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Lagrange–Noether method for solving second-order differential equations*

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The purpose of this paper is to provide a new method called the Lagrange–Noether method for solving second-order differential equations. The method is, firstly, to write the second-order differential equations completely or partially in the form of Lagrange equations, and secondly, to obtain the integrals of the equations by using the Noether theory of the Lagrange system. An example is given to illustrate the application of the result.

Keywords: differential equation, Lagrange equation, Noether theory, integral
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1. Introduction

The modern development of differential equations has mainly two directions: one is the geometrical method, such as Arnold's work;[1] the other is the Lie group method, for example, the works of Olver, Bluman, etc.[2−4] Reference [4] pointed out that symmetry and first integral are two fundamental structures for ordinary differential equations. Symmetries are mainly the Noether symmetry,[5−9] the Lie symmetry[6−8,10−18] and form invariance.[8,19−21] From the integration point of view, the Noether symmetry is easier to use. This paper completely or partially transforms a system of ordinary differential equations of second order into a Lagrange system, and finds its Noether conserved quantities by using the Noether symmetries of the above Lagrange system. So, this method is said to be the Lagrange–Noether method. On finding part of the conserved quantities of the system, it can reduce its equations; on finding all of the conserved quantities of the system, it can give the solution of the system.

2. Completely or partially Lagrangian representation of equations

Consider the integrals of a system of differential equations of second order

$$\ddot{q}_s = f_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, \ldots, n). \tag{1}$$

The necessary and sufficient condition that system (1) is self-adjoint is that f_s are linear with respect to $\dot{\boldsymbol{q}}$, i.e. they have the form

$$\ddot{q}_s - \rho_{sk}(t, \boldsymbol{q})\dot{q}_k - \sigma_s(t, \boldsymbol{q}) = 0, \tag{2}$$

and satisfy conditions[22]

$$\begin{aligned} &\rho_{sk} + \rho_{ks} = 0, \quad \frac{\partial \rho_{sk}}{\partial q_r} + \frac{\partial \rho_{kr}}{\partial q_s} + \frac{\partial \rho_{rs}}{\partial q_k} = 0, \ (s, k, r = 1, \ldots, n), \\ &\frac{\partial \rho_{sk}}{\partial t} = \frac{\partial \sigma_s}{\partial q_k} - \frac{\partial \sigma_k}{\partial q_s}. \end{aligned} \tag{3}$$

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In this case, system (1) can be written in the form of Lagrange equations whose Lagrangian is

$$L = \frac{1}{2}\dot{q}_s\dot{q}_s + D_k(t,\boldsymbol{q})\dot{q}_k + C(t,\boldsymbol{q}), \tag{4}$$

where

$$D_k = -\left[\int_0^1 \mathrm{d}\tau\tau\rho_{ks}(t,\tau\boldsymbol{q})\right] q_s, \tag{5}$$

$$C = \left[\int_0^1 \mathrm{d}\tau\left(\sigma_s + \frac{\partial D_s}{\partial t}\right)(t,\tau\boldsymbol{q})\right] q_s. \tag{6}$$

If there is only one equation in system (1), then it is self-adjoint only when it does not contain $\dot{\boldsymbol{q}}$.

If system (1) is not self-adjoint, but it can be written in the self-adjoint form

$$h_{sk}(t,\boldsymbol{q},\dot{\boldsymbol{q}})(\ddot{q}_k - f_k) = 0. \tag{7}$$

In such case, one can find the Lagrangian. The method is as follows:

Let

$$A_{sk} = h_{sk}, \quad B_s = -h_{sk}f_k. \tag{8}$$

Then the necessary and sufficient condition that Eqs.(7) are self-adjoint is

$$\begin{aligned}
&A_{sk} = A_{ks},\\
&\frac{\partial A_{sk}}{\partial\dot{q}_r} = \frac{\partial A_{rk}}{\partial\dot{q}_s},\\
&\frac{\partial B_s}{\partial\dot{q}_k} + \frac{\partial B_k}{\partial\dot{q}_s} = 2\left(\frac{\partial}{\partial t} + \dot{q}_r\frac{\partial}{\partial q_r}\right)A_{sk},\\
&\frac{\partial B_s}{\partial q_k} - \frac{\partial B_k}{\partial q_s} = \frac{1}{2}\left(\frac{\partial}{\partial t} + \dot{q}_r\frac{\partial}{\partial q_r}\right)\left(\frac{\partial B_s}{\partial\dot{q}_k} - \frac{\partial B_k}{\partial\dot{q}_s}\right),\\
&(s,k,r = 1,\ldots,n),
\end{aligned} \tag{9}$$

and the Lagrangian has the form

$$L = K(t,\boldsymbol{q},\dot{\boldsymbol{q}}) + D_k\dot{q}_k + C(t,\boldsymbol{q}), \tag{10}$$

where the $n+2$ functions K, D_k, C form a solution of the following overdetermined system of partial differential equations:

$$\begin{aligned}
&\frac{\partial^2 K}{\partial\dot{q}_s\partial\dot{q}_k} = A_{sk}(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\\
&\frac{\partial D_s}{\partial q_k} - \frac{\partial D_k}{\partial q_s} = \frac{1}{2}\left(\frac{\partial B_s}{\partial\dot{q}_k} - \frac{\partial B_k}{\partial\dot{q}_s}\right) + \frac{\partial^2 K}{\partial q_s\partial\dot{q}_k} - \frac{\partial^2 K}{\partial\dot{q}_s\partial q_k} \equiv Z_{sk}(t,\boldsymbol{q}),\\
&\frac{\partial C}{\partial q_s} = \frac{\partial D_s}{\partial t} - B_s - \frac{\partial K}{\partial q_s} + \frac{\partial^2 K}{\partial\dot{q}_s\partial t} + \left[\frac{\partial^2 K}{\partial q_s\partial\dot{q}_k} + \frac{1}{2}\left(\frac{\partial B_s}{\partial\dot{q}_k} - \frac{\partial B_k}{\partial\dot{q}_s}\right)\right]\dot{q}_k \equiv W_s(t,\boldsymbol{q}),
\end{aligned}$$

specifically,

$$\begin{aligned}
&K(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{q}_s\int_0^1 \mathrm{d}\tau'\left\{\left[\int_0^1 \mathrm{d}\tau A_{sk}(t,\boldsymbol{q},\tau\dot{\boldsymbol{q}})\right]\dot{q}_k\right\}(t,\boldsymbol{q},\tau'\dot{\boldsymbol{q}}),\\
&D_k = \left[\int_0^1 \mathrm{d}\tau\tau Z_{ks}(t,\tau\boldsymbol{q})\right] q_s,\\
&C = \left[\int_0^1 \mathrm{d}\tau W_s(t,\tau\boldsymbol{q})\right] q_s.
\end{aligned} \tag{11}$$

Thus, system (1) can be transformed into the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s} - \frac{\partial L}{\partial q_s} = 0 \quad (s = 1,\ldots,n). \tag{12}$$

If system (1) cannot be expressed or is difficult to express by using a Lagrangian, then it can be partially transformed into the Lagrange equations, namely be transformed into a general holonomic mechanical system whose equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s, \quad (s = 1,\ldots,n), \tag{13}$$

where

$$L = \frac{1}{2}\dot{q}_s\dot{q}_s, \quad Q_s = f_s. \tag{14}$$

3. Integration of equations

System (1), after being expressed by using a Lagrangian, becomes Eqs.(12). The Noether theory for the Lagrange system shows that if infinitesimal generators $\xi_0 = \xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \xi_s = \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and the gauge function $G_{\rm N} = G_{\rm N}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfy the Noether identity[5−8]

$$L\dot{\xi}_0 + X^1(L) + \dot{G}_{\rm N} = 0, \tag{15}$$

where

$$X^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s}, \tag{16}$$

then system (12) possesses the Noether conserved quantity[5−8]

$$I_{\rm N} = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\rm N} = \text{const.} \tag{17}$$

System (1), after being partially expressed by using a Lagrangian, becomes Eqs.(13). The Noether theory shows that if infinitesimal generators ξ_0, ξ_s and the gauge function $G_{\rm N}$ satisfy the Noether identity[5−8]

$$L\dot{\xi}_0 + X^{(1)}(L) + Q_s(\xi_s - \dot{q}_s\xi_0) + \dot{G}_{\rm N} = 0, \tag{18}$$

then system (13) possesses the Noether conserved quantity (17).

Clearly, the Noether conserved quantity (17) is an integral of system (1). If enough integrals can be found, then a solution of system (1) can be obtained.

4. An illustrative example

Consider a second-order differential equation

$$\ddot{q} + \frac{4}{t}\dot{q} + \frac{2}{t^2}q = 0, \quad (t > 0). \tag{19}$$

Clearly, Eq.(19) does not satisfy conditions Eqs.(3), so it is not self-adjoint. Multiplying both its sides by t^4, we have

$$t^4\ddot{q} + 4t^3\dot{q} + 2t^2q = 0, \tag{20}$$

which is self-adjoint according to conditions Eqs.(9). Expression (11) shows

$$K = \dot{q}\int_0^1 t^4\dot{q}\tau' \mathrm{d}\tau' = \frac{1}{2}t^4\dot{q}^2,$$
$$D = 0,$$
$$C = \int_0^1 (-2t^2q^2\tau)\mathrm{d}\tau = -t^2q^2.$$

Then the Lagrangian

$$L = \frac{1}{2}t^4\dot{q}^2 - t^2q^2 \tag{21}$$

and the Noether identity (15) gives

$$L\dot{\xi}_0 + (2t^3\dot{q}^2 - 2tq^2)\xi_0 - 2t^2q\xi$$
$$+t^4\dot{q}(\dot{\xi} - \dot{q}\dot{\xi}_0) + \dot{G}_{\rm N} = 0,$$

which has the following solutions

$$\xi_0 = 0, \quad \xi = t^{-2}, \quad G_{\rm N} = 2tq, \tag{22}$$

$$\xi_0 = 0, \quad \xi = t^{-1}, \quad G_{\rm N} = t^2q. \tag{23}$$

The Noether conserved quantity (17) gives respectively

$$I_{\rm N} = t^2\dot{q} + 2tq = C_1, \tag{24}$$
$$I_{\rm N} = t^3\dot{q} + t^2q = C_2, \tag{25}$$

where C_1, C_2 are arbitrary constants. Elimination of $\dot{q}$ in the above two integrals gives

$$q = C_1t^{-1} - C_2t^{-2}, \tag{26}$$

which is the solution of Eq.(19).

Otherwise, Eq.(19) can also be partially expressed by using the Lagrangian as

$$L = \frac{1}{2}\dot{q}^2, \quad Q = -\frac{4}{t}\dot{q} - \frac{2}{t^2}q. \tag{27}$$

Then the Noether identity (18) gives

$$L\dot{\xi}_0 + \dot{q}(\dot{\xi} - \dot{q}\dot{\xi}_0) - \left(\frac{4}{t}\dot{q} + \frac{2}{t^2}q\right)(\xi - \dot{q}\xi_0) + \dot{G}_{\rm N} = 0.$$

It has the following solutions:

$$\xi_0 = 0, \quad \xi = t^2, \quad G_{\rm N} = 2tq, \tag{28}$$
$$\xi_0 = 0, \quad \xi = t^3, \quad G_{\rm N} = t^2q. \tag{29}$$

The Noether conserved quantity (17) gives the integrals (24) and (25), respectively. The solution of Eq.(19) is also expression (26).

5. Conclusions

In this paper, the Lagrange–Noether method for solving second-order differential equations is presented; its essential idea is completely or partially to transform equations into a Lagrange system, and find its Noether conserved quantities, namely the integrals of the equations, by using the Noether theory of the Lagrange system. If enough conserved quantities of the equations are found, one can obtain the solution of the equations.

References

[1] Arnold V I 1983 *Geometrical Methods in the Theory of Ordinary Differential Equations* (New York: Springer-Verlag)

[2] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer-Verlag)

[3] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (New York: Springer-Verlag)

[4] Bluman G W and Anco S C 2002 *Symmetry and Integration Methods for Differential Equations* (New York: Springer-Verlag)

[5] Li Z P 1993 *Classical and Quantum Dynamics of Constrained Systems and Their Symmetrical Properties* (Beijing: Beijing Polytechnic University Press) (in Chinese)

[6] Zhao Y Y and Mei F X 1999 *Symmetries and Invariants of Mechanical Systems* (Beijing: Science Press) (in Chinese)

[7] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)

[8] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[9] Ge W K 2002 *Acta Phys. Sin.* **51** 1156 (in Chinese)

[10] Mei F X 2000 *Acta Mechanica* **141** 135

[11] Zhang H B 2002 *Chin. Phys.* **11** 1

[12] Wang S Y and Mei F X 2002 *Chin. Phys.* **11** 5

[13] Xu Z X 2002 *Acta Phys. Sin.* **51** 2423 (in Chinese)

[14] Zhang Y 2008 *Acta Phys. Sin.* **57** 2643 (in Chinese)

[15] Fang J H, Liao Y P and Zhang J 2004 *Acta Phys. Sin.* **53** 4037 (in Chinese)

[16] Fu J L, Chen L Q, Bai J H and Yang X D 2003 *Chin. Phys.* **12** 695

[17] Luo S K and Cai J L 2003 *Chin. Phys.* **12** 357

[18] Qian Y F, Li R J and Zhao S H 2004 *Chin. Phys.* **13** 1790

[19] Mei F X 2000 *J. Beijing Institute of Technology* **9** 120

[20] Xu X J, Mei F X and Qin M C 2004 *Chin. Phys.* **13** 1999

[21] Mei F X and Xu X J 2005 *Chin. Phys.* **14** 449

[22] Santilli R M 1978 *Foundations of Theoretical Mechanics I* (New York: Springer-Verlag)