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A new multi-component integrable coupling system for AKNS equation hierarchy with sixteen-potential functions*

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It is shown in this paper that the upper triangular strip matrix of Lie algebra can be used to construct a new integrable coupling system of soliton equation hierarchy. A direct application to the Ablowitz–Kaup–Newell– Segur(AKNS) spectral problem leads to a novel multi-component soliton equation hierarchy of an integrable coupling system with sixteen-potential functions. It is indicated that the study of integrable couplings when using the upper triangular strip matrix of Lie algebra is an efficient and straightforward method.

Keywords: integrable coupling system, upper triangular strip matrix, Lie algebra
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1. Introduction

Searching for the solution of the soliton equation is a piece of important and interesting work. There are many methods to obtain the new solutions of nonlinear evolution equations, such as Darboux transformation,[1] the Lie symmetry method, and so on. These soliton equations have a rich mathematic structure, for example, the Neumann, Bargmann systems, the quasi-bi-Hamiltonian systems,[2−4] etc. However, how to search for the new equation hierarchy is an important job.

Many coupled equations appear in practical problems, such as biochemistry, physics, mechanics, etc. For example, in order to systematically describe a kind of biochemistry model, Prigogine and Lefever proposed a coupled mathematical model in Ref.[5], which describes a biology-chemistry model; the well-known shallow water wave mathematical model and coupled Korteweg–de Vries(KdV) model are presented in Refs.[6] and [7]; many integrable coupled mathematical models that can solve practical problems are given in Ref.[8].

In Ref.[9], the integrable coupling system is presented by using Virasoro symmetry algebra. Specifically let

$$u_t = K(u)$$

be a known integrable system, the following system:

$$\begin{cases} u_t = K(u), \\ v_t = S(u, v) \end{cases}$$

is called the integrable coupling system, if $v_t = S(u, v)$ is also integrable and $S(u, v)$ contains explicitly u or u-derivative with respect to x. Ma and Fuchssteinen[9] first proposed a perturbation method of establishing integrable couplings. Guo and Zhang[10] presented the enlarged Lie algebra method to obtain integrable couplings.

In the present paper, we present a relation between the upper triangular strip matrix of Lie algebra and integrable couplings of the multi-component soliton equations. A feasible way to generate integrable couplings is presented through the upper triangular strip matrix of Lie algebra. The general theory is used to construct a novel class of integrable couplings of the nonlinear multi-component soliton equation hierarchy. It is indicated that the study of integrable couplings using the upper triangular strip matrix of Lie algebra is an important step towards constructing multi-component integrable systems. As an example, we present an integrable couplings of the multi-component AKNS hierarchy with sixteen-potential functions.

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2. From the upper triangular strip matrix of Lie algebra to integrable couplings

In Ref.[11], the continuous integrable coupling system is presented by using semi-direct sums of Lie algebras. In this section, we construct the integrable couplings of the multi-component AKNS equation hierarchy using the upper triangular strip matrix of Lie algebra.

Let $\boldsymbol{G}$ be a matrix Lie algebra, and assume an integrable equation of evolution type

$$\boldsymbol{U}_t = \boldsymbol{K}(u) \tag{1}$$

to be associated with $\boldsymbol{G}$. Moreover, there is a pair of Lax matrices $\boldsymbol{U}$ and $\boldsymbol{V}$, so that we obtain the matrix spectral problem

$$\boldsymbol{\phi}_x = \boldsymbol{U}\boldsymbol{\phi} = \boldsymbol{U}(u,\lambda)\boldsymbol{\phi} \tag{2}$$

and the associated matrix spectral problem

$$\boldsymbol{\phi}_t = \boldsymbol{V}\boldsymbol{\phi} = \boldsymbol{V}(u,\lambda)\boldsymbol{\phi}, \tag{3}$$

where λ is a spectral parameter. Equations (2) and (3) generate an integrable equation through their isospectral compatibility condition

$$\boldsymbol{U}_t = \boldsymbol{V}_x - [\boldsymbol{U},\boldsymbol{V}]. \tag{4}$$

To construct an integrable coupling of the nonlinear soliton equation hierarchy, we enlarge the Lie algebra $\boldsymbol{G}$ and obtain the matrix Lie algebra $\bar{\boldsymbol{G}}$ that is composed of the matrix Lie algebra $\boldsymbol{G}$ and $\boldsymbol{G}_i, i = 1,2,3$,

$$\bar{\boldsymbol{G}} = \boldsymbol{G} \oplus \boldsymbol{G}_i.$$

The notion of matrix means that $\boldsymbol{G}$ and $\boldsymbol{G}_i$ satisfy

$$[\boldsymbol{G},\boldsymbol{G}_i] \in \bar{\boldsymbol{G}}, \tag{5}$$

where $[\boldsymbol{G},\boldsymbol{G}_i] = \{[\boldsymbol{A},\boldsymbol{B}], \boldsymbol{A} \in \boldsymbol{G}, \boldsymbol{B} \in \boldsymbol{G}_i\}$, the subscript i indicates a contribution to constructing the integrable couplings. Consider a pair of new Lax pair in matrix Lie algebra $\bar{\boldsymbol{G}}$ of $\boldsymbol{G}$ and $\boldsymbol{G}_i$:

$$\bar{\boldsymbol{U}} = \boldsymbol{U} + \boldsymbol{U}_i, \quad \bar{\boldsymbol{V}} = \boldsymbol{V} + \boldsymbol{V}_i, \quad \boldsymbol{U}_i, \boldsymbol{V}_i \in \boldsymbol{G}_i. \tag{6}$$

The new Lax pair form is presented as follows:

$$\bar{\boldsymbol{U}} = \begin{pmatrix} \boldsymbol{U} & \boldsymbol{U}_1 \\ \boldsymbol{U}_2 & \boldsymbol{U}_3 \end{pmatrix}, \ \bar{\boldsymbol{V}} = \begin{pmatrix} \boldsymbol{V} & \boldsymbol{V}_1 \\ \boldsymbol{V}_2 & \boldsymbol{V}_3 \end{pmatrix}, \tag{7}$$

and by substituting a pair of enlarged matrix spectral (7) into the stationary zero curvature equation

$$\bar{\boldsymbol{V}}_x = [\bar{\boldsymbol{U}},\bar{\boldsymbol{V}}],$$

the new stationary zero curvature equations are given as

$$\begin{aligned}
&\boldsymbol{V}_x = [\boldsymbol{U},\boldsymbol{V}] + \boldsymbol{U}_1\boldsymbol{V}_2 - \boldsymbol{V}_1\boldsymbol{U}_2,\\
&\boldsymbol{V}_{1x} = \boldsymbol{U}\boldsymbol{V}_1 + \boldsymbol{U}_1\boldsymbol{V}_3 - \boldsymbol{V}\boldsymbol{U}_1 - \boldsymbol{V}_1\boldsymbol{U}_3,\\
&\boldsymbol{V}_{2x} = \boldsymbol{U}_2\boldsymbol{V} + \boldsymbol{U}_3\boldsymbol{V}_2 - \boldsymbol{V}_2\boldsymbol{U} - \boldsymbol{V}_3\boldsymbol{U}_2,\\
&\boldsymbol{V}_{3x} = \boldsymbol{U}_2\boldsymbol{V}_1 - \boldsymbol{V}_2\boldsymbol{U}_1 + [\boldsymbol{U}_3,\boldsymbol{V}_3].
\end{aligned} \tag{8}$$

Then, enlarged zero curvature equations are presented as follows:

$$\begin{aligned}
&\boldsymbol{U}_t - \boldsymbol{V}_x + [\boldsymbol{U},\boldsymbol{V}] + \boldsymbol{U}_1\boldsymbol{V}_2 - \boldsymbol{V}_1\boldsymbol{U}_2 = 0,\\
&\boldsymbol{U}_{1t} - \boldsymbol{V}_{1x} + \boldsymbol{U}\boldsymbol{V}_1 + \boldsymbol{U}_1\boldsymbol{V}_3 - \boldsymbol{V}\boldsymbol{U}_1 - \boldsymbol{V}_1\boldsymbol{U}_3 = 0,\\
&\boldsymbol{U}_{2t} - \boldsymbol{V}_{2x} + \boldsymbol{U}_2\boldsymbol{V} + \boldsymbol{U}_3\boldsymbol{V}_2 - \boldsymbol{V}_2\boldsymbol{U} - \boldsymbol{V}_3\boldsymbol{U}_2 = 0,\\
&\boldsymbol{U}_{3t} - \boldsymbol{V}_{3x} + \boldsymbol{U}_2\boldsymbol{V}_1 - \boldsymbol{V}_2\boldsymbol{U}_1 + [\boldsymbol{U}_3,\boldsymbol{V}_3] = 0.
\end{aligned} \tag{9}$$

When $\boldsymbol{U}_1 = \boldsymbol{U}_2 = 0$, the first equation here turns exactly into Eq.(4). Eq.(9) provides a coupling system for Eq.(4). To summarize, the matrix of $\bar{\boldsymbol{G}}$ with new Lie algebra provides a greater choice of candidates of integrable couplings for Eq.(9) associated with $\boldsymbol{G}$.

Next, we shed light on the above idea of constructing coupling systems by using a particular class of the upper triangular strip matrix of Lie algebra. Consider the following upper triangular strip matrix of Lie algebra:

$$\boldsymbol{G} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 1 \end{pmatrix},$$

$$\boldsymbol{G}_1 = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix}, \dots,$$

$$\dots, \boldsymbol{G}_{n-1} = \begin{pmatrix} 0 & \dots & 1 & 0 \\ \vdots & 0 & \ddots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix},$$

$$\boldsymbol{G}_n = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}, \tag{10}$$

$$\bar{\boldsymbol{G}} = \boldsymbol{G} \oplus \boldsymbol{G}_i (i = 1, \dots, n), \quad \boldsymbol{G} = \{\mathrm{diag}(1, \dots, 1)\}, \tag{11}$$

So the new enlarged spectral matrices $\bar{\boldsymbol{U}}$ and $\bar{\boldsymbol{V}}$ in $\boldsymbol{G} \oplus \boldsymbol{G}_i$ can be chosen respectively as

$$\bar{\boldsymbol{U}} = \boldsymbol{U}\boldsymbol{G} + \boldsymbol{U}_1\boldsymbol{G}_1 + \dots + \boldsymbol{U}_n\boldsymbol{G}_n$$

and

$$\bar{\boldsymbol{V}} = \boldsymbol{V}\boldsymbol{G} + \boldsymbol{V}_1\boldsymbol{G}_1 + \dots + \boldsymbol{V}_n\boldsymbol{G}_n, \tag{12}$$

i.e.

$$\bar{\boldsymbol{U}} = \begin{pmatrix} \boldsymbol{U} & \boldsymbol{U}_1 & \dots & \boldsymbol{U}_n \\ 0 & \boldsymbol{U} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{U}_1 \\ 0 & \dots & \dots & \boldsymbol{U} \end{pmatrix}$$

and

$$\bar{\boldsymbol{V}} = \begin{pmatrix} \boldsymbol{V} & \boldsymbol{V}_1 & \dots & \boldsymbol{V}_n \\ 0 & \boldsymbol{V} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{V}_1 \\ 0 & \dots & \dots & \boldsymbol{V} \end{pmatrix}.$$

Then, coupling system (9) becomes

$$\begin{aligned} &\boldsymbol{U}_t = \boldsymbol{V}_x - [\boldsymbol{U}, \boldsymbol{V}], \\ &\boldsymbol{U}_{i,t} = \boldsymbol{V}_{ix} - \sum_{k+l=i,k,l,\geq 0} [\boldsymbol{U}_k, \boldsymbol{V}_l], \quad 1 \leq i \leq n, \end{aligned} \tag{13}$$

which is similar to the equation in Ref.[11].

Define a solution

$$\bar{\boldsymbol{W}} = \begin{pmatrix} \boldsymbol{W} & \boldsymbol{W}_1 & \dots & \boldsymbol{W}_n \\ 0 & \boldsymbol{W} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{W}_1 \\ 0 & \dots & \dots & \boldsymbol{W} \end{pmatrix} \tag{14}$$

to the enlarged stationary zero curvature equation $\bar{\boldsymbol{W}}_x = [\bar{\boldsymbol{U}}, \bar{\boldsymbol{W}}]$, then for each $m \geq 0$, we will choose

$$\bar{\boldsymbol{V}} = (\lambda^m \bar{\boldsymbol{W}})_+ + \bar{\boldsymbol{\Delta}}_m, \tag{15}$$

and

$$\bar{\boldsymbol{\Delta}}_m = \begin{pmatrix} \boldsymbol{\Delta}_m & \boldsymbol{\Delta}_{1,m} & \dots & \boldsymbol{\Delta}_{n,m} \\ 0 & \boldsymbol{\Delta}_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{\Delta}_{1,m} \\ 0 & \dots & \dots & \boldsymbol{\Delta}_m \end{pmatrix}, \tag{16}$$

where $\boldsymbol{\Delta}_m$ and $\boldsymbol{\Delta}_{i,m}$ do not depend on λ, but they satisfy

$$\begin{aligned} &\boldsymbol{\Delta}_{mx} = [\boldsymbol{U}, \boldsymbol{\Delta}_m], \\ &\boldsymbol{\Delta}_{i,mx} = [\boldsymbol{U}, \boldsymbol{\Delta}_{i,m}], 1 \leq i \leq n. \end{aligned} \tag{17}$$

According to $\bar{\boldsymbol{V}}_x = [\bar{\boldsymbol{U}}, \bar{\boldsymbol{V}}]$, we can directly obtain that the zero curvature equation as

$$\bar{\boldsymbol{U}}_{t_m} = \bar{\boldsymbol{V}}_x - [\bar{\boldsymbol{U}}, \bar{\boldsymbol{V}}], \tag{18}$$

i.e.

$$\begin{aligned} &\boldsymbol{U}_{t_m} = \boldsymbol{W}_x - [\boldsymbol{U}, \boldsymbol{W}] + \boldsymbol{\Delta}_x - [\boldsymbol{U}, \boldsymbol{\Delta}], \\ &\boldsymbol{U}_{i,t_m} = \boldsymbol{W}_{i,x} - [\boldsymbol{U}_k, \boldsymbol{W}_l] + \boldsymbol{\Delta}_{i,x} - [\boldsymbol{U}_k, \boldsymbol{\Delta}_l], \\ &1 \leq i \leq n, k + l = i. \end{aligned} \tag{19}$$

We will see that this enlarged hierarchy of Lax integrable coupling systems can hold other integrable properties in the special case.

3. Integrable coupling system of the multi-component AKNS equation hierarchy

By using the theory on continuous integrable couplings,[9−14] some integrable coupling systems of the known integrable hierarchies have been obtained, such as Wadati–Konono–Ichikawa (WKI) hierarchy, Kauo–Newell (KN) hierarchy, task decomposition(TD) hierarchy, etc, in Refs.[15]–[22]. In the present paper, in order to obtain the new integrable coupling system, we apply a method using the upper triangular strip matrix of Lie algebra to nonlinear soliton equation hierarchy.

The new Lax pair, $\boldsymbol{U}_3$ and $\boldsymbol{V}_3$, is presented as follows:

$$\boldsymbol{U}_3 = \begin{pmatrix} \boldsymbol{U}_1 & \boldsymbol{U}_2 \\ 0 & \boldsymbol{U}_1 \end{pmatrix},$$

and

$$\boldsymbol{V}_3 = \begin{pmatrix} \boldsymbol{V}_1 & \boldsymbol{V}_2 \\ 0 & \boldsymbol{V}_1 \end{pmatrix}. \tag{20}$$

From the zero curvature equation it follows that

$$\begin{aligned} &\boldsymbol{U}_{1t} - \boldsymbol{V}_{1x} + [\boldsymbol{U}_1, \boldsymbol{V}_1] = 0, \\ &\boldsymbol{U}_{2t} - \boldsymbol{V}_{2x} + [\boldsymbol{U}_1, \boldsymbol{V}_2] + [\boldsymbol{U}_2, \boldsymbol{V}_1] = 0. \end{aligned} \tag{21}$$

To illustrate our method, we consider the following isospectral problem:

$$\boldsymbol{U}_1 = \begin{pmatrix} \frac{\lambda}{2} & q \\ p & -\frac{\lambda}{2} \end{pmatrix}, \quad \boldsymbol{U}_2 = \begin{pmatrix} 1 & u \\ v & 1 \end{pmatrix},$$

$$\boldsymbol{V}_1 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \boldsymbol{V}_2 = \begin{pmatrix} d & e \\ f & g \end{pmatrix}, \tag{22}$$

i.e.

$$\boldsymbol{U}_3 = \begin{pmatrix} \frac{\lambda}{2} & q & 1 & u \\ p & -\frac{\lambda}{2} & v & 1 \\ 0 & 0 & \frac{\lambda}{2} & q \\ 0 & 0 & p & -\frac{\lambda}{2} \end{pmatrix},$$

$$\boldsymbol{V}_3 = \begin{pmatrix} a & b & d & e \\ c & -a & f & g \\ 0 & 0 & a & b \\ 0 & 0 & c & -a \end{pmatrix}$$

where q and r are matrix-potentials, u, v are new dependent variables and $\bar{\boldsymbol{u}} = (q, r, u, v)^{\mathrm{T}}$.

Set

$$\begin{aligned} a &= \sum_{k=0}^{n} a_k \lambda^k, \quad b = \sum_{k=0}^{n} b_k \lambda^k, \\ c &= \sum_{k=0}^{n} c_k \lambda^k, \quad d = \sum_{k=0}^{n} d_k \lambda^k, \\ e &= \sum_{k=0}^{n} e_k \lambda^k, \quad f = \sum_{k=0}^{n} f_k \lambda^k, \\ g &= \sum_{k=0}^{n} \boldsymbol{G}_k \lambda^k. \end{aligned} \tag{23}$$

The corresponding enlarged stationary zero curvature equation is obtained as

$$\boldsymbol{V}_{3x} = [\boldsymbol{U}_3, \boldsymbol{V}_3], \tag{24}$$

i.e.

$$\begin{aligned} \boldsymbol{V}_{1x} &= [\boldsymbol{U}_1, \boldsymbol{V}_1], \\ \boldsymbol{V}_{2x} &= [\boldsymbol{U}_1, \boldsymbol{V}_2] + [\boldsymbol{U}_2, \boldsymbol{V}_1]. \end{aligned} \tag{25}$$

Obviously, we have

$$[\boldsymbol{U}_1, \boldsymbol{V}_1] = \begin{pmatrix} qc_n - b_n p & 2\lambda b_n - 2qa_n \\ -2\lambda c_n + 2pa_n & pb_n - c_n q \end{pmatrix}, \tag{26a}$$

$$[\boldsymbol{U}_1, \boldsymbol{V}_2] = \begin{pmatrix} qf_n - e_n p & \lambda e_n + qg_n - d_n q \\ pd_n - g_n p - \lambda f_n & pe_n - f_n q \end{pmatrix}, \tag{26b}$$

$$[\boldsymbol{U}_2, \boldsymbol{V}_1] = \begin{pmatrix} uc_n - b_n v & -ua_n - a_n u \\ va_n + a_n v & vb_n - c_n u \end{pmatrix}. \tag{26c}$$

According to Eq.(26), we can obtain

$$\begin{aligned} a_{nx} &= qc_n - pb_n, \\ b_{nx} &= 2b_{n+1} - 2qa_n, \\ c_{nx} &= -2c_{n+1} + 2pa_n, \\ d_{nx} &= qf_n - pe_n + uc_n - vb_n, \\ e_{nx} &= e_{n+1} + qg_n - qd_n - 2ua_n, \\ f_{nx} &= pd_n - pg_n - f_{n+1} + 2va_n, \\ g_{nx} &= pe_n - qf_n + vb_n - uc_n. \end{aligned} \tag{27}$$

Set $b_0 = c_0 = 0$, $a_0 = \frac{1}{2}$, $d_0 = e_0 = f_0 = g_0 = 0$, then we will see that all sets of functions $a_n, b_n, c_n, d_n, e_n, f_n, g_n$ are uniquely determined. In particular, the first few sets are

$$\begin{aligned} &b_1 = q, \quad c_1 = p, \quad a_1 = 0, \\ &b_2 = q_x, \quad c_2 = -p_x, \quad a_2 = -qp, \\ &b_3 = q_{xx} - 2q^2 p, \\ &c_3 = p_{xx} - 2p^2 q, \\ &d_1 = 0, \quad e_1 = u, \quad f_1 = v, \quad g_1 = 0, \\ &d_2 = -qv_x - pu, \quad e_2 = u_x, \\ &f_2 = -v_x, \quad g_2 = pu + qv. \end{aligned} \tag{28}$$

Based on Tu Guizhang's(Tu) method, it is obtained that

$$\bar{\boldsymbol{U}}_{tn} = \begin{pmatrix} q \\ p \\ u \\ v \end{pmatrix}_t = \left(\begin{array}{c|c} \boldsymbol{J}_1 & 0 \\ \hline 0 & \boldsymbol{J}_2 \end{array}\right) \begin{pmatrix} b_{n+1} \\ c_{n+1} \\ f_{n+1} \\ e_{n+1} \end{pmatrix}, \tag{29}$$

with

$$\boldsymbol{J}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{J}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are Hamiltonian operators. According to Eq.(26), we have the following reduced relation:

$$\begin{pmatrix} b_{n+1} \\ c_{n+1} \\ f_{n+1} \\ e_{n+1} \end{pmatrix} = \boldsymbol{L} \begin{pmatrix} b_n \\ c_n \\ f_n \\ e_n \end{pmatrix}, \tag{30}$$

where

$$\boldsymbol{L} = \begin{pmatrix} \partial - 2q\partial^{-1}p & 2q\partial^{-1}q & 0 & 0 \\ -2p\partial^{-1}p & -\partial + 2p\partial^{-1}q & 0 & 0 \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}$$

with

$$L_{31} = -2p\partial^{-1}v - 2v\partial^{-1}p, L_{32} = 2p\partial^{-1}u + 2v\partial^{-1}q,$$
$$L_{33} = -\partial + 2p\partial^{-1}q, L_{34} = -2p\partial^{-1}p,$$
$$L_{41} = -2q\partial^{-1}v - 2u\partial^{-1}p, L_{42} = 2q\partial^{-1}u + 2u\partial^{-1}q,$$
$$L_{43} = 2q\partial^{-1}q, L_{44} = \partial - 2q\partial^{-1}p.$$

By using Eqs.(27) and (28), when $n = 2$ the first systems in Eq.(29) are

$$\begin{aligned} q_{t_2} &= -q_{xx} + 2q^2p, \\ p_{t_2} &= p_{xx} - 2p^2q, \end{aligned}$$

$$\begin{aligned} u_{t_2} &= -u_{xx} + 4qpu + 2q^2v, \\ v_{t_2} &= v_{xx} - 4qpv - 2p^2u. \end{aligned} \tag{31}$$

When $u = v = 0$, they turn into the well-known nonlinear AKNS equation. So system (29) is a novel soliton equation hierarchy.

Substitute

$$\begin{aligned} \boldsymbol{q}(x,t) &= \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}, \\ \boldsymbol{p}(x,t) &= \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}, \\ \boldsymbol{u}(x,t) &= \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}, \\ \boldsymbol{v}(x,t) &= \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}, \end{aligned} \tag{32}$$

into integrable coupling system (31), then we will obtain the super-integrable coupling system of the AKNS equations with sixteen-potential functions as follows:

$$\begin{aligned}
q_{1t} &= -q_{1xx} + 2(q_1^2p_1 + q_2q_3p_1 + q_1q_2p_3 + q_2q_4p_3), \\
q_{2t} &= -q_{2xx} + 2(q_1^2p_2 + q_2q_3p_2 + q_1q_2p_4 + q_2q_4p_4), \\
q_{3t} &= -q_{3xx} + 2(q_1q_3p_1 + q_3q_4p_1 + q_2q_3p_3 + q_4^2p_3), \\
q_{4t} &= -q_{4xx} + 2(q_1q_3p_2 + q_3q_4p_2 + q_2q_3p_4 + q_4^2p_4), \\
p_{1t} &= p_{1xx} - 2(p_1^2q_1 + p_2p_3q_1 + p_1p_2q_3 + p_2p_4q_3), \\
p_{2t} &= p_{2xx} - 2(p_1^2q_2 + p_2p_3q_2 + p_1p_2q_4 + p_2p_4q_4), \\
p_{3t} &= p_{3xx} - 2(p_1p_3q_1 + p_3p_4q_1 + p_2p_3q_3 + p_4^2q_3), \\
p_{4t} &= p_{4xx} - 2(p_1p_3q_2 + p_3p_4q_2 + p_2p_3q_4 + p_4^2q_4), \\
u_{1t} &= -u_{1xx} + 4(q_1p_1u_1 + q_2p_3u_1 + q_1p_2u_3 + q_2p_4u_3) + 2(q_1^2v_1 + q_2q_3v_1 + q_1q_2v_3 + q_2q_4v_3), \\
u_{2t} &= -u_{2xx} + 4(q_1p_1u_2 + q_2p_3u_2 + q_1p_2u_4 + q_2p_4u_4) + 2(q_1^2v_2 + q_2q_3v_2 + q_1q_2v_4 + q_2q_4v_4), \\
u_{3t} &= -u_{3xx} + 4(q_3p_1u_1 + q_4p_3u_1 + q_3p_2u_3 + q_4p_4u_3) + 2(q_1q_3v_1 + q_3q_4v_1 + q_2q_3v_3 + q_4^2v_3), \\
u_{4t} &= -u_{4xx} + 4(q_3p_1u_2 + q_4p_3u_2 + q_3p_2u_4 + q_4p_4u_4) + 2(q_1q_3v_2 + q_3q_4v_2 + q_2q_3v_4 + q_4^2v_4),
\end{aligned}$$

$$
\begin{aligned}
v_{1t} &= v_{1xx} - 4(q_1p_1v_1 + q_2p_3v_1 + q_1p_2v_3 + q_2p_4v_3) - 2(p_1^2u_1 + p_2p_3u_1 + p_1p_2u_3 + p_2p_4u_3),\\
v_{2t} &= v_{2xx} - 4(q_1p_1v_2 + q_2p_3v_2 + q_1p_2v_4 + q_2p_4v_4) - 2(p_1^2u_2 + p_2p_3u_2 + p_1p_2u_4 + p_2p_4u_4),\\
v_{3t} &= v_{3xx} - 4(q_3p_1v_1 + q_4p_3v_1 + q_3p_2v_3 + q_4p_4v_3) - 2(p_1p_3u_1 + p_3p_4u_1 + p_2p_3u_3 + p_4^2u_3),\\
v_{4t} &= v_{4xx} - 4(q_3p_1v_2 + q_4p_3v_2 + q_3p_2v_4 + q_4p_4v_4) - 2(p_1p_3u_2 + p_3p_4u_2 + p_2p_3u_4 + p_4^2u_4). \qquad (33)
\end{aligned}
$$

We obtain the integrable couplings of AKNS equations with sixteen-potential functions by using the upper triangular strip matrix of Lie algebra. Equation(33) is referred to as multi-component AKNS equations.

4. Conclusion

A technically-feasible way to construct integrable couplings of multi-component soliton equations has been presented by using the upper triangular strip matrix of Lie algebra, and an application of the general theory to the AKNS hierarchy of soliton equations is made. The general idea in our analysis could also be applied to other types of soliton equations such as discrete soliton equations and higher-dimensional soliton equations. All integrable couplings generated in this way provide examples of multicomponent integrable systems and show a richness of integrable systems.

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