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Economical phase-covariant cloning with multiclones*

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This paper presents a very simple method to derive the explicit transformations of the optimal economical 1 to M phase-covariant cloning. The fidelity of clones reaches the theoretic bound [D'Ariano G M and Macchiavello C 2003 *Phys. Rev.* A **67** 042306]. The derived transformations cover the previous contributions [Delgado Y, Lamata L *et al*, 2007 *Phys. Rev. Lett.* **98** 150502] in which M must be odd.

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In quantum information theory, the important fundamental concept is the no-cloning theorem[1,2] which states that an unknown quantum state can not be copied perfectly. This theorem guarantees the absolute security of quantum cryptography.[3] Though perfect clones are prohibited, imperfect copies can be made in an approximate fashion.[4] Recently, a vast contribution has focused on quantum cloning.[5−15] Some schemes are proposed to implement various cloners.[16−21] In experiment, some cloners are realized.[22−26] Moreover, quantum cloning theory has important applications to quantum information processing.

For an input state $|\psi\rangle^{(\mathrm{in})} = \left(|0\rangle + \mathrm{e}^{\mathrm{i}\varphi}|1\rangle\right)/\sqrt{2}(\varphi \in [0, 2\pi)$ is unknown) to be cloned, the cloner is the so-called phase-covariant cloning. Phase-covariant cloning has an important application as the individual attack in quantum cryptography, and phase-covariant cloning itself has an intimate relation with phase estimation.[27,28] In this paper, we study phase-covariant cloning without ancilla (hence the term 'economical'). Without the presence of the ancilla, it can be observed how the phase information in the initial input state is distributed to the final system after the cloning process. In this paper, we present a very simple method to derive the explicit transformations of the optimal economical $1 \to M$ phase-covariant cloning in 2 dimensions. The fidelities are coincident with the theoretic values.[11] In Ref.[29], by using a sequential cloning process, the authors derived the optimal economical $1 \to M = 2k + 1$ phase-covariant cloning transformation. When M is odd or even, we derive the explicit transformations.

We first briefly review some previous contributions. In Ref.[11], the authors explored a complete-positive trace-preserving map to derive the fidelity of the $1 \to M$ phase-covariant cloning as

$$F_{1,M} = \frac{1}{2} + \frac{M+1}{4M} = F_{1\to M=2k+1}. \tag{1}$$

This fidelity is optimal when $M = 2k+1$. In the meantime, the authors also presented another expression of the fidelity, given by

$$\begin{aligned} F_{1,M} &= \frac{1}{2} + \frac{\sqrt{M\,(M+2)}}{4M} \\ &= F_{1\to M=2k}. \end{aligned} \tag{2}$$

In fact, Eq.(2) defines the optimal fidelity when $M = 2k$. Recently, Ref.[29] presented the explicit transformation of the optimal economical $1 \to M = 2k + 1$ phase-covariant cloning in the form of

$$\begin{aligned} |\psi\rangle^{(\mathrm{in})}|B\rangle \to \frac{1}{\sqrt{2}}[&|(k+1)\,0, k1\rangle \\ &+\mathrm{e}^{\mathrm{i}\varphi}|k0, (k+1)\,1\rangle], \end{aligned} \tag{3}$$

where the state $|(k+1)\,0, k1\rangle$ denotes the completely symmetric normalized state with $(k+1)$ qubits in the state $|0\rangle$ and k qubits in the state $|1\rangle$. When $M = 2k$,

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the authors did not derive the transformation. In the following, we completely present the explicit transformations.

For convenience in our next derivation, we use the notation $|\{k+1,0\},\{k,1\}\rangle$ instead of the state $|(k+1)\,0,k1\rangle$, and another one $|(k+1,0),(k,1)\rangle$ being completely symmetric but not normalized. The two states have the relation $|(k+1,0),(k,1)\rangle = \sqrt{C_{2k+1}^{k}}\,|\{k+1,0\},\{k,1\}\rangle$. We also denote the set $\{|(k+1,0),(k,1)\rangle\}$ (or $\{|\{k+1,0\},\{k,1\}\rangle\}$) containing C_{2k+1}^{k} elements.

Generally, quantum cloning transformation is a completely-positive trace-preserving map. For a trace-preserving matrix

$$\boldsymbol{\rho} = \begin{vmatrix} A & B\mathrm{e}^{-\mathrm{i}\varphi} \\ B\mathrm{e}^{\mathrm{i}\varphi} & 1-A \end{vmatrix} \tag{4}$$

with $A, B \geq 0$ and $A \leq 1$, the fidelity between the state $\boldsymbol{\rho}$ defined by Eq.(4) and the input state $|\psi\rangle^{(\mathrm{in})} = \big(|0\rangle + \mathrm{e}^{\mathrm{i}\varphi}\,|1\rangle\big)/\sqrt{2}(\varphi \in [0,2\pi)$ is unknown) to be cloned, can be easily calculated as

$$F = \mathrm{Tr}\left(|\psi\rangle^{(\mathrm{in})}\,\langle\psi|\,\rho\right) = 1/2 + B. \tag{5}$$

Equation (5) implies that the fidelity is optimal if and only if the off-diagonal element B is maximal. Therefore, our task is mainly to find the maximal value of the off-diagonal element B.

We want to clone one input state $|\psi\rangle^{(\mathrm{in})} = \big(|0\rangle + \mathrm{e}^{\mathrm{i}\varphi}\,|1\rangle\big)/\sqrt{2}(\varphi \in [0,2\pi)$ is unknown) and obtain M copies (the $1 \to M$ cloner) with equal and optimal fidelity. This requires the unitary transformation involving M qubits to be invariant under all permutations. We then divide the Hilbert space into $(M+1)$ subspaces $\{|(M-i,0),(i,1)\rangle\}_{i=0}^{M}$ (or the sets), and each subspace contains $C_M^i = M!/[(M-i)!i!]$ computational bases. In the case of $M = 2k+1$, the Hilbert space is divided into $(2k+2)$ subspaces. So, the most general transformation should take the following form:

$$\begin{aligned} |0\rangle &\to \sum_{i=0}^{k} a_i\,|\{2k+1-i,0\},\{i,1\}\rangle, \\ |1\rangle &\to \sum_{i=0}^{k} b_i\,|\{2k+1-i,1\},\{i,0\}\rangle \end{aligned} \tag{6}$$

with the normalization conditions $\sum_{i=0}^{k} a_i^2 = 1$ and $\sum_{i=0}^{k} b_i^2 = 1$. One can observe that the transformation contains all subspaces and thence is the most general one. By Eq.(5), we wish to find the maximal value B, and it is sufficient to calculate the fidelity of clone 1. Obviously, only two subsets $\{|\{k+1,0\},\{k,1\}\rangle\}$ and $\{|\{k+1,0\},\{k,1\}\rangle\}$ can create the off-diagonal elements after tracing out the other $(M-1)$ copies, i. e.,

$$\begin{aligned} a_k\,|\{k+1,0\},\{k,1\}\rangle &= \frac{a_k}{\sqrt{C_{2k+1}^{k}}}\left[|0\rangle_1\,|(k,0),(k,1)\rangle + |1\rangle_1\,|(k+1,0),(k-1,1)\rangle\right], \\ b_k\,|\{k+1,1\},\{k,0\}\rangle &= \frac{b_k}{\sqrt{C_{2k+1}^{k}}}\left[|1\rangle_1\,|(k,0),(k,1)\rangle + |0\rangle_1\,|(k-1,0),(k+1,1)\rangle\right]. \end{aligned} \tag{7}$$

To be precise, only the first two terms in the right-hand sides $|0\rangle_1\,|(k,0),(k,1)\rangle$ and $|1\rangle_1\,|(k,0),(k,1)\rangle$ can produce the off-diagonal element $B' = (a_k b_k)/\big(2C_{2k+1}^{k}\big)$ (the factor $1/2$ is produced by the amplitude of the input state $|\psi\rangle^{(\mathrm{in})} = \big(|0\rangle + \mathrm{e}^{\mathrm{i}\varphi}\,|1\rangle\big)/\sqrt{2}$). Since the set $\{|(k,0),(k,1)\rangle\}$ (or $\{|\{k,0\},\{k,1\}\rangle\}$) contains C_{2k}^{k} elements, we easily obtain

$$B = a_k b_k \frac{C_{2k}^{k}}{2C_{2k+1}^{k}} = a_k b_k \frac{k+1}{4k+2}. \tag{8}$$

It is obvious that the value B is maximal when $a_k = b_k = 1$ and other cloning coefficients are zero under normalization conditions. Consequently, we reach our first goal, i.e.,

$$F_{1\to M=2k+1} = \frac{1}{2} + \frac{k+1}{4k+2}. \tag{9}$$

Equation (9) is identical to Eq.(1). Thus, the transformation can be written as

$$\begin{aligned} |0\rangle &\to |\{k+1,0\},\{k,1\}\rangle, \\ |1\rangle &\to |\{k+1,1\},\{k,0\}\rangle. \end{aligned} \tag{10}$$

Equation (10) is identical to Eq.(3). When $M = 2k$, there exist $(2k+1)$ subsets to be distributed to the transformation of cloning the computational bases $|0\rangle$

and $|1\rangle$. Therefore, the most general evolution can be written as two independent transformations; one is

$$|0\rangle \to \sum_{n=0}^{k-1} a_n \left|\{2k-n,0\},\{n,1\}\right\rangle,$$
$$|1\rangle \to \sum_{m=0}^{k} b_m \left|\{2k-m,1\},\{m,0\}\right\rangle \qquad (11)$$

with the normalization conditions $\sum_{n=0}^{k-1} a_n^2 = 1$and $\sum_{m=0}^{k} b_m^2 = 1$, and the another is

$$|0\rangle \to \sum_{n=0}^{k} a_n \left|(2k-n,0),(n,1)\right\rangle,$$
$$|1\rangle \to \sum_{m=0}^{k-1} b_m \left|\{2k-m,1\},\{m,0\}\right\rangle \qquad (12)$$

with the normalization conditions $\sum_{n=0}^{k} a_n^2 = 1$ and $\sum_{m=0}^{k-1} b_m^2 = 1$. Let us solve the first case. The two subsets to create the off-diagonal elements are

$$a_{k-1}\left|\{k+1,0\},\{k-1,1\}\right\rangle = \frac{a_{k-1}}{\sqrt{C_{2k}^{k-1}}}\left[|0\rangle_1 \left|(k,0),(k-1,1)\right\rangle + |1\rangle_1 \left|(k+1,0),(k-2,1)\right\rangle\right],$$
$$b_k\left|\{k,1\},\{k,0\}\right\rangle = \frac{b_k}{\sqrt{C_{2k}^{k}}}\left[|1\rangle_1 \left|(k,0),(k-1,1)\right\rangle + |0\rangle_1 \left|(k-1,0),(k,1)\right\rangle\right]. \qquad (13)$$

Like the analysis above, two terms $|0\rangle_1 \left|(k,0),(k-1,1)\right\rangle$ and $|1\rangle_1 \left|(k,0),(k-1,1)\right\rangle$ can produce the off-diagonal element $B' = (a_{k-1}b_k)\Big/\left(2\sqrt{C_{2k}^{k-1}C_{2k}^{k}}\right)$, and the set $\{|\,(k,0),(k-1,\,1)\rangle\}$ contains C_{2k-1}^{k-1} elements. We obtain

$$B = a_{k-1}b_k \frac{C_{2k-1}^{k-1}}{2\sqrt{C_{2k}^{k-1}C_{2k}^{k}}} = a_{k-1}b_k\frac{\sqrt{k(k+1)}}{4k}. \qquad (14)$$

If and only if $a_{k-1} = b_k = 1$ and the other cloning coefficients are zero, the fidelity reaches maximal bound, i.e.,

$$F_{1\to M=2k} = \frac{1}{2} + \frac{\sqrt{k(k+1)}}{4k}. \qquad (15)$$

Equation (15) is identical to Eq.(2). The corresponding transformation can be determined as

$$|0\rangle \to \left|\{k+1,0\},\{k-1,1\}\right\rangle,$$
$$|1\rangle \to \left|\{k,1\},\{k,0\}\right\rangle. \qquad (16)$$

In the second case, we similarly determine another independent transformation if the form is

$$|0\rangle \to \left|\{k,0\},\{k,1\}\right\rangle,$$
$$|1\rangle \to \left|\{k+1,1\},\{k-1,0\}\right\rangle. \qquad (17)$$

Here, we have completed the derivation of the optimal economical $1 \to M$ phase-covariant cloning in 2 dimensions. The fidelities are coincident with the previous contributions.

In summary, we start with the most general transformation to obtain the explicit transformation of the optimal economical $1 \to M$ phase-covariant cloning in 2 dimensions. This economical phase-covariant cloning is the cloner without ancilla, and the phase information of the input state is distributed to M copies. This will facilitate the investigation of the phase estimation of a pure state in quantum information processing.

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