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Radon transforms of the Wigner operator on hyperplanes*

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The generalization of tomographic maps to hyperplanes is considered. We find that the Radon transform of the Wigner operator in multi-dimensional phase space leads to a normally ordered operator in binomial distribution—a mixed-state density operator. Reconstruction of the Wigner operator is also feasible. The normally ordered form and the Weyl ordered form of the Wigner operator are used in our derivation. The operator quantum tomography theory is expressed in terms of some operator identities, with the merit of revealing the essence of the theory in a simple and concise way.

Keywords: Radon transform, Wigner operator, hyperplane IWOP
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1. Introduction

Tomographic imaging techniques derive two-dimensional data from a three-dimensional object to obtain a slice image of the internal structure and thus have the power to observe the interior object non-invasively. It is commonly believed that the Radon transform[1] of the Wigner function[2−7] is the core of the quantum tomographic approach to the study of quantum states, and we think that the Radon transform of the Wigner operator is more essential and should receive more attention, because we can present a beautiful operator formulation in this field, which brings much convenience to deriving tomograms of quantum states. Tomography theory in quantum optics is as follows. Let $\lambda X + \mu P$ (μ and λ are real numbers and independent of each other) represent all possible linear combinations of quadratures X and P of the oscillator field mode which can be measured by the homodyne measurement just through varying the phase of the local oscillator, then the average of the random outcomes of the measurement, at a given local oscillator phase, will be connected with the marginal distribution of the Wigner function of the field, thus the homodyne measurement of light field permits the reconstruction of the Wigner function of a quantum system by varying the phase of the local oscillator. Once the distribution is obtained, one can use the inverse Radon transformation family in tomographic imaging to obtain the Wigner distribution and density matrix. Usually, the Wigner operator in the one-dimensional case is defined in the coordinate representation as

$$\Delta\left(x,p\right)=\int_{-\infty}^{\infty}\frac{\mathrm{d}u\mathrm{d}v}{2\pi}\mathrm{e}^{\mathrm{i}pu}\left|x+\frac{v}{2}\right\rangle_{11}\left\langle x-\frac{v}{2}\right|, \quad (1)$$

where $|x\rangle_1$ is the eigenvector of X_1. In our previous papers,[8,9] by the technique of integration within an ordered product (IWOP) of operators[10−12] we have derived the normally ordered form of $\Delta_1\left(x,p\right)$,

$$\Delta_1\left(x,p\right)=\frac{1}{\pi}:\mathrm{e}^{-(x-X_1)^2-(p-P_1)^2}:. \quad (2)$$

Expression (2) can be alternately obtained by noticing $|x\rangle_{11}\langle x|=\frac{1}{\sqrt{\pi}}:\mathrm{e}^{-(x-X_1)^2}:$, and $|p\rangle_{11}\langle p|=\frac{1}{\sqrt{\pi}}:\mathrm{e}^{-(p-P_1)^2}:$, where P_1 is the momentum operator with eigenvector $|p\rangle_1$ and $[X_1,P_1]=\mathrm{i}\hbar$ (in the following we adopt $\hbar=1$). Using expression (2) to perform the Radon transform leads to[13]

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$$\int_{-\infty}^{\infty} \mathrm{d}x'\mathrm{d}p'\delta\left(x-\lambda x'-\mu p'\right)\Delta_1(x',p') = \frac{1}{\sqrt{\pi(\lambda^2+\mu^2)}} : \exp\left\{-\frac{1}{\lambda^2+\mu^2}\left[x-(\lambda X_1+\mu P_1)\right]^2\right\} : . \tag{3}$$

Noting $X_1 = \left(a_1 + a_1^\dagger\right)/2$, $P_1 = \mathrm{i}\left(a_1^\dagger - a_1\right)/\sqrt{2}$, $a^\dagger$ is the bosonic creation operator, and $\left[a, a^\dagger\right] = 1$, and using $|0\rangle\langle 0| =: \mathrm{e}^{-a_1^\dagger a_1} :$, we can decompose the terms on the right-hand side of expression (3) into a pure state

$$\frac{1}{\sqrt{\pi(\lambda^2+\mu^2)}} : \exp\left\{-\frac{1}{\lambda^2+\mu^2}\left[x-(\lambda X_1+\mu P_1)\right]^2\right\} := |x,\lambda,\mu\rangle\langle x,\lambda,\mu|\,, \tag{4}$$

where $|x,\lambda,\mu\rangle$ in Fock space is expressed as

$$|x,\lambda,\nu\rangle = [\pi(\lambda^2+\mu^2)]^{-1/4}\exp\left[-\frac{x^2}{2(\lambda^2+\mu^2)} + \sqrt{2}\frac{x}{\lambda-\mathrm{i}\mu}a_1^\dagger - \frac{\lambda+\mathrm{i}\mu}{2(\lambda-\mathrm{i}\mu)}a_1^{\dagger 2}\right]|0\rangle \tag{5}$$

with $a_1|0\rangle = 0$. $|x,\lambda,\nu\rangle$ is just the eigenvector of the operator $\lambda X_1 + \mu P_1$:

$$(\lambda X_1 + \mu P_1) = x\,|x,\lambda,\mu\rangle\,. \tag{6}$$

Hence $|x,\lambda,\mu\rangle$ is inherent to the Radon transformation (with two parameters) of the Wigner operator, and the completeness relation is

$$\int_{-\infty}^{\infty}\mathrm{d}x\,|x,\lambda,\mu\rangle\langle x,\lambda,\mu| = 1. \tag{7}$$

So $|x,\lambda,\mu\rangle$ is qualified to be a new quantum mechanical representation, and

$$\begin{aligned}&|x,\lambda,\mu\rangle\langle x,\lambda,\mu|\\ &= \int_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}p'\delta\left(x-\lambda x'-\mu p'\right)\Delta(x',p'),\end{aligned} \tag{8}$$

which is the Radon transformation of tomography theory in operator form, and brings much convenience for calculating tomograms of various quantum states $\langle x,\lambda,\mu|\,\psi\rangle$.

We also find the Weyl ordered form of the Wigner operator[14]

$$\Delta(x,p) = \vdots\,\delta\left(x-X_1\right)\delta\left(p-P_1\right)\,\vdots\,, \tag{9}$$

where the symbol $\vdots\,\vdots$ denotes Weyl ordering, so the Weyl ordering form of $|x,\lambda,\mu\rangle\langle x,\lambda,\mu|$ is expressed as

$$|x,\lambda,\mu\rangle\langle x,\lambda,\mu| = \int_{-\infty}^{\infty}\mathrm{d}q'\mathrm{d}p'\delta\left(x-\lambda x'-\mu p'\right)\vdots\,\delta\left(x'-X_1\right)\delta\left(p'-P_1\right)\,\vdots = \vdots\,\delta\left(x-\lambda X_1-\mu P_1\right)\,\vdots\,. \tag{10}$$

The marginal distribution of expression (10) reads

$$\begin{aligned}|x,\mu=0,\nu=1\rangle\langle x,\mu=0,\nu=1| &= \vdots\,\delta\left(x-P\right)\,\vdots = \delta\left(x-P\right) = |p\rangle_{11}\langle p|_{p=x}\,,\\ |x,\mu=1,\nu=0\rangle\langle x,\mu=1,\nu=0| &= \vdots\,\delta\left(x-X\right)\,\vdots = \delta\left(x-X\right) = |q\rangle_{11}\langle q|_{q=x}\,.\end{aligned} \tag{11}$$

The aim of the present paper is to study how to implement tomographic maps onto hyperplanes, i.e. how to generalize the Wigner operator's Radon transform to multidimensional phase spaces. In Section 2 we show that the Radon transform of the Wigner operator on hyperplanes is no longer a pure state, but instead a mixed-state density operator which exhibits a binomial distribution in normal ordering. In Section 3 we reconstruct the Wigner operator. In our derivation, we make full use of the IWOP technique.

2. Radon transform of the Wigner operator on hyperplanes

The above construction can be generalized to higher-dimensional space by the IWOP technique. Now we consider the Wigner operator in the $2n$-dimensional phase space

$$\Delta(\boldsymbol{x},\boldsymbol{p})=\frac{1}{\pi^n}:\mathrm{e}^{-(\boldsymbol{x}-\boldsymbol{X})^2-(\boldsymbol{p}-\boldsymbol{P})^2}:. \tag{12}$$

A generic hyperplane is given by the equation

$$x-\boldsymbol{\lambda}\cdot\boldsymbol{x}'-\boldsymbol{\mu}\cdot\boldsymbol{p}'=0. \tag{13}$$

We calculate the Radon transformation of the 2n-dimesional Wigner operator $\Delta(\boldsymbol{x}',\boldsymbol{p}')$

$$\int_{-\infty}^{+\infty}\mathrm{d}^n\boldsymbol{x}'\mathrm{d}^n\boldsymbol{p}'\delta\left(x-\boldsymbol{\lambda}\cdot\boldsymbol{x}'-\boldsymbol{\mu}\cdot\boldsymbol{p}'\right)\Delta(\boldsymbol{x}',\boldsymbol{p}')\equiv I. \tag{14}$$

Let $\boldsymbol{q}=(\boldsymbol{x}',\ \boldsymbol{p}')$ be generalized coordinates, and

$$\boldsymbol{\Lambda}=(\boldsymbol{\lambda},\ \boldsymbol{\mu}),\ \boldsymbol{Q}=(\boldsymbol{X},\ \boldsymbol{P}), \tag{15}$$

we can write expression (14) in the concise form

$$I=\frac{1}{\pi^n}\int_{-\infty}^{+\infty}\mathrm{d}^{2n}\boldsymbol{q}\delta\left(x-\boldsymbol{\Lambda}\cdot\boldsymbol{q}\right):\mathrm{e}^{-(\boldsymbol{q}-\boldsymbol{Q})^2}:. \tag{16}$$

Since $\boldsymbol{\Lambda}$ is a fixed vector, we can rotate the generalized coordinate system $\boldsymbol{q}\to\boldsymbol{q}'=R\boldsymbol{q}$, $\mathrm{d}^{2n}\boldsymbol{q}\to\mathrm{d}^{2n}\boldsymbol{q}'$, such that the first coordinate axis coincides with the direction of vector $\boldsymbol{\Lambda}$, let q_1' be the first component of vector $\boldsymbol{q}$ in the new coordinate system, then

$$\boldsymbol{\Lambda}\cdot\boldsymbol{q}=|\boldsymbol{\Lambda}|\,q_1'=\Lambda q_1'=\sqrt{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}q_1'. \tag{17}$$

Correspondingly, the first component of $\boldsymbol{Q}$ in the new coordinate system is

$$Q_1'=\frac{1}{\Lambda}\Lambda Q_1'=\frac{1}{\Lambda}\boldsymbol{\Lambda}\cdot\boldsymbol{Q}. \tag{18}$$

It then follows that

$$\begin{aligned}I&=\frac{1}{\pi^n}\int_{-\infty}^{+\infty}\mathrm{d}^{2n}\boldsymbol{q}'\delta\left(x-\Lambda q_1'\right):\mathrm{e}^{-\left(\boldsymbol{q}'-\boldsymbol{Q}'\right)^2}:\\&=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}q_1'\delta\left(x-\Lambda q_1'\right):\mathrm{e}^{-\left(q_1'-Q_1'\right)^2}:\\&=\frac{1}{\Lambda\sqrt{\pi}}:\mathrm{e}^{-\left(\frac{x}{\Lambda}-Q_1'\right)^2}:=\frac{1}{\Lambda\sqrt{\pi}}:\mathrm{e}^{-\frac{1}{\Lambda^2}(x-\boldsymbol{\Lambda}\cdot\boldsymbol{Q})^2}:\\&=\frac{1}{\sqrt{\pi}\sqrt{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}}:\mathrm{e}^{-\frac{1}{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}(x-\boldsymbol{\lambda}\cdot\boldsymbol{X}-\boldsymbol{\mu}\cdot\boldsymbol{P})^2}:,\end{aligned} \tag{19}$$

which takes the form of binomial distribution within normal ordering. In the case $n=1$, expression (19) reduces to expression (4), a pure-state density operator, so the tomogram is positive definite. For $n\neq 1$, I represents a mixed-state density operator, because

$$\frac{1}{\sqrt{\pi}\sqrt{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}}\int_{-\infty}^{+\infty}\mathrm{d}x:\mathrm{e}^{-\frac{1}{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}(x-\boldsymbol{\lambda}\cdot\boldsymbol{X}-\boldsymbol{\mu}\cdot\boldsymbol{P})^2}:=1. \tag{20}$$

3. Reconstruction of the Wigner operator

Is it possible to reconstruct the Wigner operator from expression (19), where its integrals are performed over arbitrary $2(n-1)$-dimensional linear submanifolds? The answer is positive. Now we carry out the following integration:

$$\begin{aligned}&\frac{1}{\sqrt{\pi}\sqrt{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}}\int_{-\infty}^{+\infty}\mathrm{d}x:\mathrm{e}^{-\frac{1}{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}(x-\boldsymbol{\lambda}\cdot\boldsymbol{X}-\boldsymbol{\mu}\cdot\boldsymbol{P})^2}:\mathrm{e}^{-\mathrm{i}gx}\\&=\frac{1}{\sqrt{\pi}\sqrt{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}}\int_{-\infty}^{+\infty}\mathrm{d}x:\mathrm{e}^{-\frac{1}{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}\left[x-\boldsymbol{\lambda}\cdot\boldsymbol{X}-\boldsymbol{\mu}\cdot\boldsymbol{P}-\frac{\mathrm{i}g\left(\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2\right)}{2}\right]^2-\mathrm{i}g(\boldsymbol{\lambda}\cdot\boldsymbol{X}+\boldsymbol{\mu}\cdot\boldsymbol{P})-\frac{g^2}{4}\left(\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2\right)}:\\&=:\mathrm{e}^{-\mathrm{i}g(\boldsymbol{\lambda}\cdot\boldsymbol{X}+\boldsymbol{\mu}\cdot\boldsymbol{P})-\frac{g^2}{4}\left(\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2\right)}:.\end{aligned} \tag{21}$$

Introducing $\boldsymbol{\beta}=\dfrac{\boldsymbol{\lambda}-\mathrm{i}\boldsymbol{\mu}}{\sqrt{2}}$,

$$\begin{aligned}\sqrt{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}&=\sqrt{2}\,|\boldsymbol{\beta}|,\\\boldsymbol{\lambda}\cdot\boldsymbol{X}+\boldsymbol{\mu}\cdot\boldsymbol{P}&=\boldsymbol{\beta}\cdot\boldsymbol{a}+\boldsymbol{\beta}^*\cdot\boldsymbol{a}^\dagger,\end{aligned} \tag{22}$$

and using the Baker–Hausdorff formula leads the term on the right-hand side of expression (21) to the following

result:

$$\begin{aligned} &: \mathrm{e}^{-\mathrm{i}g(\boldsymbol{\lambda}\cdot\boldsymbol{X}+\boldsymbol{\mu}\cdot\boldsymbol{P})-\frac{g^2}{2}|\boldsymbol{\beta}|^2} := \mathrm{e}^{-\mathrm{i}g\boldsymbol{\beta}^*\cdot\boldsymbol{a}^\dagger}\mathrm{e}^{-\mathrm{i}g\boldsymbol{\beta}\cdot\boldsymbol{a}}\mathrm{e}^{-\frac{g^2}{2}|\boldsymbol{\beta}|^2} \\ &= \mathrm{e}^{-\mathrm{i}g\left(\boldsymbol{\beta}^*\cdot\boldsymbol{a}^\dagger+\boldsymbol{\beta}\cdot\boldsymbol{a}\right)}. \end{aligned} \tag{23}$$

Using the Weyl ordering symbol and expression (9) yields

$$\begin{aligned} \int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{\sqrt{\pi}\sqrt{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}} &: \mathrm{e}^{-\frac{1}{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}(x-\boldsymbol{\lambda}\cdot\boldsymbol{X}-\boldsymbol{\mu}\cdot\boldsymbol{P})^2} : \mathrm{e}^{-\mathrm{i}gx} \\ &= \mathrm{e}^{-\mathrm{i}g(\boldsymbol{\lambda}\cdot\boldsymbol{X}+\boldsymbol{\mu}\cdot\boldsymbol{P})} = \vdots\mathrm{e}^{-\mathrm{i}g(\boldsymbol{\lambda}\cdot\boldsymbol{X}+\boldsymbol{\mu}\cdot\boldsymbol{P})}\vdots \\ &= \int_{-\infty}^{\infty}\mathrm{d}^n\boldsymbol{x}'\mathrm{d}^n\boldsymbol{p}'\vdots\delta\left(\boldsymbol{x}'-\boldsymbol{X}\right)\delta\left(\boldsymbol{p}'-\boldsymbol{P}\right)\vdots\mathrm{e}^{-\mathrm{i}g(\boldsymbol{\lambda}\cdot\boldsymbol{x}'+\boldsymbol{\mu}\cdot\boldsymbol{p}')} \\ &= \int_{-\infty}^{\infty}\mathrm{d}^n\boldsymbol{x}'\mathrm{d}^n\boldsymbol{p}'\Delta(\boldsymbol{x}',\boldsymbol{p}')\mathrm{e}^{-\mathrm{i}g(\boldsymbol{\lambda}\cdot\boldsymbol{x}'+\boldsymbol{\mu}\cdot\boldsymbol{p}')}. \end{aligned} \tag{24}$$

Set $g=-1$ and consider the term on the right-hand side of expression (24) as a Fourier transform; its inverse transform will be

$$\begin{aligned} \frac{1}{(2\pi)^{2n}}\int_{-\infty}^{\infty}\frac{\mathrm{d}x\mathrm{d}^n\boldsymbol{\lambda}\mathrm{d}^n\boldsymbol{\mu}}{\sqrt{\pi}\sqrt{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}} &: \mathrm{e}^{-\frac{1}{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}(x-\boldsymbol{\lambda}\cdot\boldsymbol{X}-\boldsymbol{\mu}\cdot\boldsymbol{P})^2} : \mathrm{e}^{\mathrm{i}\left(x-\boldsymbol{\lambda}\cdot\boldsymbol{x}'-\boldsymbol{\mu}\cdot\boldsymbol{p}'\right)} \\ &= \frac{1}{(2\pi)^{2n}}\int_{-\infty}^{\infty}\mathrm{d}^n\boldsymbol{\lambda}\mathrm{d}^n\boldsymbol{\mu} : \mathrm{e}^{\mathrm{i}(\boldsymbol{\lambda}\cdot\boldsymbol{X}+\boldsymbol{\mu}\cdot\boldsymbol{P})-\frac{1}{4}\left(\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2\right)} : \mathrm{e}^{\mathrm{i}\left(-\boldsymbol{\lambda}\cdot\boldsymbol{x}'-\boldsymbol{\mu}\cdot\boldsymbol{p}'\right)} \\ &= \frac{1}{(2\pi)^{2n}}\int_{-\infty}^{\infty}\mathrm{d}^{2n}\boldsymbol{\Lambda} : \mathrm{e}^{-\frac{1}{4}\boldsymbol{\Lambda}^2-\mathrm{i}\boldsymbol{\Lambda}\cdot(\boldsymbol{q}-\boldsymbol{Q})} : \\ &= \frac{1}{(2\pi)^{2n}}\int_{-\infty}^{\infty}\mathrm{d}^{2n}\boldsymbol{\Lambda} : \mathrm{e}^{-\frac{1}{4}[\boldsymbol{\Lambda}+2\mathrm{i}(\boldsymbol{q}-\boldsymbol{Q})]^2-(\boldsymbol{q}-\boldsymbol{Q})^2} : \\ &= \frac{1}{\pi^n} : \mathrm{e}^{-(\boldsymbol{q}-\boldsymbol{Q})^2} := \Delta(\boldsymbol{x}',\boldsymbol{p}'). \end{aligned} \tag{25}$$

Expression (25) is the reconstruction part of tomography theory in operator form. Thus the Wigner function is derived as

$$\langle\psi|\,\Delta(\boldsymbol{x}',\boldsymbol{p}')\,|\psi\rangle = \frac{1}{(2\pi)^{2n}}\int_{-\infty}^{\infty}\frac{\mathrm{d}x\mathrm{d}^n\boldsymbol{\lambda}\mathrm{d}^n\boldsymbol{\mu}}{\sqrt{\pi}\sqrt{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}}\langle\psi| : \mathrm{e}^{-\frac{1}{\boldsymbol{\lambda}^2+\boldsymbol{\mu}^2}(x-\boldsymbol{\lambda}\cdot\boldsymbol{X}-\boldsymbol{\mu}\cdot\boldsymbol{P})^2} : |\psi\rangle\,\mathrm{e}^{\mathrm{i}\left(x-\boldsymbol{\lambda}\cdot\boldsymbol{x}'-\boldsymbol{\mu}\cdot\boldsymbol{p}'\right)}. \tag{26}$$

Finally we present a concise form of expression (21), by setting $b=\dfrac{\boldsymbol{\beta}\cdot\boldsymbol{a}}{\beta}$,

$$\left[b,b^\dagger\right]=\frac{1}{\beta^2}\sum_{i,j=1}^{n}\beta_i\beta_j^*\left[a_i,a_j^\dagger\right]=1, \tag{27}$$

so b is also an annihilation operator, expression (21) becomes

$$\begin{aligned} &\int_{-\infty}^{+\infty}\mathrm{d}^n\boldsymbol{x}'\mathrm{d}^n\boldsymbol{p}'\delta\left(x-\boldsymbol{\lambda}\cdot\boldsymbol{x}'-\boldsymbol{\mu}\cdot\boldsymbol{p}'\right)\Delta\left(\boldsymbol{x}',\boldsymbol{p}'\right) \\ &= \frac{1}{\sqrt{2\pi}\beta} : \mathrm{e}^{-\frac{1}{2\beta^2}\left(x-\boldsymbol{\beta}\cdot\boldsymbol{a}-\boldsymbol{\beta}^*\cdot\boldsymbol{a}^\dagger\right)^2} : \\ &= \frac{1}{\sqrt{2\pi}\beta} : \mathrm{e}^{-\frac{1}{2\beta^2}\left[x-\beta\left(b+b^\dagger\right)\right]^2} : \\ &= \frac{1}{\sqrt{2\pi}\beta}\mathrm{e}^{-\frac{x^2}{2\beta^2}}\mathrm{e}^{\frac{x}{\beta}b^\dagger-\frac{1}{2}b^{\dagger 2}} : \mathrm{e}^{-b^\dagger b} : \mathrm{e}^{\frac{x}{\beta}b-\frac{1}{2}b^2}. \end{aligned} \tag{28}$$

4. Summary

We present an operator tomography formalism in $2n$-dimensinal phase space which is expressed in terms of some operator identities. The Radon transform of the Wigner operator turns out to be a mixed-state density operator, exhibiting a binomial distribution within normal ordering, and has the merit of revealing the essence of the theory in a simple and concise way.

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