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Relativity stability of quantum gas in a weak magnetic field

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Based on the analytical expression of relativistic free energy for a weakly interacting Fermi gas in a weak magnetic field, by using the method of quantum statistics, the stability conditions of the system at both high and low temperatures are given, and the effects of magnetic field and interparticle interactions on the stability of the system are analysed. It is shown that at high temperatures, the stability conditions of the system are completely the same, no matter whether it is the ultrarelativistic case or nonrelativistic case. At extremely low temperatures, the mechanical stability conditions of the system show a similar rule through a comparison between the ultrarelativistic case and nonrelativistic case. At the same time, thermal stability of a relativistic Bose gas in a weak magnetic field is discussed, and the influence of the effect of relativity on the thermal stability of the system is investigated.

Keywords: quantum gas, effect of relativity, stability
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1. Introduction

The investigation of the thermodynamic properties of interacting quantum gas has recently been very fruitful,[1−9] and thereby some achievements about the stability of the Fermi and Bose system have also been obtained. For example, inter-particle interactions can make the condensate of Bose gas disintegrate and collapse.[10−13] It was indicated in Ref.[14] that degenerate Fermi gas might collapse under certain conditions. In Refs.[15] and [16] the effects of interparticle interactions on the thermodynamic stabilization of the Bose and Fermi system were investigated respectively, and the particle-number-density conditions of the instability of the system were presented. In Ref.[17] the instability conditions of a weakly interacting Fermi gas in a weak magnetic field were given. In Ref.[18] the thermodynamic stability of a weak interacting Fermi gas trapped in a harmonic potential was also investigated.

However, both theoretical and experimental studies on the stability of the quantum system were still in the initial stages, thus it is very important to investigate the thermodynamic stability of the system under different conditions. The stability of the system can be affected by many factors including external potential, the interparticle interaction and the effect of relativity. It is well known that many systems in the universe consist of fermions which have very small rest mass. For the system which consists of these small particles, the rest energies of the particles are comparable with their kinetic energies, thus, the effect of relativity on the stability of the system needs to be considered.

In the present paper, based on the analytical expression of relativistic free energy of the Fermi system, we study the stability of the system, and the effects of magnetic field and interparticle interactions on the stability. At the same time, we will still discuss the thermal stability of a relativistic Bose gas in a weak magnetic field.

2. Stability of Fermi gas with the consideration of relativistic effect

We consider a weak interacting Fermi gas of spin-1/2 fermions, with volume V and particle number N, confined in an even weaker magnetic field $B = B_Z$. The relativistic energy of a single particle is expressed

as

$$\varepsilon_p = mc^2\sqrt{1+\frac{p^2}{m^2c^2}} - mc^2,$$

where p and m are, respectively, the momentum and the rest mass of the particle. Here, we only consider the kinetic energy of particles. Following the means in Ref.[17], we obtain an analytical expression of the relativistic free energy of the system as

$$F \approx 2A_0\left(\frac{N}{2}\right) + \frac{\alpha}{4}nN - N\mu B r_0 - N\mu B\alpha n\frac{r_0}{D}, \quad (1)$$

where μ is the magnetic moment $\mu_{\rm B}$ of fermion, $\alpha = 4\pi a\hbar^2/m$ is the parameter of interaction, a is the s-wave scattering length, n is particle-number density,

$$r_0 = \left.\frac{2\mu B}{\partial u_0(xN)/\partial x}\right|_{x=\frac{1}{2}},$$

$$D = \left.\partial u_0(xN)/\partial x\right|_{x=1/2},$$

u_0 is the chemical potential corresponding to A_0, $A_0(N/2)$ is the free energy of the $N/2$ free fermions system, in which the relativistic effect is taken into account.

In order to maintain the stability of the system, the conditions of mechanical and thermal stabilities must be satisfied at the same time

$$\left(\frac{\partial P}{\partial V}\right)_T < 0, \quad C_V > 0. \quad (2)$$

where

$$P = -\left(\frac{\partial F}{\partial V}\right)_{N,T} = -\left[-N\mu B\left(\frac{\partial r_0}{\partial V}\right)_{N,T} + 2\left.\frac{\partial A_0(N/2)}{\partial V}\right|_{N,T} - \frac{1}{4}\alpha n^2 - N\mu B\alpha\frac{\partial}{\partial V}\left(\frac{nr_0}{D}\right)\right], \quad (3)$$

$$\left(\frac{\partial P}{\partial V}\right)_T = -\left\{-N\mu B\left(\frac{\partial^2 r_0}{\partial V^2}\right)_{N,T} + 2\left.\frac{\partial^2 A_0(N/2)}{\partial V^2}\right|_{N,T} + \frac{1}{2}\frac{\alpha n^2}{V} - N\mu B\alpha\left[\frac{\partial^2}{\partial V^2}\left(\frac{nr_0}{D}\right)\right]_{N,T}\right\}, \quad (4)$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{\partial}{\partial T}\left[\frac{\partial}{\partial\beta}(\beta F)\right] = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_{N,V}. \quad (5)$$

The stability conditions can be expressed as

$$\left.2\frac{\partial^2 A_0(N/2)}{\partial V^2}\right|_{N,T} + \frac{1}{2}\frac{\alpha n^2}{V} - N\mu B\left(\frac{\partial^2 r_0}{\partial V^2}\right)_{N,T} - N\mu B\alpha\left[\frac{\partial^2}{\partial V^2}\left(\frac{nr_0}{D}\right)\right]_{N,T} > 0. \quad (6)$$

$$\left.2\frac{\partial^2 A_0(N/2)}{\partial T^2}\right|_{N,V} - N\mu B\left(\frac{\partial^2 r_0}{\partial T^2}\right)_{N,V} - N\mu B\alpha n\left[\frac{\partial^2}{\partial T^2}\left(\frac{r_0}{D}\right)\right]_{N,V} < 0. \quad (7)$$

2.1. Case 1. $T \gg T_{\rm F}$

In the case of high temperatures, i.e. $T \gg T_{\rm F}$, we have[19]

$$u_0 = u_0(T,V,N/2) = -kT\ln\left\{\frac{8\pi}{n}\left(\frac{mc}{h}\right)^3\frac{K_2(\phi)}{\phi}\right\} - mc^2. \quad (8)$$

The corresponding free energy can be expressed as

$$A_0(T,V,N/2) = -kT\int\ln\left\{\frac{8\pi}{n}\left(\frac{mc}{h}\right)^3\frac{K_2(\phi)}{\phi}\right\}{\rm d}\left(\frac{N}{2}\right) - \frac{N}{2}mc^2, \quad (9)$$

where $K_n(\phi)$ is the correct Bessel function, $\phi = \beta mc^2$, k is the Boltzmann constant, and h is the Planck constant. Substituting expressions (8) and (9) into expressions (6) and (7), we have the thermodynamic stability conditions of the system at high temperatures.

Especially, considering the ultrarelativistic case, i.e.,$\phi \ll 1$, we have

$$u_0\left(\frac{N}{2},V,T\right) \approx kT\ln\left[\frac{n}{16\pi}\left(\frac{hc}{kT}\right)^3\right], \quad (10)$$

$$A_0\left(\frac{N}{2},V,T\right)\approx\int kT\ln\left[\frac{n}{16\pi}\left(\frac{hc}{kT}\right)^3\right]\mathrm{d}\left(\frac{N}{2}\right)=\frac{NkT}{2}\left\{\ln\left[\frac{n}{16\pi}\left(\frac{hc}{kT}\right)^3\right]-1\right\}. \tag{11}$$

Via calculations, we can obtain the stability conditions:

$$kT+\frac{1}{2}\alpha n\left[1-2\left(\frac{\mu B}{kT}\right)^2\right]>0, \tag{12}$$

$$1+\frac{2}{3}\left(\frac{\mu B}{kT}\right)^2\left[1+\frac{3}{2}\alpha n\right]>0. \tag{13}$$

When $\phi\gg 1$, i.e. under the nonrelativistic condition, there are

$$u_0\left(\frac{N}{2},V,T\right)\approx kT\ln\left(\frac{n}{2}\lambda^3\right), \tag{14}$$

$$A_0\left(\frac{N}{2},V,T\right)=\int kT\ln\left(\frac{n}{2}\lambda^3\right)\mathrm{d}\left(\frac{N}{2}\right)=\frac{NkT}{2}\left[\ln\frac{n\lambda^3}{2}-1\right]. \tag{15}$$

Via similar calculations, we obtain the stability conditions

$$kT+\frac{1}{2}\alpha n\left[1-2\left(\frac{\mu B}{kT}\right)^2\right]>0, \tag{16}$$

$$1+\frac{4}{3}\left(\frac{\mu B}{kT}\right)^2\left[1+\frac{3}{2}\alpha n\right]>0. \tag{17}$$

2.2. Case 2. $T\to 0$

At extremely low temperatures, i.e. $T\to 0$, we have[19]

$$u_0\left(\frac{N}{2},V,T\right)=mc^2\sqrt{1+\left(\frac{3h^3N}{8\pi V}\right)^{2/3}\frac{1}{(mc)^2}}-mc^2. \tag{18}$$

The corresponding free energy can be written as

$$A_0\left(\frac{N}{2},V,T\right)=\int mc^2\sqrt{1+\left(\frac{3h^3N}{8\pi V}\right)^{2/3}\frac{1}{(mc)^2}}\mathrm{d}\left(\frac{N}{2}\right)-\frac{N}{2}mc^2. \tag{19}$$

Substituting expressions (18) and (19) into inequality (6) yields the mechanical stability conditions at $T=0$. Especially, in the ultrarelativistic case, there are

$$u_0\left(\frac{N}{2},V,T\right)\approx hc\left(\frac{3n}{8\pi}\right)^{1/3}, \tag{20}$$

$$A_0\left(\frac{N}{2},V,T\right)=\int hc\left(\frac{3n}{8\pi}\right)^{1/3}\mathrm{d}\left(\frac{N}{2}\right)=\frac{3}{8}hcN\left(\frac{3n}{8\pi}\right)^{1/3}. \tag{21}$$

Therefore, we have the mechanical stability condition

$$1+2\left(\frac{\mu B}{hc}\right)^2\left(\frac{8\pi}{3n}\right)^{2/3}>\frac{3}{hc}\left(\frac{8\pi}{3}\right)^{1/3}\alpha\left[2\left(\frac{\mu B}{hc}\right)^2\left(\frac{8\pi}{3}\right)^{2/3}-\frac{n^{2/3}}{2}\right]. \tag{22}$$

In the nonrelativistic case, there are

$$u_0\left(\frac{N}{2},V,T\right)\approx\frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{2/3}, \tag{23}$$

$$A_0\left(\frac{N}{2},V,T\right)=\int\frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{2/3}\mathrm{d}\left(\frac{N}{2}\right)=\frac{3}{10}\frac{h^2}{2m}N\left(\frac{3n}{8\pi}\right)^{2/3}. \tag{24}$$

Consequently, we have the mechanical stability condition

$$1+2\left(\frac{\mu Bm}{h^2}\right)^2\left(\frac{8\pi}{3n}\right)^{4/3}>-\frac{3m}{2h^2}\left(\frac{8\pi}{3}\right)^{2/3}\alpha\left[n^{1/3}+2\left(\frac{\mu Bm}{h^2}\right)^2\left(\frac{8\pi}{3}\right)^{4/3}\frac{1}{n}\right]. \tag{25}$$

2.3. Analysis and discussion

Through the comparison between inequalities (12) and (13) and expressions (16) and (17), we know that at high temperatures, the thermodynamic stability conditions of the system are identical, no matter whether it is an ultrarelativistic case or nonrelativistic case. When the conditions of the weak magnetic field and the weak interaction are satisfied, i.e. $\left(\frac{\mu B}{kT}\right)^2 \ll 1$ and $\frac{\alpha n}{kT} \ll 1$, the system always keeps stable. Inequalities (12) and (13) indicate that the thermodynamic stability of the system is related to temperature, magnetic field, particle-number density and inter-particle interaction. The mechanical stability will rise, but thermal stability will fall when temperature goes up. The mechanical stability will weaken, but thermal stability will strengthen when the magnetic field increases. The repulsive interaction increases the thermodynamic stability of the system, moreover, the stability increases more obviously with the density of the particle-number also increasing. The attractive interaction debases the thermodynamic stability of the system, moreover, the stability is distinctly depressed due to the increasing of particle-number density.

Inequalities (22) and (25) indicate that at zero degree temperature, the mechanical stability of the system is correlated with magnetic field, particle-number density and interparticle interactions. Under the ultrarelativistic condition, the system must satisfy inequality (22) for keeping the mechanical stability. Because of

$$2\left(\frac{\mu B}{hc}\right)^2\left(\frac{8\pi}{3}\right)^{2/3} - \frac{n^{2/3}}{2} < 0,$$

when the interactions are repulsive, the system always maintains mechanical stability, moreover, the stability will rise along with magnetic field strengthening. Whereas, when the interactions are attractive, particle-number density may attain a very large value under the ultrarelativistic condition, and attractive interactions also may increase to a certain degree. Thus inequality (22) may not be satisfied and the mechanical instability of the system may appear. Furthermore, the mechanical instability will weaken, when the magnetic field strengthens.

At extremely low temperatures, i.e. $T \to 0$, when the nonrelativistic case is considered, the mechanical stability possesses similar conclusions by comparing the nonrelativistic case with the ultrarelativistic case.

3. Thermal stability of Bose gas with consideration of the relativistic effect

We consider an ideal relativistic Bose gas in a weak magnetic field. The single-particle Hamiltonian can be expressed as

$$H(p,r) = \sqrt{p^2c^2 + m^2c^4} + U(r), \qquad (26)$$

where c is the speed of light and $U(r)$ is the external potential, r is the coordinate of a particle.

In the present paper, we study a Bose system of the particles with nonzero rest mass, and choose $D = 1$ and $t_k = 1$, where D and t_k are the dimensionality of space and the strength of the external potential respectively. The system studied here satisfies the condition for the BEC occurrence, i.e. $\sum_k 1//t_k + D/2 > 1$. In Ref.[20] were investigated the features of the heat capacity of a relativistic Bose gas trapped in a general external potential such as a D-dimensional box, a D-dimensional harmonic potential. For the system that we studied, we may deduce from the studied results of the heat capacity in Ref.[20] that, compared with the nonrelativistic case, when $T < T_{\rm C}$, the relativistic effect decreases the thermal stability below a certain temperature but results in the increase in the thermal stability above this temperature; when $T > T_{\rm C}$, the effect of relativity increases the thermal stability.

4. Conclusions

Based on the consideration of the effect of relativity, in the present paper the method of quantum statistics is used to study the stability of a weakly interacting Fermi gas in a weak magnetic field, the analytical expressions of stability conditions at both high and extremely low temperatures are presented, and analyses of the effects of many factors on the stability are performed. These factors include temperature, magnetic field, interparticle interaction and particle-number density. The results show that the thermodynamic stability conditions of the system are identical at high temperatures, no matter whether the case is ultrarelativistic or nonrelativistic. At extremely low temperatures, the mechanical stability conditions of the system show a similar rule through a comparison between the ultrarelativistic case and nonrelativistic

case. For an ideal relativistic Bose gas in a weak magnetic field, compared with the nonrelativistic case, it is found that when $T < T_{\rm C}$, the effect of relativity reduces the thermal stability below a certain temperature but results in the increase in thermal stability above this temperature; when $T > T_{\rm C}$, the effect of relativity increases the thermal stability.

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