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The scattering states of the generalized Hulthén potential with an improved new approximate scheme for the centrifugal term

Wei Gao-Feng(卫高峰)[a)†], Chen Wen-Li(陈文利)[b)‡],
Wang Hong-Ying(王红英)[a)§], and Li Yuan-Yuan(李院院)[a)¶]

[a)] *Department of Physics, Xi'an University of Arts and Science, Xi'an* 710065, *China*

[b)] *Faculty of Science, Xi'an University of Architecture and Technology, Xi'an* 710055, *China*

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This paper finds the approximate analytical scattering state solutions of the arbitrary l-wave Schrödinger equation for the generalized Hulthén potential by taking an improved new approximate scheme for the centrifugal term. The normalized analytical radial wave functions of the l-wave Schrödinger equation for the generalized Hulthén potential are presented and the corresponding calculation formula of phase shifts is derived. Some useful figures are plotted to show the improved accuracy of the obtained results and two special cases for the standard Hulthén potential and Woods–Saxon potential are also studied briefly.

Keywords: generalized Hulthén potential, scattering states, approximate analytical solutions
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1. Introduction

There is growing interest in searching for exact solutions within the framework of non-relativistic quantum mechanics and relativistic quantum mechanics since they contain all the necessary information regarding the quantum system under consideration. For example, the exact solutions of the Schrödinger equation for a hydrogen atom and for a harmonic oscillator in three dimensions are an important milestone at the beginning stage of quantum mechanics, which provides strong evidence for supporting the correctness of quantum theory.[1−3] However, the exact solutions are rare such that many quantum systems have to be treated by approximate methods. The typical example is the rotating Morse potential by the Pekeris approximation,[4,5] and other typical potentials have also been approximately solved such as the Manning–Rosen potential,[6] the Eckart potential,[7] Hulthén potential,[8] the Pöschl–Teller potential,[9] etc. It should be mentioned that almost all the contributions above are made on the study of bound states rather than that of scattering states. Nevertheless, we have to study both of them in order to understand the studied quantum system completely. Thus, based on previous work,[4−9] we have applied a proper approximation to the centrifugal term to obtain approximate analytical l-wave scattering solutions for the Schrödinger equation with the Manning–Rosen potential[10] and the Eckart potential.[11]

The Hulthén potential is one of the most important exponential model potential in physics, and extensively used to describe interaction systems in a number of areas such as nuclear and particle physics,[12] atomic physics,[13,14] molecular physics,[15,16] and chemical physics.[17] It is natural that research into this problem has attracted much attention and interest within the framework of non-relativistic and relativistic quantum mechanics. Unfortunately, Schrödinger and Klein–Gordon and Dirac equations with the Hulthén potential cannot be analytically solved except for s-state[18,19] due to the centrifugal term. In order to obtain arbitrary l-state solutions, several techniques are used to solve the Schrödinger equation with this potential to obtain the approximate analytical solution for

†Corresponding author. E-mail: fgwei_2000@163.com
‡E-mail: chenwenli882002@163.com
§E-mail: why@opt.ac.cn
¶E-mail: liyynxcn@yahoo.com.cn

non-zero angular momentum. These methods include numerical,[20,21] quasi-analytical,[22] shifted $1/N$ expansion,[23,24] NU,[25,26] perturbation,[27] AIM,[28] and SUSYQM methods.[29−31] In addition, the arbitrary l-state relativistic solutions of the Klein–Gordon equation for zero-spin and Dirac equation for 1/2-spin with this potential have also been obtained approximately by taking a proper approximate scheme for the centrifugal term.[32−34] As mentioned above, the Hulthén potential cannot be solved exactly for arbitrary l-states except for a proper approximation to the centrifugal term $1/r^2 \approx \alpha^2 \mathrm{e}^{-\alpha r}/(1-\mathrm{e}^{-\alpha r})^2$. However, it should be noted that the results obtained previously by this approximation[7−9] are in good agreement with those obtained by using the program based on a numerical integration procedure[35] for the short-range potential, but a difference between them appears for the long-range potential. Therefore, it is necessary and of considerable interest to find a new approximation approach to deal with the bound state and scattering state problems for the Hulthén potential. The generalized form of the Hulthén potential has more general significance and important application than the Hulthén potential. For example, the generalized Hulthén potential can be reduced to the standard Hulthén potential and the Woods–Saxon potential by transforming the corresponding parameters; the latter plays an essential role in microscopic physics and nuclear physics since it can be used to describe the interaction of a nucleon with a heavy nucleus.[36] Based on this consideration and motivated by recent work about the bound states of the Hulthén potential carried out by Jia *et al*,[37] we attempt to study the scattering states of the arbitrary l-wave Schrödinger equation with the generalized Hulthén potential, which has not been performed to the best of our knowledge. The generalized Hulthén potential is given by[38]

$$V(r) = -\frac{V_0 \mathrm{e}^{-\alpha r}}{1-q\mathrm{e}^{-\alpha r}}, \tag{1}$$

where α is the screening parameter, which determines the range of the generalized Hulthén potential, parameter V_0 can be seen as the depth of the potential well, and q is a deformation parameter which can take any real value different from zero.

The purpose of this work is two-fold. Firstly, we study the scattering states of the l-wave Schrödinger equation for the generalized Hulthén potentials with the centrifugal term. Secondly, we try to show the relation between the energy levels of the continuum states and those of bound states at the poles of scattering amplitude. Undoubtedly, this study will provide a good reference to interpret theoretically quantum systems with arbitrary l states.

This paper is organized as follows. In Section 2 we obtain approximate analytical scattering state solutions of the Schrödinger equation for the generalized Hulthén potential with the centrifugal term. Section 3 is devoted to briefly studying two special cases for the standard Hulthén potential and Woods–Saxon potential. Some useful figures are plotted to show the improved accuracy in Section 4. The conclusions are given finally in Section 5.

2. The scattering states of the arbitrary l-wave Schrödinger equation

The Schrödinger equation with the generalized Hulthén potential is given by ($\hbar = \mu = 1$)

$$-\frac{1}{2}\nabla^2 \Psi(r,\theta,\phi) + V(r)\,\Psi(r,\theta,\phi) = E\,\Psi(r,\theta,\phi). \tag{2}$$

Here, the potential is taken as the generalized Hulthén potential, i.e., Eq.(1). Taking $\Psi(r,\theta,\varphi) = r^{-1}u(r) \times Y_{lm}(\theta,\varphi)$ and substituting it into Eq.(2), we obtain the radial Schrödinger equation as

$$\frac{\mathrm{d}^2 u(r)}{\mathrm{d}r^2} + \left[2E + \frac{2V_0 \mathrm{e}^{-\alpha r}}{1-q\mathrm{e}^{-\alpha r}} - \frac{l(l+1)}{r^2}\right] u(r) = 0. \tag{3}$$

Obviously, Eq.(3) cannot be solved analytically except for the s-state due to the centrifugal term. Therefore, we must use a proper approximation for the centrifugal term similar to other authors. Unlike the following approximation used in the previous work,[6−11]

$$\frac{1}{r^2} \approx \frac{\alpha^2 \mathrm{e}^{-\alpha r}}{(1-q\mathrm{e}^{-\alpha r})^2}, \tag{4}$$

here we apply an improved new approximate scheme to the centrifugal term[37]

$$\frac{1}{r^2} \approx \alpha^2 \left[\frac{\omega \mathrm{e}^{-\alpha r}}{1-q\mathrm{e}^{-\alpha r}} + \frac{q\mathrm{e}^{-2\alpha r}}{(1-q\mathrm{e}^{-\alpha r})^2}\right], \tag{5}$$

which reduces to Eq.(4) when $\omega = 1$.

Now, let us solve Eq.(3). Inserting Eq.(5) into Eq.(3) we obtain

$$\frac{\mathrm{d}^2u\left(r\right)}{\mathrm{d}r^2}+\left[2E+\frac{2V_0\mathrm{e}^{-\alpha r}}{1-q\mathrm{e}^{-\alpha r}}-\frac{\alpha^2\omega l(l+1)\mathrm{e}^{-\alpha r}}{1-q\mathrm{e}^{-\alpha r}}-\frac{q\alpha^2 l(l+1)\mathrm{e}^{-2\alpha r}}{(1-q\mathrm{e}^{-\alpha r})^2}\right]u\left(r\right)=0. \tag{6}$$

Introducing a new variable $z=1-q\mathrm{e}^{-\alpha r}$, we can rearrange Eq.(6) as

$$\frac{\mathrm{d}^2u\left(z\right)}{\mathrm{d}z^2}-\frac{1}{1-z}\frac{\mathrm{d}u\left(z\right)}{\mathrm{d}z}+\left(\frac{2E}{\alpha^2\left(1-z\right)^2}+\frac{2V_0}{q\alpha^2z\left(1-z\right)}+\frac{\left(1-\omega\right)l\left(l+1\right)}{qz\left(1-z\right)}-\frac{l\left(l+1\right)}{qz^2\left(1-z\right)}\right)u\left(z\right)=0. \tag{7}$$

We take the trial wave function of the form

$$u\left(z\right)=z^{\lambda}\left(1-z\right)^{-\mathrm{i}\,k/\alpha}F\left(z\right), \tag{8}$$

where

$$\lambda=\frac{1}{2}\left[1+\sqrt{\frac{4l^2+4l+q}{q}}\right],\quad k=\sqrt{2E}. \tag{9}$$

Inserting Eq.(8) into Eq.(7), we have

$$\left(1-z\right)z\frac{\mathrm{d}^2F\left(z\right)}{\mathrm{d}z^2}+\left[2\lambda-\left(2\lambda+1-\frac{2\,\mathrm{i}\,k}{\alpha}\right)z\right]\frac{\mathrm{d}F\left(z\right)}{\mathrm{d}z}+\left[\frac{2\,\mathrm{i}\,k\lambda}{\alpha}+\frac{2V_0}{q\alpha^2}-\lambda^2+\frac{\left(1-\omega\right)l\left(l+1\right)}{q}\right]F\left(z\right)=0, \tag{10}$$

whose solutions are nothing but the hypergeometric function[39]

$$F\left(z\right)={}_2F_1\left(a;b;c;z\right)=\sum_{\gamma=0}^{\infty}\frac{\left(a\right)_{\gamma}\left(b\right)_{\gamma}}{\left(c\right)_{\gamma}}\frac{z^{\gamma}}{\gamma!}, \tag{11}$$

where

$$\begin{aligned}&a=\lambda-\frac{\mathrm{i}\,k}{\alpha}+\xi,\quad b=\lambda-\frac{\mathrm{i}\,k}{\alpha}-\xi,\quad c=2\lambda,\\&\xi=\sqrt{\frac{2V_0}{q\alpha^2}-\frac{k^2}{\alpha^2}+\frac{\left(1-\omega\right)l\left(l+1\right)}{q}},\quad \left(z\right)_{\gamma}=\frac{\Gamma\left(z+\gamma\right)}{\Gamma\left(z\right)}.\end{aligned} \tag{12}$$

From Eqs.(12), (11) and (8), we can write the normalized analytical radial wave functions of arbitrary l-wave scattering states for the generalized Hulthén potential as

$$u\left(r\right)=N\left(1-q\mathrm{e}^{-\alpha r}\right)^{\lambda}q^{-\frac{\mathrm{i}\,k}{\alpha}}\mathrm{e}^{\mathrm{i}\,kr}{}_2F_1\left(a;b;c;1-q\mathrm{e}^{-\alpha r}\right), \tag{13}$$

where N is a normalized constant to be determined later.

Before deriving the phase shifts, let us recall a recurrence relation of the hypergeometric function,[39] which is used to analyse the asymptotic behaviour of the wave function and present the normalized constant and phase shifts,

$$\begin{aligned}{}_2F_1\left(a;b;c;z\right)=&\frac{\Gamma\left(a\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}{}_2F_1\left(a;b;a+b-c+1;1-z\right)\\&+\left(1-z\right)^{c-a-b}\frac{\Gamma\left(c\right)\Gamma\left(a+b-c\right)}{\Gamma\left(a\right)\Gamma\left(b\right)}{}_2F_1\left(c-a;c-b;c-a-b+1;1-z\right),\end{aligned} \tag{14}$$

with which, and considering ${}_2F_1\left(a;b;c;0\right)=1$, we have

$$\begin{aligned}{}_2F_1\left(a;b;c;1-q\mathrm{e}^{-\alpha r}\right)\overset{r\to\infty}{\longrightarrow}&\Gamma\left(c\right)\left\{\frac{\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}+q^{\frac{2\,\mathrm{i}\,k}{\alpha}}\mathrm{e}^{-2\,\mathrm{i}\,kr}\left[\frac{\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}\right]^*\right\}\\=&\Gamma\left(c\right)q^{\frac{\mathrm{i}\,k}{\alpha}}\left\{\frac{\Gamma\left(c-a-b\right)q^{-\frac{\mathrm{i}\,k}{\alpha}}}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}+\mathrm{e}^{-2\,\mathrm{i}\,kr}\left[\frac{\Gamma\left(c-a-b\right)q^{-\frac{\mathrm{i}\,k}{\alpha}}}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}\right]^*\right\}.\end{aligned} \tag{15}$$

Taking

$$\frac{\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}=\left|\frac{\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}\right|\mathrm{e}^{\mathrm{i}\,\delta}, \tag{16}$$

and

$$q^{-\frac{\mathrm{i}k}{\alpha}} = \left|q^{-\frac{\mathrm{i}k}{\alpha}}\right| \mathrm{e}^{\mathrm{i}\delta'}, \tag{17}$$

and inserting them into Eq.(15) we have

$$\begin{aligned}
{}_2F_1\left(a;b;c;1-q\mathrm{e}^{-\alpha r}\right) &\overset{r\to\infty}{\longrightarrow} \Gamma(c)\, q^{\frac{\mathrm{i}k}{\alpha}} \left|\frac{\Gamma(c-a-b)\, q^{-\frac{\mathrm{i}k}{\alpha}}}{\Gamma(c-a)\,\Gamma(c-b)}\right| \left[\mathrm{e}^{\mathrm{i}\left(\delta+\delta'\right)} + \mathrm{e}^{-2\,\mathrm{i}\,kr} \times \mathrm{e}^{-\,\mathrm{i}\left(\delta+\delta'\right)}\right] \\
&= \Gamma(c)\, q^{\frac{\mathrm{i}k}{\alpha}} \left|\frac{\Gamma(c-a-b)\, q^{-\frac{\mathrm{i}k}{\alpha}}}{\Gamma(c-a)\,\Gamma(c-b)}\right| \mathrm{e}^{-\,\mathrm{i}\,kr} \left[\mathrm{e}^{\mathrm{i}\left(kr+\delta+\delta'\right)} + \mathrm{e}^{-\,\mathrm{i}\left(kr+\delta+\delta'\right)}\right] \\
&= 2\Gamma(c)\, q^{\frac{\mathrm{i}k}{\alpha}} \left|\frac{\Gamma(c-a-b)\, q^{-\frac{\mathrm{i}k}{\alpha}}}{\Gamma(c-a)\,\Gamma(c-b)}\right| \mathrm{e}^{-\,\mathrm{i}\,kr} \sin\left(kr+\delta+\delta'+\frac{\pi}{2}\right),
\end{aligned} \tag{18}$$

which is inserted into Eq.(13), so we have the asymptotic form of Eq.(13) for $r \to \infty$ as follows:

$$u(r) \overset{r\to\infty}{\longrightarrow} 2N\Gamma(c) \left|\frac{\Gamma(c-a-b)\, q^{\frac{-\,\mathrm{i}\,k}{\alpha}}}{\Gamma(c-a)\,\Gamma(c-b)}\right| \sin\left(kr+\frac{\pi}{2}+\delta+\delta'\right). \tag{19}$$

Comparing Eq.(19) with the boundary condition[2] $r \to \infty \Rightarrow u(\infty) \to 2\sin\left(kr - \frac{1}{2}l\pi + \delta_l\right)$, we get the phase shifts and the normalized constant

$$\begin{aligned}
\delta_l = \frac{\pi}{2}(l+1) + \delta + \delta' &= \frac{\pi}{2}(l+1) + \arg\Gamma(c-a-b) - \arg\Gamma(c-a) - \arg\Gamma(c-b) + \arg\left(q^{\frac{-\,\mathrm{i}\,k}{\alpha}}\right) \\
&= \frac{\pi}{2}(l+1) + \arg\Gamma\left(\frac{2\,\mathrm{i}\,k}{\alpha}\right) - \arg\Gamma\left(\frac{\mathrm{i}\,k}{\alpha} + \lambda + \sqrt{\frac{2V_0}{q\alpha^2} - \frac{k^2}{\alpha^2} + \frac{(1-\omega)\,l\,(l+1)}{q}}\right) \\
&\quad + \arg\Gamma\left(\frac{\mathrm{i}\,k}{\alpha} + \lambda - \sqrt{\frac{2V_0}{q\alpha^2} - \frac{k^2}{\alpha^2} + \frac{(1-\omega)\,l\,(l+1)}{q}}\right) + \arg\left(q^{\frac{-\,\mathrm{i}\,k}{\alpha}}\right),
\end{aligned} \tag{20}$$

$$\begin{aligned}
N &= \frac{1}{\Gamma(2\lambda)} \left|\frac{\Gamma(c-a)\,\Gamma(c-b)}{\Gamma(c-a-b)\, q^{\frac{-\,\mathrm{i}\,k}{\alpha}}}\right| \\
&= \frac{\left|\Gamma\left(\frac{\mathrm{i}\,k}{\alpha} + \lambda - \sqrt{\frac{2V_0}{q\alpha^2} - \frac{k^2}{\alpha^2} + \frac{(1-\omega)\,l\,(l+1)}{q}}\right)\right|}{\Gamma(2\lambda)} \times \left|\frac{\Gamma\left(\frac{\mathrm{i}k}{\alpha} + \lambda + \sqrt{\frac{2V_0}{q\alpha^2} - \frac{k^2}{\alpha^2} + \frac{(1-\omega)\,l\,(l+1)}{q}}\right)}{\Gamma\left(\frac{2\,\mathrm{i}\,k}{\alpha}\right) q^{\frac{-\,\mathrm{i}\,k}{\alpha}}}\right|.
\end{aligned} \tag{21}$$

In the above derivation, we have used the following relations:

$$\begin{aligned}
c-a-b &= \frac{2\,\mathrm{i}\,k}{\alpha} = (a+b-c)^*, \\
c-a &= \frac{\mathrm{i}\,k}{\alpha} + \lambda - \sqrt{\frac{2V_0}{q\alpha^2} - \frac{k^2}{\alpha^2} + \frac{(1-\omega)\,l\,(l+1)}{q}} = b^*, \\
c-b &= \frac{\mathrm{i}\,k}{\alpha} + \lambda + \sqrt{\frac{2V_0}{q\alpha^2} - \frac{k^2}{\alpha^2} + \frac{(1-\omega)\,l\,(l+1)}{q}} = a^*.
\end{aligned} \tag{22}$$

Before concluding this section, let us study the properties of the scattering amplitude. We shall find that the poles of the S-matrix in the complex energy plane correspond to bound states for real poles and scattering states for complex poles in the lower half of the energy plane.[40,41] For this purpose, based on Eq.(20), we discuss the properties of the gamma function $\Gamma\left(\mathrm{i}\,k/\alpha + \lambda + \sqrt{2V_0/q\alpha^2 - k^2/\alpha^2 + (1-\omega)\,l\,(l+1)/q}\right)$. From the definition of the gamma function

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \ldots, \tag{23}$$

we know that $z = 0, -1, -2, \ldots$ are the first order poles of the $\Gamma(z)$, i.e., the first order poles of

$$\Gamma\left(\mathrm{i}\,k/\alpha + \lambda + \sqrt{2V_0/q\alpha^2 - k^2/\alpha^2 + (1-\omega)\,l\,(l+1)\,/q}\right)$$

are situated at

$$\frac{\mathrm{i}\,k}{\alpha} + \lambda + \sqrt{\frac{2V_0}{q\alpha^2} - \frac{k^2}{\alpha^2} + \frac{(1-\omega)\,l\,(l+1)}{q}} = 0,\ -1,\ -2,\ -3, \ldots = -n_r, \quad (n_r = 0, 1, 2\ldots)\,. \tag{24}$$

At the poles of the scattering amplitude, the corresponding energy levels are given by

$$E = -\frac{1}{2q}\left\{\frac{\alpha^2\left[(1+2n_r+2n_r)^2\,q + (1+2n_r)\,\sqrt{4ql\,(l+1)+q^2} + 2l\,(l+1)\,\omega\right] - 4V_0}{2\alpha\left[(1+2n_r)\,\sqrt{q} + \sqrt{4l\,(l+1)+q}\right]}\right\}^2, \tag{25}$$

which are the bound state energy levels of the generalized Hulthén potential. For the standard Hulthén potential, i.e., $q = 1$, Eq.(25) coincides with that given in Ref.[37]. It is well shown that the poles of the S-matrix in the complex energy plane correspond to bound states for real poles and scattering states for complex poles in the lower half of the energy plane.[40,41] That is to say, the energy levels of the continuum states reduce to those of the bound states at the poles of the scattering amplitude.

3. Discussion

In this section we discuss two special cases. Firstly, let us discuss the special case $q = 1$. Obviously, in this case, we have $\lambda = l + 1$, and the generalized Hulthén potential reduces to the standard Hulthén potential. If so, from Eqs.(20), (21) and (13), these obtained results are simplified as

$$\begin{aligned}\delta_l = {} & \frac{\pi}{2}\,(l+1) + \arg\Gamma\left(\frac{2\,\mathrm{i}\,k}{\alpha}\right) - \arg\Gamma\left(\frac{\mathrm{i}\,k}{\alpha} + l + 1 - \sqrt{l\,(l+1)\,(1-\omega) + \frac{2V_0}{\alpha^2} - \frac{k^2}{\alpha^2}}\right) \\ & + \arg\Gamma\left(\frac{\mathrm{i}\,k}{\alpha} + l + 1 + \sqrt{l\,(l+1)\,(1-\omega) + \frac{2V_0}{\alpha^2} - \frac{k^2}{\alpha^2}}\right).\end{aligned} \tag{26}$$

$$N = \frac{\left|\Gamma\left(\frac{\mathrm{i}\,k}{\alpha} + l + 1 - \sqrt{l\,(l+1)\,(1-\omega) + \frac{2V_0}{\alpha^2} - \frac{k^2}{\alpha^2}}\right)\right|}{(2l+1)!}\left|\frac{\Gamma\left(\frac{\mathrm{i}\,k}{\alpha} + l + 1 + \sqrt{l\,(l+1)\,(1-\omega) + \frac{2V_0}{\alpha^2} - \frac{k^2}{\alpha^2}}\right)}{\Gamma\left(\frac{2\,\mathrm{i}\,k}{\alpha}\right)}\right|, \tag{27}$$

$$u\,(r) = N\left(1 - \mathrm{e}^{-\alpha r}\right)^{\lambda}\mathrm{e}^{\mathrm{i}\,kr}{}_2F_1\left(a; b; c; 1 - \mathrm{e}^{-\alpha r}\right), \tag{28}$$

which are the same as those given in Ref.[42] in the case of letting $\omega = 1$ in the above equations.

Secondly, we discuss another typically special case, i.e., $\alpha = 1/R$, $q = \mathrm{e}^{\theta/R}$ and $V_0 \to V_0 q = V_0\mathrm{e}^{\theta/R}$. In this case, the generalized Hulthén potential reduces to the standard Woods–Saxon potential[34,43] as follows:

$$V\,(r) = \frac{V_0}{1 - \exp\left[(r-\theta)/R\right]}, \tag{29}$$

where V_0 is the potential depth, θ is the width of the potential, and R is the surface thickness which is usually adjusted to the experimental values of ionization energies. If so, the corresponding results given in Eqs.(20), (21) and (13) become arbitrary l-wave scattering solutions of the Schrödinger equation for the standard Woods–Saxon potential as follows:

$$\begin{aligned}\delta_l = {} & \frac{\pi}{2}\,(l+1) + \arg\Gamma\,(2\,\mathrm{i}\,Rk) - \arg\Gamma\left(\mathrm{i}\,Rk + \lambda + \sqrt{2V_0R^2 - k^2R^2 - \frac{(1-\omega)\,l\,(l+1)}{\exp\,(\theta/R)}}\right) \\ & + \arg\Gamma\left(\mathrm{i}\,Rk + \lambda - \sqrt{2V_0R^2 - k^2R^2 - \frac{(1-\omega)\,l\,(l+1)}{\exp\,(\theta/R)}}\right) + \arg\left(\left(\exp\,(\theta/R)\right)^{-\,\mathrm{i}\,Rk}\right),\end{aligned} \tag{30}$$

$$N=\frac{\left|\Gamma\left(\mathrm{i}\,Rk+\lambda-\sqrt{2V_0R^2-k^2R^2-\dfrac{(1-\omega)\,l\,(l+1)}{\exp\left(\theta/R\right)}}\right)\right|}{\Gamma\left(2\lambda\right)}\times\left|\frac{\Gamma\left(\mathrm{i}\,Rk+\lambda+\sqrt{2V_0R^2-k^2R^2-\dfrac{(1-\omega)\,l\,(l+1)}{\exp\left(\theta/R\right)}}\right)}{\Gamma\left(2\,\mathrm{i}\,Rk\right)\left[\exp\left(\theta/R\right)\right]^{-\,\mathrm{i}\,Rk}}\right|, \tag{31}$$

$$u\left(r\right)=N\left[1-\exp\left(\left(\theta-r\right)/R\right)\right]^{\lambda}\left[\exp\left(\theta/R\right)\right]^{-\,\mathrm{i}\,Rk}\mathrm{e}^{\mathrm{i}\,kr}{}_2F_1\left(a;b;c;1-\exp\left(\left(\theta-r\right)/R\right)\right). \tag{32}$$

The above results have not been obtained before.

4. Accuracy analysis

To show the improved accuracy of our results, the plots of the centrifugal term (solid line), the improved new approximate scheme for it $1/r^2\approx\alpha^2\left[\omega\mathrm{e}^{-\alpha r}/(1-q\mathrm{e}^{-\alpha r})+q\mathrm{e}^{-2\alpha r}/(1-q\mathrm{e}^{-\alpha r})^2\right]$ (dashed) and the conventional approximation for it $1/r^2\approx\alpha^2\mathrm{e}^{-\alpha r}/\left(1-q\mathrm{e}^{-\alpha r}\right)^2$ (long dashed) as functions of the variable r are displayed in Fig.1(a) and 1(b) with deformation parameter $q=0.8$ and different potential range parameter $\alpha=0.85$, 2.65, and corresponding parameter $\omega=1.295$, 4.750. It is obvious that the conventional approximation given in Eq.(4) is a good approximation to the centrifugal term for short potential range, i.e., small α (for example, $\alpha=0.85$), but not a good approximation to the centrifugal term for long potential range, i.e., large α (for example, $\alpha=2.65$). In order to improve the accuracy in studying the scattering states of the generalized Hulthén potential in the whole potential range, the improved new approximate scheme given in Eq.(5) has been applied in the present work. It should be mentioned that parameter ω is an effectively adjustable parameter and should be very close to 1 for short potential range, i.e., small α, and bigger than 1 for long potential range, i.e., large α as shown in Figs.1(a) and 1(b). Although the parameter ω could not make a good adjustable function when variable $r>4$, the improved new approximate scheme given in Eq.(5) is obviously much better than the conventional approximation given in Eq.(4) as shown in Figs.1(a) and 1(b), especially when variable $r<4$.

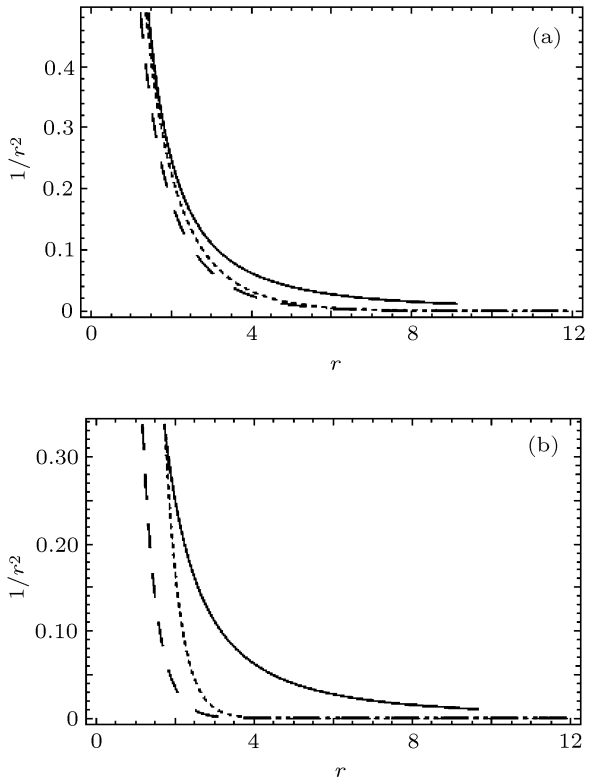


Fig.1. The centrifugal term $1/r^2$ (solid line), the improved new approximation for it $1/r^2\approx\alpha^2\left[\omega\mathrm{e}^{-\alpha r}/(1-q\mathrm{e}^{-\alpha r})+q\mathrm{e}^{-2\alpha r}/(1-q\mathrm{e}^{-\alpha r})^2\right]$ (dashed) and the conventional approximation for it $1/r^2\approx\alpha^2\mathrm{e}^{-\alpha r}/\left(1-q\mathrm{e}^{-\alpha r}\right)^2$ (long dashed) as functions of the variable r with deformation parameter $q=0.8$. (a) $\alpha=0.85,\ \omega=1.295$; (b) $\alpha=2.65,\ \omega=4.750$.

5. Conclusions

This work has presented approximate analytical scattering state solutions of the arbitrary l-wave Schrödinger equation with the generalized Hulthén potentials by taking an improved new approximation to the centrifugal term. It obtains normalized analytical radial wave functions of arbitrary l-wave scattering state and derives the corresponding calculation formula of phase shifts, and shows that the energy levels of the continuum states reduce to those of the bound states at the poles of the scattering amplitude. Also, two special cases for the standard Hulthén potential and Woods–Saxon potential are also studied briefly. Finally, some useful figures are plotted to show the improved accuracy of our results.

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