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The homotopic mapping solution for the solitary wave for a generalized nonlinear evolution equation*

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This paper studies a generalized nonlinear evolution equation. Using the homotopic mapping method, it constructs a corresponding homotopic mapping transform. Selecting a suitable initial approximation and using homotopic mapping, it obtains an approximate solution with an arbitrary degree of accuracy for the solitary wave. From the approximate solution obtained by using the homotopic mapping method, it possesses a good accuracy.

Keywords: evolution equation, nonlinear, soliton, approximate method
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1. Introduction

The solitary wave is a very attractive object in the nonlinear equation of evolution. A lot of new methods have appeared, for example, the hyperbolic tangent function method,[1] the homogeneous equilibrium method,[2] the Jacobi elliptic function method,[3] the auxiliary equation method,[4] etc. Many scholars have done a great deal of work, such as on shock waves,[5,6] scattering light waves[7] quantum mechanics,[8] atmospheric physics,[9] networks of neurons[10] and so on, studying the theorem of the solitary wave. The asymptotic method for a nonlinear theory of the solitary wave is a new study. The essentials of this method are that the studied nonlinear problem deals with the linear problem by using asymptotic expansion. The homotopic mapping method[11,12] is namely such a new method.

During the past decade many approximate methods have been developed and refined, including the method of averaging, the boundary layer method, methods of matched asymptotic expansion and multiple scales. Recently, many scholars such as Ni and Wei,[13] Bartier,[14] Llibre, Silva and Teixeira[15] and Guarguaglini and Natalini[16] have done a great deal of work. Using the method of differential inequalities and others, Mo *et al* also considered a class of reaction diffusion problems,[17] shock waves,[18] ecological environments,[19] soliton waves,[20−22] laser pulses,[23] ocean science,[24] atmospheric physics,[25−30] etc. In this paper, we consider the nonlinear evolution equation, and obtain an approximate solution of the solitary wave.

2. Model and homotopic mapping

Consider the following generalized nonlinear evolution equation:[31]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - pu + qu^3 = f(u), \tag{1}$$

where p and q are positive constants, $f(u)$ is a sufficiently smooth function.

Equation (1) as $f(u) = 0$ is

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} - pv + qv^3 = 0. \tag{2}$$

From Ref.[32], Eq.(2) possesses the following single solitary wave solution:

$$v(x,t) = \sqrt{\frac{p}{q}} \tanh \sqrt{\frac{p}{2(1-\beta^2)}}(x - x_0 + \beta\, t), \tag{3}$$

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where x_0, β $(\beta^2 < 1)$ are arbitrary constants, which can be decided by the conditions of the original problem.

In order to find the approximate solution to Eq.(1), we introduce homotopic mapping[11,12] $H(v,s): R \times I \to R$:

$$H(u,s) = L(u) - L(w) + s(L(w) + qu^3 - f(u)), \quad (4)$$

where w is an undetermined function which is determined below, $R = (-\infty, +\infty)$, $I = [0,1]$, and the linear operator L is

$$L(u) = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - pu. \quad (5)$$

Obviously, from Eq.(4), $H(u,1) = 0$, which is the same as Eq(1). Thus the solution $u(x,t)$ for Eq.(1) is the solution of $H(u,s) = 0$ as $s \to 1$.

3. Approximate solution for the solitary wave

Let

$$u = \sum_{i=0}^{\infty} u_i(x,t)s^i. \quad (6)$$

Considering Eq.(6), from Eq.(4), we compare the coefficients of equation $H(u,s) = 0$ for the same powers in s. From the coefficients of 0-th power in s for $H(u,s) = 0$, we have

$$L(u_0) = L(w). \quad (7)$$

Selecting w to be u_0, then from Eqs.(7), (5) and (3), we can obtain

$$u_0(x,t) = \sqrt{\frac{p}{q}} \tanh \sqrt{\frac{p}{2(1-\beta^2)}}(x - x_0 + \beta\, t). \quad (8)$$

From the coefficients of the first power in s for $H(u,s) = 0$, we have

$$L(u_1) = f(u_0) - L(w) - q(u_0)^3.$$

That is

$$L(u_1) = f(u_0), \quad (9)$$

where u_0 is represented by Eq.(8). Using the Fourier transform method, we obtain the solution of Eq.(9) as zero' initial value

$$u_1(x,t) = \frac{1}{\pi}\int_0^t \mathrm{d}t_1 \int_0^{\infty} \frac{1}{\sqrt{\lambda^2 - p}}\mathrm{d}\lambda \int_{-\infty}^{\infty} f(u_0(\xi,t_1)) \sin\sqrt{\lambda^2 - p}(t - t_1)\cos\lambda\xi \mathrm{d}\xi. \quad (10)$$

From the coefficients of the second power in s for $H(u,s) = 0$, we have

$$L(u_2) = (-3qu_0^2 + f_u(u_0))u_1. \quad (11)$$

The solution of Eq.(11) as zero' initial value is

$$\begin{aligned} u_2(x,t) = \frac{1}{\pi}\int_0^t \mathrm{d}t_1 \int_0^{\infty} \frac{1}{\sqrt{\lambda^2 - p}} \int_{-\infty}^{\infty} [-3qu_0^2(\xi,t_1) \\ + f_u(u_0(\xi,t_1))u_1(\xi,t_1)[\sin\sqrt{\lambda^2 - p}(t - t_1)\cos\lambda\xi \mathrm{d}\xi \mathrm{d}\lambda, \end{aligned} \quad (12)$$

where u_0 and u_1 are denoted by Eqs.(8) and (10), respectively.

From Eqs.(6), (8), (10), (12) and the homotopic theory, we obtain a second order approximation of the solitary wave solution for the nonlinear evolution equation:

$$\begin{aligned} u(x,t) = \sqrt{\frac{p}{q}} \tanh \sqrt{\frac{p}{2(1-\beta^2)}}(x - x_0 + \beta\, t) \\ + \frac{1}{\pi}\int_0^t \mathrm{d}t_1 \int_0^{\infty} \frac{1}{\sqrt{\lambda^2 - p}}\mathrm{d}\lambda \int_{-\infty}^{\infty} f(u_0(\xi,t_1)) \sin\sqrt{\lambda^2 - p}(t - t_1)\cos\lambda\xi \mathrm{d}\xi \\ + \frac{1}{\pi}\int_0^t \mathrm{d}t_1 \int_0^{\infty} \frac{1}{\sqrt{\lambda^2 - p}} \int_{-\infty}^{\infty} [-3qu_0^2(\xi,t_1) \\ + f_u(u_0(\xi,t_1))u_1(\xi,t_1)[\sin\sqrt{\lambda^2 - p}(t - t_1)]\cos\lambda\xi \mathrm{d}\xi \mathrm{d}\lambda, \end{aligned} \quad (13)$$

where u_0 and u_1 are denoted by Eqs.(8) and (10), respectively.

Continuously, using the same method, we can also obtain a higher order approximation of the solitary wave solution for the nonlinear evolution equation.

4. Comparison of accuracy

In order to explain the accuracy of the expressions of the approximate solution represented by Eq.(13), we consider the small perturbation term $f(u) = \varepsilon \sin u$, $0 < \varepsilon \ll 1$ in Eq.(1).

Firstly, from Eq.(13), we can obtain the approximate solution $u(x,t)$ by using the homotopic mapping method as follows:

$$\begin{aligned} u(x,t) = & \sqrt{\frac{p}{q}} \tanh \sqrt{\frac{p}{2(1-\beta^2)}}(x - x_0 + \beta\, t) \\ & + \frac{1}{\pi} \int_0^t \mathrm{d}t_1 \int_0^\infty \frac{1}{\sqrt{\lambda^2 - p}} \int_{-\infty}^\infty [-3qu_0^2(\xi, t_1)\; u_1(\xi, t_1) \sin \sqrt{\lambda^2 - p}(t - t_1) \cos \lambda\xi] \mathrm{d}\xi \mathrm{d}\lambda \\ & + \frac{\varepsilon}{\pi} \int_0^t \mathrm{d}t_1 \int_0^\infty \frac{1}{\sqrt{\lambda^2 - p}} \int_{-\infty}^\infty [\sin u_0(\xi, t_1)\; u_1(\xi, t_1) \sin \sqrt{\lambda^2 - p}(t - t_1) \cos \lambda\xi] \mathrm{d}\xi \mathrm{d}\lambda, \end{aligned} \tag{14}$$

where u_0 and u_1 are denoted by Eqs.(8) and (10), respectively.

Secondly, by using the perturbation method we construct the asymptotic solution $\bar{u}(x,t)$ of Eq.(1) as $f(u) = \varepsilon \sin u$. Let

$$\bar{u} = \sum_{i=0}^{\infty} \bar{u}_i(x,t)\varepsilon^i, \quad 0 < \varepsilon \ll 1.$$

Substituting the above expansion into Eq.(1) as $f(u) = \varepsilon \sin u$, and using the perturbation method, we can obtain

$$\begin{aligned} \bar{u}_0(x,t) = & \sqrt{\frac{p}{q}} \tanh \sqrt{\frac{p}{2(1-\beta^2)}}(x - x_0 + \beta\, t) \\ & + \frac{1}{\pi} \int_0^t \mathrm{d}t_1 \int_0^\infty \frac{1}{\sqrt{\lambda^2 - p}} \int_{-\infty}^\infty [-3qu_0^2(\xi, t_1)\; u_1(\xi, t_1) \sin \sqrt{\lambda^2 - p}(t - t_1) \cos \lambda\xi] \mathrm{d}\xi \mathrm{d}\lambda, \\ \bar{u}_1(x,t) = & \frac{1}{\pi} \int_0^t \mathrm{d}t_1 \int_0^\infty \frac{1}{\sqrt{\lambda^2 - p}} \int_{-\infty}^\infty [\sin u_0(\xi, t_1)\; u_1(\xi, t_1) \sin \sqrt{\lambda^2 - p}(t - t_1) \cos \lambda\xi] \mathrm{d}\xi \mathrm{d}\lambda. \end{aligned}$$

Thus the perturbation asymptotic solution $\bar{u}(x,t)$ of Eq.(1) as $f(u) = \varepsilon \sin u$ is

$$\begin{aligned} \bar{u}(x,t) = & \sqrt{\frac{p}{q}} \tanh \sqrt{\frac{p}{2(1-\beta^2)}}(x - x_0 + \beta\, t) \\ & + \frac{1}{\pi} \int_0^t \mathrm{d}t_1 \int_0^\infty \frac{1}{\sqrt{\lambda^2 - p}} \int_{-\infty}^\infty [-3qu_0^2(\xi, t_1)\; u_1(\xi, t_1) \sin \sqrt{\lambda^2 - p}(t - t_1) \cos \lambda\xi] \mathrm{d}\xi \mathrm{d}\lambda \\ & + \frac{\varepsilon}{\pi} \int_0^t \mathrm{d}t_1 \int_0^\infty \frac{1}{\sqrt{\lambda^2 - p}} \int_{-\infty}^\infty [\sin u_0(\xi, t_1)\; u_1(\xi, t_1) \sin \sqrt{\lambda^2 - p}(t - t_1) \cos \lambda\xi] \mathrm{d}\xi \mathrm{d}\lambda \\ & + O(\varepsilon^2), \qquad 0 < \varepsilon \ll 1. \end{aligned} \tag{15}$$

From the above two methods, as $f(u) = \varepsilon \sin u$, we find that the obtained approximate solutions Eqs.(14) and (15) of Eq.(1) have the same accuracy. Therefore, by using the homotopic mapping method we may see that the approximate solution of the solitary wave for the nonlinear evolution equation (1) possesses a good accuracy.

5. Discussion

The solitary wave is a complicated natural phenomenon. Hence we need to reduce the basic models. We solve them by using an approximate method. The homotopic mapping method is a simple and valid method.

The homotopic mapping method is an approximate analytic method, which is different from the general numerical method. The expansions of the solution using the homotopic mapping method can be a continuous analytic operation. Thus, from Eq.(13), we can study the qualitative and quantitative behavior of the solitary wave further.

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