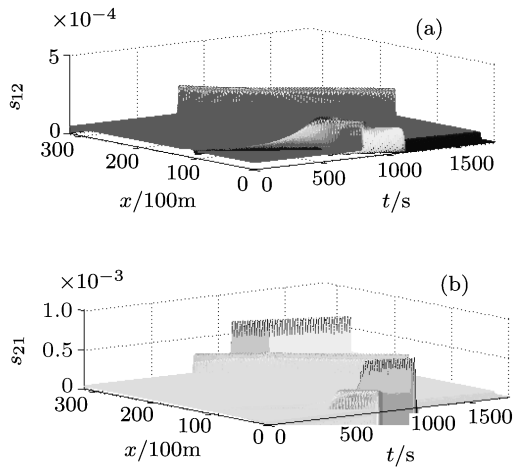


Fig.14. $k_{10} = 0.068$ and $k_{20} = 0.075$: (a) s_{12} is the lane changing rate from lane 1 to lane 2 and (b) s_{21} is the lane changing rate from lane 2 to lane 1.

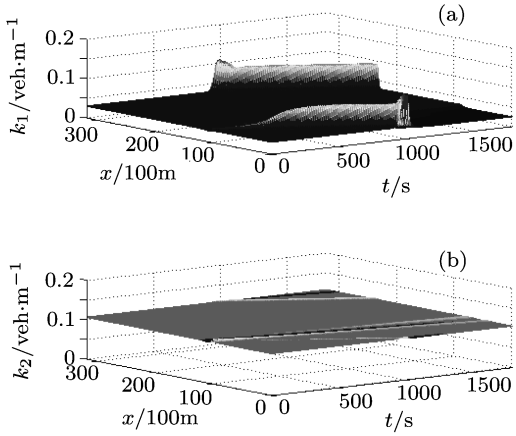


Fig.15. $k_{10} = 0.03$ and $k_{20} = 0.105$.

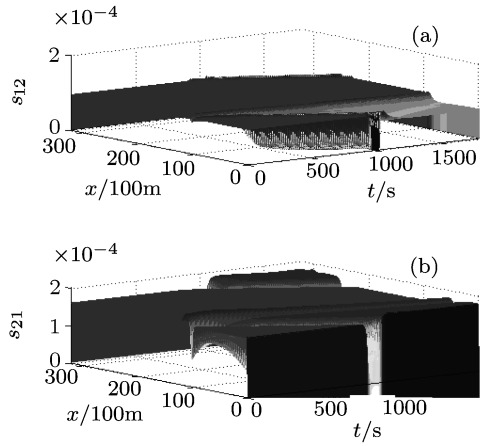


Fig.16. $k_{10} = 0.03$ and $k_{20} = 0.105$: (a) s_{12} is the lane changing rate from lane 1to lane 2 and (b) s_{21} is the lane changing rate from lane 2 to lane 1.

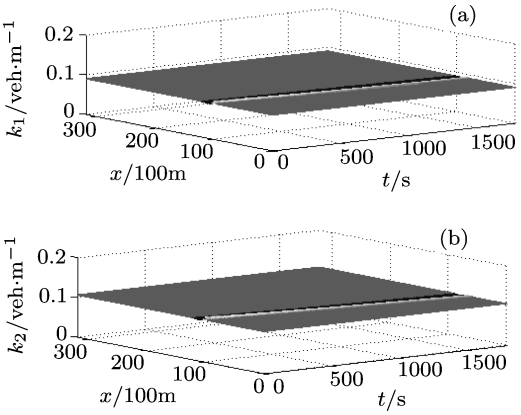


Fig.17. $k_{10} = 0.09$ and $k_{20} = 0.105$.

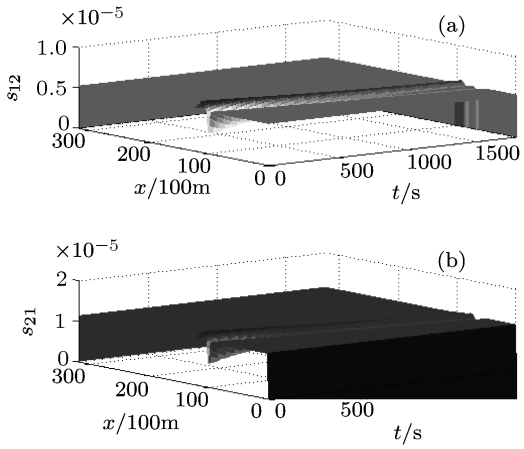


Fig.18. $k_{10} = 0.09$ and $k_{20} = 0.105$: (a) s_{12} is the lane changing rate from lane 1 to lane 2 and (b) s_{21} is the lane changing rate from lane 2 to lane 1.

5. Conclutions

In this paper, the viscous continuum traffic flow model has been extended to two-lane traffic flow. Lane changing behaviour is incorporated into the extended model by introducing a source term of lane changing rate and coupling effect between two lanes in the continuity equation and two corresponding terms in the speed dynamic equations. Our model is different from previous ones in that the source term and the coupling effect are modelled separately. In this way, the lane changing rate can be investigated.

The proposed model is a higher-order continuum model considering the coupling effect and lane changing behaviour of vehicles between two adjacent lanes. Moreover, the model has non-anisotropy, which is reasonable to describe two-lane traffic flow. Therefore, our proposed model can provide an intuitively acceptable explanation of some common traffic behaviour in multi-lane traffic.

References

[1] Lighthill M J and Whitham G B 1955 *Proc. R. Soc.* Ser A **229** 317
[2] Richards P I 1956 *Operations Res.* **4** 42
[3] Payne H J 1971 In: Bekey G A (eds.) *Mathematical Models of Public Systems. Simulation Councils Proceedings* Series **1** p51
[4] Newell G 1961 *Operations Res.* **9** 209
[5] Daganzo C F 1995 *Transpn. Res.* B **29** 277
[6] Del Castillo J M, Pintado P and Benitez F G 1993 *Transportation and Traffic Theory* (Amsterdam: Elsevier) p387
[7] Kihne R D 1984 *Proceedings of 9th International Symposium on Transportation and Traffic Theory* (Netherlands: VNU Science Press) p21
[8] Ross P 1988 *Transpn. Res.* B **22** 421
[9] Papageorgiou M, Blosseville J M and Hadj Salem H 1989 *Transpn. Res.* B **23** 29
[10] Michaloupolos R, Yi P and Lyrintzis A S 1993 *Transpn. Res.* B **27** 315
[11] Zhang H M 1998 *Transpn. Res.* B **32** 485
[12] Jiang R, Wu Q S and Zhu Z J 2002 *Transpn. Res.* B **36** 405
[13] Jiang R, Wu Q S and Zhu Z J 2001 *Phys. Rev.* E **64** 017101
[14] Xue Y 2002 *Chin. Phys.* **11** 1128
[15] Xue Y, Dong L Y, Yuan Y W and Dai S Q 2002 *Acta Phys. Sin.* **51** 492 (in Chinese)
[16] Xue Y 2003 *Acta Phys. Sin.* **52** 2750 (in Chinese)

[17] Wang T, Gao Z Y and Zhao X M 2006 *Acta Phys. Sin.* **55** 634 (in Chinese)
[18] Li L and Shi P F 2005 *Chin. Phys.* **14** 0576
[19] Li Z P and Liu Y C 2006 *Chin. Phys.* **15** 1570
[20] Li X L, Song T, Kuang H and Dai S Q 2008 *Chin. Phys.* B **17** 3014
[21] Peng G H, Sun D H and He H P 2008 *Acta Phys. Sin.* **57** 7541 (in Chinese)
[22] Mo Y L, He H D, Xue Y, Shi W and Lu W Z 2008 *Chin. Phys.* B **17** 4446
[23] Jiang R and Wu Q S 2004 *Acta Mech. Sin.* **20** 106
[24] Jiang R and Wu Q S 2003 *Transpn. Res.* B **37** 85
[25] Zhang H M 2003 *Transpn. Res.* B **37** 101
[26] Xue Y and Dai S Q 2003 *Phys. Rev.* E **68** 066123
[27] Zhang H M 2003 *Transpn. Res.* B **37** 561
[28] Ge H X and Han X L 2006 *Physica* A **371** 667
[29] Lei L, Xue Y and Dai S Q 2003 *Acta Phys. Sin.* **52** 2121 (in Chinese)
[30] Xue Y 2004 *Acta Phys. Sin.* **53** 25 (in Chinese)
[31] Li K P and Gao Z Y 2005 *Chin. Phys.* **14** 930
[32] Ge H X, Zhu H B and Dai S Q 2005 *Acta Phys. Sin.* **54** 4621 (in Chinese)
[33] Guo S L, Wei Y F and Xue Y 2006 *Acta Phys. Sin.* **55** 3336 (in Chinese)
[34] Li J W, Lin B L and Huang Y C 2007 *Chin. Phys.* **16** 753
[35] Han X L, Jiang C Y, Ge H X and Dai S Q 2007 *Acta Phys. Sin.* **56** 4383 (in Chinese)
[36] Tang T Q, Huang H J and Xu G and Xue Y 2008 *Acta Phys. Sin.* **57** 56 (in Chinese)
[37] Li J W, Lin B L and Huang Y C 2008 *Chin. Phys.* B **17** 473
[38] Jiang R, Jin W L and Wu Q S 2008 *Chin. Phys.* B **17** 829
[39] Guo R Y and Huang H J 2008 *Chin. Phys.* B **17** 1689
[40] Daganzo C F 1997 *Transpn. Res.* B **31** 83
[41] Wu Z 1994 *Acta Mech. Sin.* **26** 149 (in Chinese)
[42] Tang T Q and Huang H J 2004 *Chin. Sci. Bull.* **49** 1937
[43] Tang T Q and Huang H J 2005 *J. Beijing University of Aero. and Astro.* **31** 1121
[44] Huang H J, Tang T Q and Gao Z Y 2006 *Acta Mech. Sin.* **22** 131 (in Chinese)
[45] Tang C F, Jiang R and Wu Q S 2007 *Chin. Phys.* **16** 1570
[46] Peng L, Tan H L, Wu D Y, Liu M R and Kong L J 2004 *Acta Phys. Sin.* **53** 2899 (in Chinese)
[47] Zheng R S, Tan H L, Kong L J and Liu M R 2005 *Acta Phys. Sin.* **54** 4614 (in Chinese)
[48] Tang T Q, Huang H J and Xue Y 2006 *Acta Phys. Sin.* **55** 4027 (in Chinese)
[49] Wu K F, Kong L J and Liu M R 2006 *Acta Phys. Sin.* **55** 6275 (in Chinese)
[50] Chen X, Gao Z Y, Jia B and Zhao X M 2007 *Acta Phys. Sin.* **56** 2024 (in Chinese)
[51] Teng Y F, Gao Z Y, Jia B and Li F 2008 *Acta Phys. Sin.* **57** 1365 (in Chinese)
[52] Liu G Q 1996 *Appl. Math. Model* **20** 459
[53] Del Castillo J M and Benitez F G 1995 *Transpn. Res.* B **29** 373
[54] Herrmann M and Kerner B S 1998 *Physica* A **255** 163
[55] Kerner B S and Konhäuser P 1993 *Phys. Rev.* E **48** R2335
[56] Kerner B S and Konhäuser P 1994 *Phys. Rev.* E **50** 54