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Initial-boundary value problems for a class of nonlinear thermoelastic plate equations*

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This paper studies initial-boundary value problems for a class of nonlinear thermoelastic plate equations. Under some certain initial data and boundary conditions, it obtains an existence and uniqueness theorem of global weak solutions of the nonlinear thermoelstic plate equations, by means of the Galerkin method. Moreover, it also proves the existence of strong and classical solutions.

Keywords: thermoelastic, hinged plate, weak solution, classical solution
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1. Introduction

With the development of nonlinear dynamics, research on the dynamical behaviour[1,2] of nonlinear elastic beams and rods has become common. For example, Ball[3] studied the initial-boundary value problems for an extensible beam, Gao[4] studied the global existence and decay of the solution of the thermoelastic rod equation. But studies on the problems of nonlinear plates are relatively few.

Hugo and Zoraida[5] studied the following nonlinear system

$$\begin{aligned}&\ddot{w}+\Delta^2 w+\alpha\Delta\theta=f_1(t,w,\theta), \quad t\geq 0, \quad x\in\Omega,\\ &\dot{\theta}-\beta\Delta\theta-\alpha\Delta\dot{w}=f_2(t,w,\theta), \quad t\geq 0, \quad x\in\Omega,\\ &w=\Delta w=\theta=0, \quad t\geq 0, \quad x\in\partial\Omega,\end{aligned}$$

where Ω is a regular domain in $R^n\,(n\geq 1)$; they proved the existence, stability and smoothness of a bounded solution by the analytic semi-group method. Santos *et al*[6] studied the initial boundary value problems of a von Karman plate system with free boundary and boundary conditions of memory type in R^2.

In this paper, we study the initial-boundary value problems of the following nonlinear thermoelastic plate equations:

$$\ddot{u}+\Delta^2 u+\alpha\Delta\theta+N_1(u,\theta)=0,\ t\geq 0, x,y\in\Omega, \tag{1}$$

$$\dot{\theta}-\beta\Delta\theta-\alpha\Delta\dot{u}+N_2(u,\theta)=0,\ t\geq 0, x,y\in\Omega \tag{2}$$

with the initial conditions

$$\begin{aligned}&u(x,y,0)=u_0(x,y),\\ &\dot{u}(x,y,0)=u_1(x,y),\\ &\theta(x,y,0)=\theta_0(x,y),\end{aligned} \tag{3}$$

and the hinged boundary conditions

$$u(0,y,t)=u(l,y,t)=0, \tag{4}$$

$$\begin{aligned}&u(x,0,t)=u(x,l,t)=0,\\ &u_{xx}(0,y,t)=u_{xx}(l,y,t)=0,\end{aligned} \tag{5}$$

$$\begin{aligned}&\theta(0,y,t)=\theta(l,y,t)=0,\\ &\theta(x,0,t)=\theta(x,l,t)=0,\end{aligned} \tag{6}$$

where u,θ denote respectively the vertical deflection and the temperature of the plate. $\Omega=(0,l)\times(0,l)$ is a rectangular domain of R^2, $\Delta=\dfrac{\partial}{\partial x^2}+\dfrac{\partial}{\partial y^2}$, α,β are positive constants. For brevity of notation, dots above symbols denote differentiation with respect to time t, for example, $\dot{z}=\dfrac{\partial z}{\partial t}$, $\ddot{z}=\dfrac{\partial^2 z}{\partial t^2}$.

2. Preliminaries

Let $Q=\Omega\times(0,T)$, where $T>0$ is fixed.

Now we note that

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$$S_2 = H_0^1(\Omega) \cap H^2(\Omega),$$
$$S_1 = \left\{u \in H^4(\Omega) \cap S_2 \middle| \ u_{xx}(0,y,t) = u_{xx}(l,y,t) = 0, u_{yy}(x,0,t) = u_{yy}(x,l,t) = 0\right\},$$
$$S_0 = \left\{u \in H^6(\Omega) \cap S_1 \middle| \ u_{x^4}(0,y,t) = u_{x^4}(l,y,t) = 0, u_{y^4}(x,0,t) = u_{y^4}(x,l,t) = 0\right\}.$$

Then S_2, S_1 and S_0 are Hilbert space and $S_0 \subset S_1 \subset S_2 \subset L^2(\Omega)$.

In the following arguments, the nonlinear terms $N_1(u,\theta)$ and $N_2(u,\theta)$ satisfy

$$N_1(u,\theta) = N_{11}(u) + N_{12}(\theta),$$
$$N_{11}(u) \in C^1(R), |N'_{11}(u)| \le M_1, N_{11}(0) = 0,$$
$$N_{12}(\theta) \in C^1(R), |N'_{12}(\theta)| \le M_2, N_{12}(0) = 0; \quad (7)$$
$$N_2(u,\theta) = N_{21}(u) + N_{22}(\theta),$$
$$N_{21}(u) \in C^1(R), |N'_{21}(u)| \le M_3, N_{21}(0) = 0,$$
$$N_{22}(\theta) \in C^1(R), |N'_{22}(\theta)| \le M_4, N_{22}(0) = 0; \quad (8)$$
$$|N''_{11}(u)| < M_5, \quad |N''_{12}(\theta)| = 0,$$
$$|N''_{21}(u)| < M_6, \quad |N''_{22}(\theta)| = 0; \quad (9)$$
$$|N'''_{11}(u)| = |N'''_{12}(\theta)| = |N'''_{21}(u)| = |N'''_{22}(\theta)| = 0, \quad (10)$$

where M_i $(i = 1, \ldots, 6)$ are positive constants.

Lemma 1[3] Let $f \in H^1(\Omega)$ and suppose $f(\xi) = 0$ for some $\xi \in \bar{\Omega}$. Then

$$\|f\| \le \frac{l}{\sqrt{2}} \left\|f^{(1)}\right\|.$$

3. Existence and uniqueness of the weak solution

Theorem 1 Supposing that $N_1(u,\theta)$ and $N_2(u,\theta)$ satisfy Eqs.(7) and (8), and $(u_0, u_1, \theta_0) \in S_2 \times L^2(\Omega) \times L^2(\Omega)$, then for any $T > 0$, there exist weak solutions u and θ satisfying the initial conditions Eq.(3) and the following equations:

$$\begin{cases} (\ddot{u}, \varphi) + (\Delta u, \Delta\varphi) + \alpha(\theta, \Delta\varphi) + (N_1(u,\theta), \varphi) = 0, \\ \left(\dot{\theta}, \varphi\right) - \beta(\theta, \Delta\varphi) - \alpha(\dot{u}, \Delta\varphi) + (N_2(u,\theta), \varphi) = 0, \end{cases} \quad \forall \varphi \in S_2$$

with $u \in L^\infty(0,T;S_2)$, $\dot{u} \in L^\infty\left(0,T;L^2(\Omega)\right)$, $\theta \in L^\infty\left(0,T;L^2(\Omega)\right)$.

Proof Let $\{\omega_i(x,y)\}$ be a basis of S_2 and $L^2(\Omega)$. For any fixed natural number m, set

$$V_m = [\omega_1, \omega_2, \ldots, \omega_m],$$

then $V_m \subset S_2$. Now the Galerkin approximating solutions $u_m(t)$ and $\theta_m(t)$ of Eqs.(1)–(6) can be denoted by

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t)\,\omega_i(x,y),$$
$$\theta_m(t) = \sum_{i=1}^{m} h_{im}(t)\,\omega_i(x,y),$$

where $g_{im}(t)$ and $h_{im}(t)$ are sufficiently regular functions depending on t. Moreover, u_m and θ_m satisfy the following generalized equations

$$(\ddot{u}_m, \varphi) + (\Delta u_m, \Delta\varphi) + \alpha(\theta_m, \Delta\varphi) + (N_1(u_m,\theta_m), \varphi) = 0, \quad \forall \varphi \in V_m, \quad (11)$$
$$(\dot{\theta}_m, \varphi) - \beta(\theta_m, \Delta\varphi) - \alpha(\dot{u}_m, \Delta\varphi) + (N_2(u_m,\theta_m), \varphi) = 0, \quad \forall \varphi \in V_m \quad (12)$$

with the initial conditions:

$$u_m(0) = u_{0m} = \sum_{i=1}^{m} \alpha_i \omega_i, \quad u_{0m} \to u_0(x,y) \text{ (in } S_2),$$

$$\dot{u}_m(0) = u_{1m} = \sum_{i=1}^{m} \beta_i \omega_i, \quad u_{1m} \to u_1(x,y) \text{ (in } L^2(\Omega)),$$
$$\theta_m(0) = \theta_{0m} = \sum_{i=1}^{m} \gamma_i \omega_i, \quad \theta_{0m} \to \theta_0(x,y) \text{ (in } L^2(\Omega)). \tag{13}$$

Now Eqs.(11)–(13) become a Cauchy problem of ordinary differential equations. Using the theory of ordinary differential equation, we know that there exists $t_m > 0$, such that $u_m(t)$ and $\theta_m(t)$ are existent and unique in $[0, t_m]$ $(0 \le t \le t_m)$. The extension of the solutions to the interval $[0, T]$ is a consequence of the a priori estimate which we are going to prove below.

Estimate We first consider $\varphi = \dot{u}_m$ in Eq.(11), then we have

$$(\ddot{u}_m, \dot{u}_m) + (\Delta u_m, \Delta \dot{u}_m) + \alpha(\theta_m, \Delta \dot{u}_m) + (N_1(u_m, \theta_m), \dot{u}_m) + (u_m, \dot{u}_m) - (\Delta u_m, \dot{u}_m) = (u_m, \dot{u}_m) - (\Delta u_m, \dot{u}_m). \tag{14}$$

And considering $\varphi = \theta_m$ in Eq.(12), we obtain

$$\left(\dot{\theta}_m, \theta_m\right) - \beta(\theta_m, \Delta\theta_m) - \alpha(\dot{u}_m, \Delta\theta_m) + (N_2(u_m, \theta_m), \theta_m) = 0. \tag{15}$$

Adding Eqs.(14) and (15), we can obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\dot{u}_m\|^2 + \|\Delta u_m\|^2 + \|u_m\|^2 + \left\|\frac{\partial u_m}{\partial x}\right\|^2 + \left\|\frac{\partial u_m}{\partial y}\right\|^2 + \|\theta_m\|^2\right) + \beta\left(\left\|\frac{\partial \theta_m}{\partial x}\right\|^2 + \left\|\frac{\partial \theta_m}{\partial y}\right\|^2\right)$$
$$= (u_m, \dot{u}_m) - (\Delta u_m, \dot{u}_m) - (N_{11}(u_m), \dot{u}_m) - (N_{12}(\theta_m), \dot{u}_m) - (N_{21}(u_m), \theta_m) - (N_{22}(\theta_m), \theta_m).$$

Let

$$\tilde{E}(t; u_m, \theta_m) = \|\dot{u}_m\|^2 + \|\Delta u_m\|^2 + \|\theta_m\|^2.$$

From Eqs.(7) and (8) and applying the Schwarz inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}(t; u_m, \theta_m) \le C\tilde{E}(t; u_m, \theta_m),$$

where C is a positive constant, while C denotes different constants in different parts of this paper.

Using the Gronwall inequality, we have

$$\tilde{E}(t; u_m, \theta_m) \le \tilde{E}(0; u_{0m}, \theta_{0m})\mathrm{e}^{CT} \qquad (0 < t < T).$$

From Eq.(13), it follows that $\tilde{E}(0; u_{0m}, \theta_{0m})$ is bounded and independent of m. Then we obtain that $E_m(t)$ is uniformly bounded and independent of m and $t \in [0, T]$. That is,

$$\|\dot{u}_m\| \le C, \quad \|\Delta u_m\| \le C, \quad \|\theta_m\| \le C. \tag{16}$$

From $\|\Delta u_m\| \le C$ and according to the boundary conditions, we have

$$\left\|\frac{\partial^2 u_m}{\partial x^2}\right\| \le C, \quad \left\|\frac{\partial^2 u_m}{\partial x \partial y}\right\| \le C, \quad \left\|\frac{\partial^2 u_m}{\partial y^2}\right\| \le C. \tag{17}$$

As above, $u_m(t)$ and $\theta_m(t)$ exist in $[0, T]$, and satisfy the bounds of Eqs.(16) and (17).

Convergence From the bounds of Eqs.(16) and (17), it is easily to show that there exist a subsequence $\{u_\mu\}$ of $\{u_m\}$ and a subsequence $\{\theta_\mu\}$ of $\{\theta_m\}$, such that

$$u_\mu \to u \text{ weak - star in } L^\infty(0, T; S_2);$$
$$\dot{u}_\mu \to \dot{u} \text{ weak - star in } L^\infty\left(0, T; L^2(\Omega)\right);$$
$$\theta_\mu \to \theta \text{ weak - star in } L^\infty\left(0, T; L^2(\Omega)\right);$$
$$u_\mu \to u \text{ strongly in } L^2(Q) \text{ and a.e.}$$

Since $u_\mu \to u$ weak-star in $L^\infty(0, T; S_2)$, when $\mu \to \infty$, we have

$$\left|\int_0^T \int_0^l \int_0^l (u_\mu - u)\zeta_1 \mathrm{d}x\mathrm{d}y\mathrm{d}t\right| \to 0, \quad \forall \zeta_1 \in L^1(0, T; S_2'). \tag{18}$$

About $N_{11}(u_\mu)$, by Eq.(7) and $u_m \in L^\infty(0, T; S_2)$, we know

$$N_{11}(u_\mu) \to \chi_1 \text{ weak - star in } L^\infty\left(0, T; L^2(\Omega)\right), \text{ as } \mu \to \infty.$$

Since $|N'_{11}(u)| \leq M_1$, by Eq.(18) we arrive at

$$\begin{aligned}&\left|\int_0^T\int_0^l\int_0^l (N_{11}(u_\mu)-N_{11}(u))\zeta_1 \mathrm{d}x\mathrm{d}y\mathrm{d}t\right|\\ =&\left|\int_0^T\int_0^l\int_0^l N'_{11}(\xi_1)(u_\mu-u)\zeta_1 \mathrm{d}x\mathrm{d}y\mathrm{d}t\right|\\ \leq& M_1\left|\int_0^T\int_0^l\int_0^l (u_\mu-u)\zeta_1 \mathrm{d}x\mathrm{d}y\mathrm{d}t\right|\to 0,\end{aligned}$$

as $\mu\to\infty$, where ξ_1 is between u_μ and u. By the uniqueness, we conclude that

$N_{11}(u_\mu)\to N_{11}(u)$ weak-star in $L^\infty(0,T;S_2)$.

Using similar arguments to those above, we can easily verify

$N_{12}(\theta_\mu)\to N_{12}(\theta)$ weak-star in $L^\infty\left(0,T;L^2(\Omega)\right)$,

$N_{21}(u_\mu)\to N_{21}(u)$ weak-star in $L^\infty(0,T;S_2)$,

$N_{22}(\theta_\mu)\to N_{22}(\theta)$ weak-star in $L^\infty\left(0,T;L^2(\Omega)\right)$.

As above, when $\mu\to\infty$, we obtain the weak solutions u and θ which satisfy Theorem 1.

Theorem 2 The weak solutions (u,θ) satisfying Theorem 1 are unique.

Proof Suppose (u,θ) and (v,η) are two solutions satisfying Theorem 1 with

$$\begin{aligned}&u,v\in L^\infty(0,T;S_2),\\ &\dot{u},\dot{v}\in L^\infty\left(0,T;L^2(\Omega)\right),\\ &\theta,\eta\in L^\infty\left(0,T;L^2(\Omega)\right)\end{aligned}$$

and suppose that (u,θ) and (v,η) satisfy the initial conditions

$$\begin{aligned}&u(0)=v(0)=u_0,\\ &\dot{u}(0)=\dot{v}(0)=u_1,\\ &\theta(0)=\eta(0)=\theta_0\end{aligned}$$

with $(u_0,u_1,\theta_0)\in S_2\times L^2(\Omega)\times L^2(\Omega)$.

Then for all $\varphi\in S_2$, we have

$$\begin{aligned}&(\ddot{u},\varphi)+(\Delta u,\Delta\varphi)+\alpha(\theta,\Delta\varphi)\\ &+(N_1(u,\theta),\varphi)=0,\end{aligned}\tag{19}$$

$$\begin{aligned}&\left(\dot{\theta},\varphi\right)-\beta(\theta,\Delta\varphi)-\alpha(\dot{u},\Delta\varphi)\\ &+(N_2(u,\theta),\varphi)=0,\end{aligned}\tag{20}$$

and

$$\begin{aligned}&(\ddot{v},\varphi)+(\Delta v,\Delta\varphi)+\alpha(\eta,\Delta\varphi)\\ &+(N_1(v,\eta),\varphi)=0,\end{aligned}\tag{21}$$

$$\begin{aligned}&(\dot{\eta},\varphi)-\beta(\eta,\Delta\varphi)-\alpha(\dot{v},\Delta\varphi)\\ &+(N_2(v,\eta),\varphi)=0.\end{aligned}\tag{22}$$

Now subtracting Eq.(21) from Eq.(19) and also Eq.(22) from Eq.(20), we obtain

$$\begin{aligned}&(\ddot{w},\varphi)+(\Delta w,\Delta\varphi)+\alpha(v,\Delta\varphi)\\ &+(N_1(u,\theta)-N_1(v,\eta),\varphi)=0,\end{aligned}\tag{23}$$

$$\begin{aligned}&(\dot{\sigma},\varphi)-\beta(\sigma,\Delta\varphi)-\alpha(\dot{w},\Delta\varphi)\\ &+(N_2(u,\theta)-N_2(v,\eta),\varphi)=0,\end{aligned}\tag{24}$$

where $w=u-v$, $\sigma=\theta-\eta$, then $w_0=0$, $\sigma_0=0$.

Since S_2 can be continuously and densely embedded into $L^2(\Omega)$, thus φ in Eqs.(23) and (24) can be replaced by $\dot{w}$ and σ (where $\dot{w},\ \sigma\in L^2(\Omega)$), respectively. Then we obtain

$$\begin{aligned}&(\ddot{w},\dot{w})+(\Delta w,\Delta\dot{w})+\alpha(\sigma,\Delta\dot{w})\\ &+(N_1(u,\theta)-N_1(v,\eta),\dot{w})=0,\\ &(\dot{\sigma},\sigma)-\beta(\sigma,\Delta\sigma)-\alpha(\dot{w},\Delta\sigma)\\ &+(N_2(u,\theta)-N_2(v,\eta),\sigma)=0.\end{aligned}$$

Set

$$E(t)=\|\dot{w}\|^2+\|\Delta w\|^2+\|w\|^2+\left\|\frac{\partial w}{\partial x}\right\|^2+\left\|\frac{\partial w}{\partial y}\right\|^2+\|\sigma\|^2.$$

Using similar arguments to the proof of Theorem 1, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)\leq CE(t).$$

By the Gronwall inequality, we have

$$E(t)\leq E(0)\mathrm{e}^{CT}\quad(0<t<T).$$

From the assumptions on (u,θ) and (v,η), it implies that $E(0)=0$. Hence

$$\|\dot{w}\|^2+\|\Delta w\|^2+\|w\|^2+\left\|\frac{\partial w}{\partial x}\right\|^2+\left\|\frac{\partial w}{\partial y}\right\|^2+\|\sigma\|^2=0,$$

that is $u=v$, $\theta=\eta$. Now Theorem 2 is proved.

4. Existence of the strong solution

Theorem 3 Suppose that $N_1(u,\theta)$ and $N_2(u,\theta)$ satisfy Eqs.(7) and (8), $(u_0,u_1,\theta_0)\in S_1\times S_2\times S_2$, then there exist strong solutions u and θ satisfying the following equations:

$$\begin{cases} \ddot{u} + \Delta^2 u + \alpha\Delta\theta + N_1(u,\theta) = 0, \\ \dot{\theta} - \beta\Delta\theta - \alpha\Delta\dot{u} + N_2(u,\theta) = 0, \end{cases} \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right)$$

with $u \in L^\infty(0,T;S_1)$, $\dot{u} \in L^\infty(0,T;S_2)$, $\ddot{u} \in L^\infty\left(0,T;L^2(\Omega)\right)$, $\theta \in L^\infty(0,T;S_2)$, $\dot{\theta} \in L^\infty\left(0,T;L^2(\Omega)\right)$.

Proof Letting $\{\omega_i(x,y)\}$ be a basis of S_1, similar to the proof of Theorem 1, we can obtain Eqs.(16) and (17). The approximate solutions $u_m(t)$ and $\theta_m(t)$ satisfy

$$\ddot{u}_m + \Delta^2 u_m + \alpha\Delta\theta_m + N_1(u_m,\theta_m) = 0, \quad (25)$$

$$\dot{\theta}_m - \beta\Delta\theta_m - \alpha\Delta\dot{u}_m + N_2(u_m,\theta_m) = 0. \quad (26)$$

Estimate Letting $t = 0$ in Eq.(25), from the initial data and Eq.(7), we know

$$\|\ddot{u}_m(0)\| \le \left\|\Delta^2 u_{0m}\right\| + \alpha\|\Delta\theta_{0m}\| + \|N_1(u_{0m},\theta_{0m})\| \le C. \quad (27)$$

Letting $t = 0$ in Eq.(26), from the initial data and Eq.(8), we know

$$\begin{aligned} &\|\dot{\theta}_m(0)\| \\ &\le \beta\|\Delta\theta_{0m}\| + \alpha\|\Delta\dot{u}_{0m}\| + \|N_2(u_{0m},\theta_{0m})\| \\ &\le C. \end{aligned} \quad (28)$$

Now differentiating Eq.(25) with respect to t, and taking the inner product with $\ddot{u}_m$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\ddot{u}_m\|^2 + \left\|\frac{\partial^2\dot{u}_m}{\partial x^2}\right\|^2 + 2\left\|\frac{\partial^2\dot{u}_m}{\partial x\partial y}\right\|^2 + \left\|\frac{\partial^2\dot{u}_m}{\partial y^2}\right\|^2\right) + \alpha\left(\Delta\dot{\theta}_m,\ddot{u}_m\right) + \left(N_{11}'(u_m)\dot{u}_m,\ddot{u}_m\right) \\ &+ \left(N_{12}'(\theta_m)\dot{\theta}_m,\ddot{u}_m\right) = 0. \end{aligned} \quad (29)$$

Also, differentiating Eq.(26) with respect to t, and taking the inner product with $\dot{\theta}_m$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\dot{\theta}_m\right\|^2 + \beta\left(\left\|\frac{\partial\dot{\theta}_m}{\partial x}\right\|^2 + \left\|\frac{\partial\dot{\theta}_m}{\partial y}\right\|^2\right) - \alpha\left(\ddot{u}_m,\Delta\dot{\theta}_m\right) + \left(N_{21}'(u_m)\dot{u}_m,\dot{\theta}_m\right) + \left(N_{22}'(\theta_m)\dot{\theta}_m,\dot{\theta}_m\right) = 0. \quad (30)$$

Add Eqs.(29) and (30), and note

$$E_1(t;u_m,\theta_m) = \|\ddot{u}_m\|^2 + \left\|\frac{\partial^2\dot{u}_m}{\partial x^2}\right\|^2 + 2\left\|\frac{\partial^2\dot{u}_m}{\partial x\partial y}\right\|^2 + \left\|\frac{\partial^2\dot{u}_m}{\partial y^2}\right\|^2 + \left\|\dot{\theta}_m\right\|^2.$$

Then we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_1(t;u_m,\theta_m) + \beta\left(\left\|\frac{\partial\dot{\theta}_m}{\partial x}\right\|^2 + \left\|\frac{\partial\dot{\theta}_m}{\partial y}\right\|^2\right) \\ &= -\left(N_{11}'(u_m)\dot{u}_m,\ddot{u}_m\right) - \left(N_{12}'(\theta_m)\dot{\theta}_m,\ddot{u}_m\right) - \left(N_{21}'(u_m)\dot{u}_m,\dot{\theta}_m\right) - \left(N_{22}'(\theta_m)\dot{\theta}_m,\dot{\theta}_m\right). \end{aligned}$$

By Eqs.(7), (8) and the Schwarz inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t;u_m,\theta_m) \le CE_1(t;u_m,\theta_m).$$

By the Gronwall inequality, Eqs.(27), (28) and the initial data, it follows that

$$\|\ddot{u}_m\| \le C, \quad \left\|\frac{\partial^2\dot{u}_m}{\partial x^2}\right\| \le C, \quad \left\|\frac{\partial^2\dot{u}_m}{\partial x\partial y}\right\| \le C, \quad \left\|\frac{\partial^2\dot{u}_m}{\partial y^2}\right\| \le C, \quad \left\|\dot{\theta}_m\right\| \le C. \quad (31)$$

From Eqs.(8), (26) and (31), we have

$$\beta\|\Delta\theta_m\| \le \left\|\dot{\theta}_m\right\| + \alpha\|\Delta\dot{u}_m\| + \|N_2(u_m,\theta_m)\| \le C.$$

Since β is a positive constant, therefore

$$\|\Delta\,\theta_m\| \le C.$$

Moreover, we have

$$\left\|\frac{\partial^2\theta_m}{\partial x^2}\right\| \le C, \quad \left\|\frac{\partial^2\theta_m}{\partial x\partial y}\right\| \le C, \quad \left\|\frac{\partial^2\theta_m}{\partial y^2}\right\| \le C. \tag{32}$$

From Eqs.(7), (25), (31) and (32), we obtain as follows that

$$\left\|\Delta^2 u_m\right\| \le \|\ddot{u}_m\| + \alpha\,\|\Delta\theta_m\| + \|N_1\,(u_m,\theta_m)\| \le C.$$

Moreover, we have

$$\left\|\frac{\partial^4 u_m}{\partial x^4}\right\| \le C, \quad \left\|\frac{\partial^4 u_m}{\partial x^3\partial y}\right\| \le C, \quad \left\|\frac{\partial^4 u_m}{\partial x^2\partial y^2}\right\| \le C, \quad \left\|\frac{\partial^4 u_m}{\partial x\partial y^3}\right\| \le C, \quad \left\|\frac{\partial^4 u_m}{\partial y^4}\right\| \le C. \tag{33}$$

Now letting Eq.(25) take the inner product with $-\Delta\dot{u}_m$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{\partial\dot{u}_m}{\partial x}\right\|^2 + \left\|\frac{\partial\dot{u}_m}{\partial y}\right\|^2 + \left\|\frac{\partial^3 u_m}{\partial x^3}\right\|^2 + 3\left\|\frac{\partial^3 u_m}{\partial x^2\partial y}\right\|^2 + 3\left\|\frac{\partial^3 u_m}{\partial x\partial y^2}\right\|^2 + \left\|\frac{\partial^3 u_m}{\partial y^3}\right\|^2\right)\\ &+\alpha\,(\Delta\theta_m, -\Delta\dot{u}_m) + (N_1\,(u_m,\theta_m)\,, -\Delta\dot{u}_m) = 0.\end{aligned} \tag{34}$$

Also, letting Eq.(26) take the inner product with $-\Delta\theta_m$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{\partial\theta_m}{\partial x}\right\|^2 + \left\|\frac{\partial\theta_m}{\partial y}\right\|^2\right) + \beta\,\|\Delta\theta_m\|^2 - \alpha\,(\Delta\dot{u}_m, -\Delta\,\theta_m) + (N_2\,(u_m,\theta_m)\,, -\Delta\,\theta_m) = 0. \tag{35}$$

Add Eqs.(34) and (35), and note

$$\begin{aligned}E_2(t;u_m,\theta_m) = &\left\|\frac{\partial\dot{u}_m}{\partial x}\right\|^2 + \left\|\frac{\partial\dot{u}_m}{\partial y}\right\|^2 + \left\|\frac{\partial^3 u_m}{\partial x^3}\right\|^2 + 3\left\|\frac{\partial^3 u_m}{\partial x^2\partial y}\right\|^2 + 3\left\|\frac{\partial^3 u_m}{\partial x\partial y^2}\right\|^2 + \left\|\frac{\partial^3 u_m}{\partial x^3}\right\|^2\\ &+\left\|\frac{\partial\theta_m}{\partial x}\right\|^2 + \left\|\frac{\partial\theta_m}{\partial y}\right\|^2.\end{aligned}$$

From Eqs.(7), (8), (31), (32) and the Schwarz inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t;u_m,\theta_m) \le C.$$

By the Gronwall inequality, we obtain

$$E_2(t;u_m,\theta_m) \le E_2(0;u_{0m},\theta_{0m}) + CT.$$

From the initial data, it is easy to know that $E_2(0;u_{0m},\theta_{0m})$ is bounded. Thus

$$\begin{aligned}&\left\|\frac{\partial\dot{u}_m}{\partial x}\right\| \le C, \quad \left\|\frac{\partial\dot{u}_m}{\partial y}\right\| \le C,\\ &\left\|\frac{\partial^3 u_m}{\partial x^3}\right\| \le C, \quad \left\|\frac{\partial^3 u_m}{\partial x^2\partial y}\right\| \le C,\\ &\left\|\frac{\partial^3 u_m}{\partial x^2\partial y}\right\| \le C, \quad \left\|\frac{\partial^3 u_m}{\partial y^3}\right\| \le C,\\ &\left\|\frac{\partial\theta_m}{\partial x}\right\| \le C, \quad \left\|\frac{\partial\theta_m}{\partial x}\right\| \le C.\end{aligned} \tag{36}$$

Convergence From the bounds of Eqs.(16), (17), (31), (32), (33) and (36), we find that there exist a subsequence $\{u_\mu\}$ of $\{u_m\}$ and a subsequence $\{\theta_\mu\}$ of $\{\theta_m\}$, such that

$u_\mu \to u$ weak-star in $L^\infty(0,T;S_1)$,

$\dot{u}_\mu \to \dot{u}$ weak-star in $L^\infty(0,T;S_2)$,

$\ddot{u}_\mu \to \ddot{u}$ weak-star in $L^\infty\left(0,T;L^2(\Omega)\right)$,

$\theta_\mu \to \theta$ weak-star in $L^\infty(0,T;S_2)$,

$\dot{\theta}_\mu \to \dot{\theta}$ weak-star in $L^\infty\left(0,T;L^2(\Omega)\right)$,

$u_\mu \to u$ strongly in $H^1(Q)$ and a.e.

Using similar arguments as in the proof of Theorem 1, we can easily verify

$N_{11}(u_\mu) \to N_{11}(u)$ weak-star in $L^\infty(0,T;S_1)$,
$N_{12}(\theta_\mu) \to N_{12}(\theta)$ weak-star in $L^\infty(0,T;S_2)$,
$N_{21}(u_\mu) \to N_{21}(u)$ weak-star in $L^\infty(0,T;S_1)$,
$N_{22}(\theta_\mu) \to N_{22}(\theta)$ weak-star in $L^\infty(0,T;S_2)$.

As above, when $\mu \to \infty$, we obtain the strong solutions u and θ which satisfy Theorem 3.

5. Existence of the classical solution

Theorem 4 Suppose that $N_1(u,\theta)$ and $N_2(u,\theta)$ satisfy Eqs.(7)–(10), $(u_0,u_1,\theta_0) \in S_0 \times S_1 \times S_1$, then there exist classical solutions u and θ satisfying Eqs.(1)–(6) with $u \in L^\infty(0,T;S_1)$, $\dot{u} \in L^\infty(0,T;S_2)$, $\ddot{u} \in L^\infty(0,T;L^2(\Omega))$, $\theta \in L^\infty(0,T;S_2)$, $\dot{\theta} \in L^\infty(0,T;L^2(\Omega))$, and $u \in C^1(\bar{Q}) \cap [C^4(\bar{\Omega}) \times C^2[0,T]]$, $\theta \in C^2(\Omega) \times C^1[0,T]$.

Proof Letting $\{\omega_j(x,y)\}$ be a basis of S_0, similar to the proof of Theorem 1 and Theorem 3, the approximating solutions $u_m(t)$ and $\theta_m(t)$ satisfy Eqs.(16), (17), (31), (32), (33) and (36).

Estimate Differentiating Eq.(25) twice with respect to x and letting $t=0$, we have

$$\frac{\partial^2 \ddot{u}_m}{\partial x^2}(0) + \frac{\partial^6 u_m}{\partial x^6}(0) + 2\frac{\partial^6 u_m}{\partial x^4 \partial y^2}(0) + \frac{\partial^6 u_m}{\partial x^2 \partial y^4}(0) + \alpha\left(\frac{\partial^4 \theta_m}{\partial x^4}(0) + \frac{\partial^4 \theta_m}{\partial x^2 \partial y^2}(0)\right) + \frac{\partial^2 N_1(u_{0m},\theta_{0m})}{\partial x^2} = 0.$$

Then from Eqs.(7), (9) and the initial data, we obtain that

$$\left\| \frac{\partial^2 \ddot{u}_m}{\partial x^2}(0) \right\| \le C. \tag{37}$$

Similarly, we have

$$\left\| \frac{\partial^2 \ddot{u}_m}{\partial y^2}(0) \right\| \le C, \quad \left\| \frac{\partial^2 \ddot{u}_m}{\partial x \partial y}(0) \right\| \le C. \tag{38}$$

Differentiating Eq.(26) twice with respect to x and letting $t=0$, we have

$$\frac{\partial^2 \dot{\theta}_m}{\partial x^2}(0) - \beta\left(\frac{\partial^4 \theta_m}{\partial x^4}(0) + \frac{\partial^4 \theta_m}{\partial x^2 \partial y^2}(0)\right) - \alpha\left(\frac{\partial^4 \dot{u}_m}{\partial x^4}(0) + \frac{\partial^4 \dot{u}_m}{\partial x^2 \partial y^2}(0)\right) + \frac{\partial^2 N_2(u_{0m},\theta_{0m})}{\partial x^2} = 0.$$

Then from Eqs.(8), (9) and the initial data, we have

$$\left\| \frac{\partial^2 \dot{\theta}_m}{\partial x^2}(0) \right\| \le C. \tag{39}$$

Similarly, we have

$$\left\| \frac{\partial^2 \dot{\theta}_m}{\partial y^2}(0) \right\| \le C, \quad \left\| \frac{\partial^2 \dot{\theta}_m}{\partial x \partial y}(0) \right\| \le C. \tag{40}$$

From $\{\omega_j\} \in S_0$, Eqs.(25) and (26), we obtain the compatibility conditions

$$u_{x^4}(x,0,t) = u_{x^4}(x,l,t) = 0,$$
$$u_{y^4}(0,y,t) = u_{y^4}(l,y,t) = 0,$$
$$\theta_{xx}(0,y,t) = \theta_{xx}(l,y,t) = 0,$$
$$\theta_{yy}(x,0,t) = \theta_{yy}(x,l,t) = 0.$$

Now differentiating Eq.(25) with respect to t, and taking the inner product with $\Delta^2 \ddot{u}_m$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{\partial^2 \ddot{u}_m}{\partial x^2}\right\|^2 + 2\left\|\frac{\partial^2 \ddot{u}_m}{\partial x \partial y}\right\|^2 + \left\|\frac{\partial^2 \ddot{u}_m}{\partial y^2}\right\|^2 + \|\Delta^2 \dot{u}_m\|^2\right) + \alpha\left(\Delta\dot{\theta}_m, \Delta^2 \ddot{u}_m\right) + \left(N_{11}'(u_m)\dot{u}_m, \Delta^2 \ddot{u}_m\right) + \left(N_{12}'(\theta_m)\dot{\theta}_m, \Delta^2 \ddot{u}_m\right) = 0. \tag{41}$$

Also, differentiating Eq.(26) with respect to t, and taking the inner product with $\Delta^2 \dot{\theta}_m$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{\partial^2 \dot{\theta}_m}{\partial x^2}\right\|^2 + 2\left\|\frac{\partial^2 \dot{\theta}_m}{\partial x \partial y}\right\|^2 + \left\|\frac{\partial^2 \dot{\theta}_m}{\partial y^2}\right\|^2\right) + \beta\left(\left\|\frac{\partial^3 \dot{\theta}_m}{\partial x^3}\right\|^2 + 3\left\|\frac{\partial^3 \dot{\theta}_m}{\partial x^2 \partial y}\right\|^2 + 3\left\|\frac{\partial^3 \dot{\theta}_m}{\partial x \partial y^2}\right\|^2 + \left\|\frac{\partial^3 \dot{\theta}_m}{\partial y^3}\right\|^2\right) - \alpha\left(\Delta^2 \ddot{u}_m, \Delta\dot{\theta}_m\right) + \left(N_{21}'(u_m)\dot{u}_m, \Delta^2 \dot{\theta}_m\right) + \left(N_{22}'(\theta_m)\dot{\theta}_m, \Delta^2 \dot{\theta}_m\right) = 0. \tag{42}$$

Plus Eqs.(41) and (42), and note

$$E_3(t;u_m,\theta_m)=\left\|\frac{\partial^2\ddot{u}_m}{\partial x^2}\right\|^2+2\left\|\frac{\partial^2\ddot{u}_m}{\partial x\partial y}\right\|^2+\left\|\frac{\partial^2\ddot{u}_m}{\partial y^2}\right\|^2+\left\|\Delta^2\dot{u}_m\right\|^2+\left\|\frac{\partial^2\dot{\theta}_m}{\partial x^2}\right\|^2+2\left\|\frac{\partial^2\dot{\theta}_m}{\partial x\partial y}\right\|^2+\left\|\frac{\partial^2\dot{\theta}_m}{\partial y^2}\right\|^2.$$

Then from Eqs.(7), (8), (9), (10), (16) and (31), together with the Gagliardo–Nirenberg inequality[5] and the Schwarz inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_3(t;u_m,\theta_m)\le CE_3(t;u_m,\theta_m).$$

Using the Gronwall inequality, we obtain

$$E_3(t;u_m,\theta_m)\le E_3(0;u_{0m},\theta_{0m})\mathrm{e}^{CT}.$$

From Eqs.(37), (38), (39), (40) and the initial data, we find that $E_3(0;u_m,\theta_m)$ is bounded. Thus we have

$$\left\|\frac{\partial^2\ddot{u}_m}{\partial x^2}\right\|\le C,\left\|\frac{\partial^2\ddot{u}_m}{\partial x\partial y}\right\|\le C,\left\|\frac{\partial^2\ddot{u}_m}{\partial y^2}\right\|\le C,\left\|\Delta^2\dot{u}_m\right\|\le C,\quad\left\|\frac{\partial^2\dot{\theta}_m}{\partial x^2}\right\|\le C,\left\|\frac{\partial^2\dot{\theta}_m}{\partial x\partial y}\right\|\le C,\left\|\frac{\partial^2\dot{\theta}_m}{\partial y^2}\right\|\le C.\quad(43)$$

From $\left\|\Delta^2\dot{u}_m\right\|\le C$, we obtain

$$\left\|\frac{\partial^4\dot{u}_m}{\partial x^4}\right\|\le C,\quad\left\|\frac{\partial^4\dot{u}_m}{\partial x^3\partial y}\right\|\le C,\quad\left\|\frac{\partial^4\dot{u}_m}{\partial x^2\partial y^2}\right\|\le C,\quad\left\|\frac{\partial^4\dot{u}_m}{\partial x\partial y^3}\right\|\le C,\quad\left\|\frac{\partial^4\dot{u}_m}{\partial y^4}\right\|\le C.\quad(44)$$

Now applying the Laplace operator Δ to Eq.(25), we have

$$\Delta\ddot{u}_m+\Delta^3u_m+\alpha\Delta^2\theta_m+N'_{11}(u_m)\Delta u_m+N''_{11}(u_m)\left(\left(\frac{\partial u_m}{\partial x}\right)^2+\left(\frac{\partial u_m}{\partial y}\right)^2\right)$$
$$+N'_{12}(\theta_m)\Delta\theta_m+N''_{12}(\theta_m)\left(\left(\frac{\partial\theta_m}{\partial x}\right)^2+\left(\frac{\partial\theta_m}{\partial y}\right)^2\right)=0.\quad(45)$$

Applying the Laplace operator Δ to Eq.(26), we have

$$\Delta\dot{\theta}_m-\beta\Delta^2\theta_m-\alpha\Delta^2\dot{u}_m+N'_{21}(u_m)\Delta u_m+N''_{21}(u_m)\left(\left(\frac{\partial u_m}{\partial x}\right)^2+\left(\frac{\partial u_m}{\partial y}\right)^2\right)$$
$$+N'_{22}(\theta_m)\Delta\theta_m+N''_{22}(\theta_m)\left(\left(\frac{\partial\theta_m}{\partial x}\right)^2+\left(\frac{\partial\theta_m}{\partial y}\right)^2\right)=0.\quad(46)$$

From Eqs.(46), (8), (9), (16), (32), (43), (44) and the Gagliardo–Nirenberg inequality,[5] we have

$$\left\|\Delta^2\theta_m\right\|\le C.$$

Then it follows that

$$\left\|\frac{\partial^4\theta_m}{\partial x^4}\right\|\le C,\quad\left\|\frac{\partial^4\theta_m}{\partial x^3\partial y}\right\|\le C,\quad\left\|\frac{\partial^4\theta_m}{\partial x^2\partial y^2}\right\|\le C,$$
$$\left\|\frac{\partial^4\theta_m}{\partial x\partial y^3}\right\|\le C,\quad\left\|\frac{\partial^4\theta_m}{\partial y^4}\right\|\le C.\quad(47)$$

From Eqs.(45), (7), (9), (16), (32), (43), (44) and the Gagliardo–Nirenberg inequality,[5] we obtain

$$\left\|\Delta^3u_m\right\|\le C.$$

Then it follows that

$$\left\|\frac{\partial^6u_m}{\partial x^6}\right\|\le C,\quad\left\|\frac{\partial^6u_m}{\partial x^5\partial y}\right\|\le C,$$
$$\left\|\frac{\partial^6u_m}{\partial x^4\partial y^2}\right\|\le C,\quad\left\|\frac{\partial^6u_m}{\partial x^3\partial y^3}\right\|\le C,$$
$$\left\|\frac{\partial^6u_m}{\partial x^2\partial y^4}\right\|\le C,\quad\left\|\frac{\partial^6u_m}{\partial x\partial y^5}\right\|\le C,$$
$$\left\|\frac{\partial^6u_m}{\partial y^6}\right\|\le C.\quad(48)$$

Now differentiating Eq.(25) with respect to t,

from Eq.(7) and the derived bounds, we have

$$\|\dddot{u}_m\| \leq C. \tag{49}$$

Also, differentiate Eq.(26) with respect to t, then from Eq.(8) and the derived bounds, we have

$$\left\|\ddot{\theta}_m\right\| \leq C. \tag{50}$$

Convergence From the bounds Eqs.(43), (44), (47), (48), (49), (50) and (50), and using Lemma 1, we may extract a subsequence $\{u_\mu\}$ of $\{u_m\}$ and a subsequence $\{\theta_\mu\}$ of $\{\theta_m\}$, such that

$u_\mu \to u$ weak-star in $L^\infty(0,T;S_0)$,
$\dot{u}_\mu \to \dot{u}$ weak-star in $L^\infty(0,T;S_1)$,
$\ddot{u}_\mu \to \ddot{u}$ weak-star in $L^\infty(0,T;S_2)$,
$\dddot{u}_\mu \to \dddot{u}$ weak-star in $L^\infty\left(0,T;L^2(\Omega)\right)$,
$\theta_\mu \to \theta$ weak-star in $L^\infty(0,T;S_1)$,
$\dot{\theta}_\mu \to \dot{\theta}$ weak-star in $L^\infty(0,T;S_2)$,
$u_\mu \to u$ strongly in $H^2(Q)$ and a.e.
$\theta_\mu \to \theta$ strongly in $H^1(Q)$ and a.e.

Using the similar arguments as in the proof of Theorem 1, we can easily verify

$N_{11}(u_\mu) \to N_{11}(u)$ weak-star in $L^\infty(0,T;S_0)$,
$N_{12}(\theta_\mu) \to N_{12}(\theta)$ weak-star in $L^\infty(0,T;S_1)$,
$N_{21}(u_\mu) \to N_{21}(u)$ weak-star in $L^\infty(0,T;S_0)$,
$N_{22}(\theta_\mu) \to N_{22}(\theta)$ weak-star in $L^\infty(0,T;S_1)$.

By the Sobolev embedding theorem, we can obtain classical solutions u and θ which satisfy Theorem 4 as $\mu \to \infty$.

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