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Asymptotic solution for a perturbed mechanism of western boundary undercurrents in the Pacific*

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This paper consider a class of perturbed mechanism for the western boundary undercurrents in the Pacific. The model of generalized governing equations is studied. Using the perturbation method, it constructs the asymptotic solution of the model. And the accuracy of asymptotic solution is proved by the theory of differential inequalities. Thus the uniformly valid asymptotic expansions of the solution are obtained.

Keywords: undercurrents, perturbation, asymptotic solution
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1. Introduction

The existence of the western boundary undercurrents suggests the baroclinity of the western boundary currents (WBC), so results of study on their baroclinic structure and dynamics should have strong bearing on the baroclinic theory of oceanic circulation. However, because of their late discovery, baroclinic WBCs have not yet become an important part in the baroclinic theory of oceanic circulation.[1] A continuous solution of the baroclinic WBCs matching the interior has not been found.[2] It is noteworthy that few models of the abyssal circulation included the compensator counter to the deep western boundary currents near the bottom.[3] Moreover, the results of the above authors are similarity solutions which cannot satisfy all essential boundary conditions for the physics and thus they are valid when strict restrictions are imposed. So far the barotropic theory of the WBC has been successfully tested, but the baroclinic theory is still far from perfection, as present knowledge on the baroclinic structure of the WBC is still only based on observations.[4]

Recently, many scholars have investigated the relevant nonlinear perturbation problem. They have developed and improved approximate methods including the method of averaging, boundary layer method, matched asymptotic expansion method and multiple scales method.[5−10] Using the differential inequalities and other methods, Mo *et al* also considered a class of singularly perturbed nonlinear boundary value problems for ordinary differential equations,[11] reaction diffusion equations,[12,13] the shock layer solution of nonlinear equations for singularly perturbed problems[14,15] and problems of atmospheric physics.[16−31] In this paper we consider the asymptotic solution for a perturbed mechanism of the western boundary undercurrents in the Pacific by using a simple and special method for perturbation theory of the nonlinear problems.

2. The model of generalized governing equations

A two-dimensional ideal fluid dynamic model is presented on a vertical plane with x and z as the horizontal and vertical coordinates, respectively. Assume that the flow is horizontal and independent in the downstream direction. Let the z-axis point upward with the origin on the undisturbed sea surface and the x-axis be cross-stream. In this paper we consider the following weakly nonlinear perturbed problem of the generalized governing equations:[4]

$$\frac{1}{\rho_0}\frac{\partial p}{\partial x} - fv = \varepsilon\, F(x, z, v, p, \varepsilon), \tag{1}$$

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$$\frac{\partial p}{\partial z} + \rho g = \varepsilon\, G(x, z, v, p, \varepsilon), \tag{2}$$

$$v = \bar{v}_0, \ \text{ and } \ p = 0, \quad \text{at } \ z = \eta(x), \tag{3}$$

where ε is a small positive parameter, p is pressure, v is velocity perpendicular to the bounded region of model plane, ρ is density, ρ_0 is mean value of ρ, f is the Coriolis parameter, g is the acceleration due to gravity, the right-hand sides of Eqs.(1) and (2) are weak perturbation terms, $z = \eta(x)$ is the sea surface height, $\bar{v}_0 = \dfrac{g\rho(x, \eta(x))}{f\rho_0}\dfrac{\mathrm{d}\eta}{\mathrm{d}x}$, and F, G, ρ and η are sufficiently smooth functions with regard to their variables.

3. Construction of the asymptotic solution

The reduced equations of Eqs.(1) and (2) are

$$\frac{1}{\rho_0}\frac{\partial p}{\partial x} - fv = 0, \tag{4}$$

$$\frac{\partial p}{\partial z} + \rho g = 0. \tag{5}$$

It is easy to see that a set of the solution (p_0, v_0) of Eqs.(4), (5) and (3) is

$$p_0(x, z) = \int_z^{\eta(x)} \rho(x, t) g \mathrm{d}t, \tag{6}$$

$$v_0(x, z) = \frac{1}{f\rho_0}\int_z^{\eta(x)} \left[\frac{\partial}{\partial x}\rho(x, t)\right] g\mathrm{d}t + \bar{v}_0. \tag{7}$$

Let the formal expansion of a set of the solution $(p,\ v)$ for the original problem Eqs.(1)–(3) be

$$p \sim \sum_{i=0}^{\infty} p_i \varepsilon^i, \tag{8}$$

$$v \sim \sum_{i=0}^{\infty} v_i \varepsilon^i, \tag{9}$$

where p_0 and v_0 are presented by Eqs.(6), (7) respectively. Substituting Eqs.(8), (9) into Eqs.(1)–(3), developing F and G in ε and equating coefficients of the same powers of ε, for $i = 1$ we obtain

$$\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} - fv_1 = \ F(x, z, v_0, p_0, 0), \tag{10}$$

$$\frac{\partial p_1}{\partial z} + \rho g = \ G(x, z, v_0, p_0, 0), \tag{11}$$

$$v_1 = 0, \ \text{ and } \ p_1 = 0, \quad \text{at } \ z = \eta(x). \tag{12}$$

From Eqs.(10)–(12), we have

$$p_1(x, z) = \int_z^{\eta(x)} [\rho(x, t) g - G(x, t, v_0, p_0, 0)]\mathrm{d}t, \tag{13}$$

$$v_1(x, z) = \frac{1}{f}\left[\int_z^{\eta(x)} \frac{1}{\rho_0}\left[\frac{\partial}{\partial x}[\rho(x, t) g - G(x, t, v_0, p_0, 0)]\mathrm{d}t\right] - F(x, z, v_0, p_0, 0)\right]. \tag{14}$$

Thus we obtain asymptotic expansions of the solution for the model Eqs.(1)–(3):

$$p(x, z) = \int_z^{\eta(x)} [\rho(x, t)(1 + \varepsilon) g - \varepsilon G(x, t, v_0, p_0, 0)]\mathrm{d}t + O(\varepsilon^2), \quad 0 < \varepsilon << 1, \tag{15}$$

$$v(x, z) = \frac{1}{f\rho_0}\int_z^{\eta(x)} \left[\frac{\partial}{\partial x}\rho(x, t)\right](1 + \varepsilon) g\mathrm{d}t + \bar{v}_0 - \frac{\varepsilon}{f}\left[\int_z^{\eta(x)} \frac{1}{\rho_0}\frac{\partial}{\partial x}[G(x, t, v_0, p_0, 0)]\mathrm{d}t\right] + F(x, z, v_0, p_0, 0)] + O(\varepsilon^2), \quad 0 < \varepsilon << 1, \tag{16}$$

where v_0 and p_0 are expressed by Eqs.(6) and (7) respectively.

Using the same method, we can obtain a higher order asymptotic solution of the model Eqs.(1)–(3).

4. Accuracy of asymptotic solution

We can prove that Eqs.(15) and (16) are the uniformly valid asymptotic expansions.

In fact, we first construct the auxiliary functions α_i and β_i,

$$\alpha_i(x, z) = Y_i(x, z) - r_i\varepsilon^2,$$

$$\beta_i(x, z) = Y_i(x, z) + r_i\varepsilon^2, \quad i = 1, 2, \tag{17}$$

where r_i are positive constants, which are large enough, and chosen below, and

$$Y_1(x, z) = \int_z^{\eta(x)} [\rho(x, t)(1 + \varepsilon) g - \varepsilon G(x, t, v_0, p_0, 0)]\mathrm{d}t,$$

$$Y_2(x, z) = \frac{1}{f\rho_0}\int_z^{\eta(x)} \left[\frac{\partial}{\partial x}\rho(x, t)\right](1 + \varepsilon) g\mathrm{d}t + \bar{v}_0 - \frac{\varepsilon}{f}\left[\int_z^{\eta(x)} \frac{1}{\rho_0}\frac{\partial}{\partial x}[G(x, t, v_0, p_0, 0)]\mathrm{d}t] + F(x, z, v_0, p_0, 0)\right] + O(\varepsilon^2).$$

Obviously, we have

$$\alpha_i(x,z) \le \beta_i(x,z), \quad i=1,2. \tag{18}$$

And we have

$$\alpha_1 \big|_{z=\eta(x)} = v_0 - r_1\varepsilon^2, \quad \beta_1 \big|_{z=\eta(x)}\ v_0 + r_1\varepsilon^2.$$

That is

$$\alpha_1 \big|_{z=\eta(x)} \le \beta_1 \big|_{z=\eta(x)}\ . \tag{19}$$

Analogously, we show that

$$\alpha_2 \big|_{z=\eta(x)} \le \beta_2 \big|_{z=\eta(x)}\ . \tag{20}$$

Now we prove that

$$\begin{aligned} &\frac{1}{\rho_0}\frac{\partial \beta_1}{\partial x} - f\beta_2 - \varepsilon\, F(x,z,\beta_1,\alpha_2,\varepsilon) \\ &\le 0 \le \frac{1}{\rho_0}\frac{\partial \alpha_1}{\partial x} - f\alpha_2 - \varepsilon\, F(x,z,\alpha_1,\alpha_2,\varepsilon), \end{aligned} \tag{21}$$

$$\begin{aligned} &\frac{\partial \beta_1}{\partial z} + \rho g - \varepsilon\, G(x,z,\beta_1,\beta_2,\varepsilon) \\ &\le 0 \le \frac{\partial \alpha_1}{\partial z} + \rho g - \varepsilon\, G(x,z,\alpha_1,\alpha_2,\varepsilon). \end{aligned} \tag{22}$$

It follows that for ε small enough there exists a positive constant M such that

$$\begin{aligned} &\frac{1}{\rho_0}\frac{\partial \alpha_1}{\partial x} - f\alpha_2 - \varepsilon\, F(x,z,\alpha_1,\alpha_2,\varepsilon) \\ &= \frac{1}{\rho_0}\frac{\partial Y_2}{\partial x} - f(Y_1 - r_1\varepsilon^2) \\ &\quad -\varepsilon F(x,z,Y_1 - r_1\varepsilon^2, Y_2 - r_2\varepsilon^2,\varepsilon) \\ &\ge \frac{1}{\rho_0}\frac{\partial p_0}{\partial x} - fv_0 + \frac{1}{\rho_0}\frac{\partial p_1}{\partial x} - fv_1 \\ &\quad -F(x,z,v_0,p_0,0) + (f_1 r_1 - M)\varepsilon^2 \\ &= (f_1 r_1 - M)\varepsilon^2 \ge 0, \end{aligned}$$

where $r_1 \ge M/f$. Similarly, we can prove all the inequalities of Eqs.(21) and (22). Thus from Eqs.(18)–(22) and using the theorem of differential inequalities,[32] we obtain a set of solutions $(p(x,z,\varepsilon),\ \ v(x,z,\varepsilon))$, such that

$$\begin{aligned} &\alpha_1(x,z,\varepsilon) \le p(x,z,\varepsilon) \le \beta_1(x,z,\varepsilon), \\ &\alpha_2(x,z,\varepsilon) \le v(x,z,\varepsilon) \le \beta_2(x,z,\varepsilon). \end{aligned}$$

Thus from Eq.(17), we know that Eqs.(15) and (16) are the uniformly valid asymptotic expansions.

5. Conclusions

1) The western boundary undercurrents in the Pacific are a complicated natural phenomenon. Hence we need to reduce basic models and solve them by using an approximate method. The perturbation method is a simple and valid method.

2) The perturbation method is an approximate method, which differs from a general numerical method. The expansions of the solution through the perturbation method can be kept in the analytic operation. Thus, from Eqs.(15) and (16) we can further study the fixed qualitative and quantitative behaviours of the perpendicular velocity and pressure of the western boundary undercurrents in the Pacific. But we do not discuss this any further in this paper.

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