Vol 18 No 9, September 2009
1674-1056/2009/18(09)/3616-08

Chinese Physics B

© 2009 Chin. Phys. Soc.
and IOP Publishing Ltd

The symplectic eigenfunction expansion theorem and its application to the plate bending equation*

Huang Jun-Jie(黄俊杰)[a], Alatancang(阿拉坦仓)[a], and Wang Hua(王 华)[a)b)†]

[a)] *School of Mathematical Sciences, Neimongol University, Hohhot* 010021, *China*

[b)] *Department of Mathematics, College of Sciences, Neimongol University of Technology, Hohhot* 010051, *China*

(Received 7 November 2008; revised manuscript received 24 February 2009)

This paper deals with the bending problem of rectangular plates with two opposite edges simply supported. It is proved that there exists no normed symplectic orthogonal eigenfunction system for the associated infinite-dimensional Hamiltonian operator H and that the two block operators belonging to Hamiltonian operator H possess two normed symplectic orthogonal eigenfunction systems in some space. It is demonstrated by using the properties of the block operators that the above bending problem can be solved by the symplectic eigenfunction expansion theorem, thereby obtaining analytical solutions of rectangular plates with two opposite edges simply supported and the other two edges supported in any manner.

Keywords: plate bending equation, symplectic eigenfunction expansion theorem, infinite dimensional Hamiltonian operator, analytical solution

PACC: 0200, 0340D, 0420J, 4630C

1. Introduction

The Hilbert–Schmidt theorem, also known as the eigenfunction expansion theorem, is the theoretical foundation of traditional separation of variables, and it is very useful in solving self-adjoint problems in mathematical physics. As is well known, the expansion theorem has two requirements: the compactness and self-adjointness of the corresponding operators in Hilbert spaces, which guarantee the completeness of their eigenfunction system and the calculation of coefficients of the eigenfunction expansion. However, most systems encountered in elasticity and relevant fields do not possess self-adjointness. So, traditional separation of variables does not work for these problems.

In 1991, by using the simulation theory between structural mechanics and optimal control and by combining infinite dimensional Hamiltonian systems with elasticity, the separation of variables based on Hamiltonian systems of solving problems in elasticity was introduced.[1] In later publications, a large number of non-self-adjoint problems in applied mechanics have been considered (see, for example, Refs.[2]–[5]). This method extends the traditional separation of variables, and its mathematical foundation, i.e. the symplectic eigenfunction expansion theorem, belongs to the eigenvalue problem of infinite dimensional Hamiltonian operators. In general, Hamiltonian operators are non-self-adjoint, and their eigenfunction expansion problem is far from being understood. In this direction, we mention the cases of the Sturm–Liouville equation and plane magnetoelectroelastic solids (see, for example, Refs.[6] and [7] and references therein). For more details about the symplectic eigenfunction expansion, readers may refer to Refs.[8] and [9].

In the present paper, we propose a symplectic eigenfunction expansion method of the plate bending equation. To be precise, for the bending problem of rectangular plates with two opposite edges simply supported, we consider the existence and completeness of the normed symplectic orthogonal eigenfunction system of the associated infinite-dimensional Hamiltonian operator, and give analytical solutions of the problem.

*Project supported by the National Natural Science Foundation of China (Grant No 10562002), the Natural Science Foundation of Inner Mongolia, China (Grants No 200508010103 and 200711020106), and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No 20070126002).

†Corresponding author. E-mail: hrenly@163.com

2. Preliminaries

This section is devoted to reviewing some basic notions in mathematics. Let X be a Hilbert space unless otherwise stated in particular. Throughout this paper, an operator or operator matrix is always linear (not necessarily bounded).

Definition 2.1.[10] Let $H : \mathcal{D}(H) \subseteq X \times X \to X \times X$,

$$H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} \tag{1}$$

be a closed densely defined operator. If A is a closed densely defined operator, and B and C are self-adjoint operators, then H is called an infinite-dimensional Hamiltonian operator or simply a Hamiltonian operator. Note that for an operator T, T^* denotes its adjoint operator.

Obviously, the infinite dimensional Hamiltonian operator is a generalization of the finite dimensional case (i.e. Hamiltonian matrix). So, we call $\dot{u} = Hu$ an infinite dimensional Hamiltonian system. When H is a differential operator matrix in some energy space, by the Vainberg theorem about the criteria of potential operators,[11] this definition in general coincides with the traditional definition $\dot{u} = J^{-1}\dfrac{\delta \mathcal{H}}{\delta u}$, where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{2}$$

is the structure operator matrix on $X \times X$, and $\mathcal{H}$ is the corresponding Hamiltonian functional.

The domains of the operators A, B and C are denoted by $\mathcal{D}(A)$, $\mathcal{D}(B)$ and $\mathcal{D}(C)$, respectively. Clearly,

$$\mathcal{D}(H) = (\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(B) \cap \mathcal{D}(A^*)) \tag{3}$$

serves as the natural domain of the Hamiltonian operator H.

Definition 2.2. Let X be a Hilbert space of functions. Then, the operator $M : \mathcal{D}(M) \subseteq X \to X$ is said to have a normed symplectic orthogonal eigenfunction system, if there is a sequence of functions $\{U_k \,|\, k = \pm 1, \pm 2, \dots\}$ such that

$$MU_k = \lambda_k U_k, \tag{4}$$

$$\langle U_k, JU_{-j} \rangle_X = c_{kj}\delta_{kj} \tag{5}$$

for each $k, j = \pm 1, \pm 2, \dots$. Here for the Hilbert space G, $\langle \cdot, \cdot \rangle_G$ represents the inner product of G, c_{kj} ($c_{kj} \neq 0$ for $k = j$) is a constant associated with the functions U_k and U_{-j}, and δ_{kj} is the common Kronecker symbol.

3. Plate bending equation and its Hamiltonian properties

3.1. Hamiltonian form

Without losing the generality of the symplectic eigenfunction expansion method, we consider here the case of rectangular plates with two opposite edges simply supported, i.e.

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \omega = f(x, y) \tag{6}$$

with $f(x, y)$ being the lateral load in the region $\{(x, y) \mid 0 \leq x \leq h, 0 \leq y \leq 1\}$, and the boundary conditions for simply supported edges $y = 0$ and $y = 1$ are

$$\omega = 0, \frac{\partial^2 \omega}{\partial y^2} = 0, \quad \text{for } y = 0 \text{ and } y = 1. \tag{7}$$

Introducing the state functions

$$\begin{aligned} \theta &= \frac{\partial \omega}{\partial x}, \\ q &= D\left(\frac{\partial^3 \omega}{\partial x^3} + \frac{\partial^3 \omega}{\partial x \partial y^2}\right), \\ m &= -D\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \end{aligned} \tag{8}$$

(q is a Lagrange parametric function, and m represents the bending moment), we obtain

$$\begin{bmatrix} \dot{\omega} \\ \dot{\theta} \\ \dot{q} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\partial^2/\partial y^2 & 0 & 0 & -1/D \\ 0 & 0 & 0 & \partial^2/\partial y^2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ \theta \\ q \\ m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f \\ 0 \end{bmatrix}, \tag{9}$$

which is an infinite dimensional Hamiltonian system derived from Eq.(6), and the corresponding Hamiltonian operator is

$$H = \begin{bmatrix} 0 & I & 0 & 0 \\ -\mathrm{d}^2/\mathrm{d}y^2 & 0 & 0 & -I/D \\ 0 & 0 & 0 & \mathrm{d}^2/\mathrm{d}y^2 \\ 0 & 0 & -I & 0 \end{bmatrix}. \tag{10}$$

In Eq.(9), we have used the x-coordinate to simulate time coordinate t, to be precise, $\dot{\omega} = \partial\omega/\partial x$, $\dot{\theta} = \partial\theta/\partial x$, $\dot{q} = \partial q/\partial x$ and $\dot{m} = \partial m/\partial x$.

For the Hamiltonian operator (10), a suitable choice of state space which directly influences the solving of the problem is necessary. Here, we introduce the Hilbert space $L^2[0,1]$ and set

$$W = L^2[0,1] \times L^2[0,1] \times L^2[0,1] \times L^2[0,1]. \tag{11}$$

According to the boundary conditions (7), we have

$$\mathcal{D}(H) = \left\{ \begin{bmatrix} \omega \\ \theta \\ q \\ m \end{bmatrix} \in W \;\middle|\; \begin{array}{c} \omega(0) = \omega(1) = 0,\ m(0) = m(1) = 0,\ \omega', m' \text{ are} \\ \text{absolutely continuous},\ \omega'', m'' \in L^2[0,1] \end{array} \right\}. \tag{12}$$

Thus, the plate bending problem (Eq.(6) and boundary condition (7)) can be written in the following Hamiltonian form

$$\dot{u} = Hu. \tag{13}$$

3.2. Eigenvalues and eigenfunction systems

In Ref.[12], for the bending problem of rectangular plates with two opposite edges simply supported, the distribution of eigenvalues of the corresponding Hamiltonian operator is given. Further, we now study the properties of the Hamiltonian operator.

Direct calculations show that the eigenvalues and the associated eigenfunctions of the Hamiltonian operator H are given by

$$\lambda_k = k\pi, \qquad Y_k = \begin{bmatrix} \sin(k\pi y) \\ k\pi \sin(k\pi y) \\ 0 \\ 0 \end{bmatrix}, \tag{14}$$

where $k = \pm 1, \pm 2, \ldots$.

The normed orthogonality of the eigenfunction system plays an especially important role in the Hilbert–Schmidt theorem, and so is the normed symplectic orthogonality of the eigenfunction system in the symplectic eigenfunction expansion theorem. However, unfortunately, the preceding eigenfunction system does not possess this kind of property.

Theorem 3.1. The infinite dimensional Hamiltonian operator H has no normed symplectic orthogonal eigenfunction system.

Proof For the space W defined in expression (11) and the Hamiltonian operator H, the structure operator matrix is clearly

$$J = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{bmatrix} \tag{15}$$

with I being the identity operator on $L^2[0,1]$. Then, for each pair of numbers $(k, -j)$, we always have that

$$\langle Y_k, JY_{-j} \rangle_W = \int_0^1 \begin{bmatrix} \sin(k\pi y) \\ k\pi \sin(k\pi y) \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{bmatrix} \begin{bmatrix} \sin(-j\pi y) \\ -j\pi \sin(-j\pi y) \\ 0 \\ 0 \end{bmatrix} \mathrm{d}y = 0. \tag{16}$$

In particular, set $j = k\,(k = \pm1, \pm2, \ldots)$ in Eq.(16), then the above formula still holds true.

On the other hand, apart from a multiple of a constant, every eigenfunction system of H coincides with $\{Y_k \,|\, k = \pm1, \pm2, \ldots\}$ defined in expression (14). Therefore, by Eq.(5) in Definition 2.2, the claim of the Theorem is proven.

4. Analytical solutions of the plate bending problem

4.1. Theoretical aspects

Many problems in elasticity can be solved by using the symplectic eigenfunction expansion theorem: first, write the problem under consideration in a suitable infinite-dimensional Hamiltonian form Eq.(13); next, separate variables for Eq.(13) and obtain the solutions of separated variable type. As we mentioned, the essential point of this method is the existence and the completeness of the normed symplectic orthogonal eigenfunction system. However, from Theorem 3.1, we see that the Hamiltonian operator H derived from the plate bending problem (Eq.(6) and boundary condition (7)) has no any normed symplectic orthogonal eigenfunction system. This is a pathological case that leads to the failure of the common symplectic eigenfunction expansion method. Fortunately, we can in general remedy various kinds of pathologies through taking full advantage of the inherent structure of the Hamiltonian operator.

Indeed, the infinite dimensional Hamiltonian operator H can be written as the following block operator representation

$$H = \left[\begin{array}{cc|cc} 0 & I & 0 & 0 \\ -\mathrm{d}^2/\mathrm{d}y^2 & 0 & 0 & -I/D \\ \hline 0 & 0 & 0 & \mathrm{d}^2/\mathrm{d}y^2 \\ 0 & 0 & -I & 0 \end{array}\right] \tag{17}$$

with the block operators

$$\left.\begin{array}{r} A = \left[\begin{array}{cc} 0 & I \\ -\mathrm{d}^2/\mathrm{d}y^2 & 0 \end{array}\right], -A^* = \left[\begin{array}{cc} 0 & \mathrm{d}^2/\mathrm{d}y^2 \\ -I & 0 \end{array}\right] \\ B = \left[\begin{array}{cc} 0 & 0 \\ 0 & -I/D \end{array}\right] \\ C = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right] \end{array}\right\}. \tag{18}$$

Then, the remaining important thing is how to take the domain of the operators in expression (18). The selection of the domain is dependent not only on the problem itself but also on whether it can reduce complexities in the treatment. Since the domain of the Hamiltonian operator H defined as expression (17) is actually written as expression (3), the various choices of the domains of the operator matrices in expression (18) are admitted if the domain of the Hamiltonian operator H satisfies expression (12). These choices are equivalent to each other, but their effects could be completely different. Based on the above considerations, we set

$$\mathcal{D}(A) = \left\{ \left[\begin{array}{c} \omega \\ \theta \end{array}\right] \in \widehat{W} \,\middle|\, \begin{array}{l} \omega' \text{ is absolutely continuous,} \\ \omega(0) = \omega(1) = 0, \omega'' \in L^2[0,1] \end{array} \right\}, \tag{19}$$

$$\mathcal{D}(A^*) = \left\{ \left[\begin{array}{c} q \\ m \end{array}\right] \in \widehat{W} \,\middle|\, \begin{array}{l} m' \text{ is absolutely continuous,} \\ m(0) = m(1) = 0, m'' \in L^2[0,1] \end{array} \right\}, \tag{20}$$

where

$$\widehat{W} = L^2[0,1] \times L^2[0,1]. \tag{21}$$

From expressions (18), (19) and (20), we see that A and $-A^*$ are both infinite dimensional Hamiltonian operators. We endeavour to solve the plate bending problem (Eq.(6) and boundary condition (7)) by using Hamiltonian operators A and $-A^*$, which are needed to study their properties.

Theorem 4.1. The infinite dimensional Hamiltonian operator A has a normed symplectic orthogonal eigenfunction system in the symplectic space $\widehat{W}$. In particular, apart from a multiple of a constant, the eigenvalues and the associated eigenfunctions of A are given by

$$\mu_k = k\pi, \quad U_k = \begin{bmatrix} \sin(k\pi y) \\ k\pi \sin(k\pi y) \end{bmatrix}, \quad k \in \Lambda, \tag{22}$$

where $\Lambda = \{ \pm 1, \pm 2, \ldots, \pm n \ldots \}$.

Proof Straightforward calculations show that the sequence $\{\mu_k \,|\, k \in \Lambda\}$ consists of all eigenvalues of the Hamiltonian operator A, and the sequence of functions $\{U_k \,|\, k \in \Lambda\}$ defined in expression (22) is an eigenfunction system.

Set

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{23}$$

with I being the identity operator on $L^2[0,1]$, then it will be the structure operator matrix related to the space $\widehat{W}$ and the Hamiltonian operator A. Thus,

$$\langle U_k, JU_{-j} \rangle_{\widehat{W}} = \int_0^1 \begin{bmatrix} \sin(k\pi y) \\ k\pi \sin(k\pi y) \end{bmatrix}^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \sin(-j\pi y) \\ -j\pi \sin(-j\pi y) \end{bmatrix} \mathrm{d}y = k\pi\delta_{kj} \tag{24}$$

for each $k, j \in \Lambda$. This implies by Definition 2.2 that the eigenfunction system $\{U_k \,|\, k \in \Lambda\}$ possesses normed symplectic orthogonality.

Theorem 4.2. The eigenfunction system $\{U_k \,|\, k \in \Lambda\}$ defined in the expression (22) of the Hamiltonian operator A is complete in the symplectic space $\widehat{W}$, i.e., for each $\begin{bmatrix} f \\ g \end{bmatrix} \in \widehat{W}$, there exist sequences $\{C_k\}_{k=1}^{+\infty}$ and $\{C_{-k}\}_{k=1}^{+\infty}$ such that

$$\begin{bmatrix} f \\ g \end{bmatrix} = \sum_{k=1}^{+\infty} C_k U_k + C_{-k} U_{-k}. \tag{25}$$

Moreover, the above formula is called the symplectic eigenfunction expansion, and, in general, it is also called the symplectic Fourier expansion of $\begin{bmatrix} f \\ g \end{bmatrix}$ in terms of $\{U_k \,|\, k \in \Lambda\}$.

Proof For $k \in \Lambda$, setting

$$C_k = \frac{\displaystyle\int_0^1 U_{-k}^T J \begin{bmatrix} f \\ g \end{bmatrix} \mathrm{d}y}{\displaystyle\int_0^1 U_{-k}^T J U_k \, \mathrm{d}y} = -\frac{1}{k\pi} \int_0^1 (-k\pi f \sin(k\pi y) - g \sin(k\pi y)) \, \mathrm{d}y, \tag{26}$$

and substituting it into expansion (25), we have

$$\sum_{k=1}^{+\infty} C_k U_k + C_{-k} U_{-k} = \begin{bmatrix} \sum_{k=1}^{+\infty} 2(\int_0^1 f \sin(k\pi y) \, \mathrm{d}y) \sin(k\pi y) \\ \sum_{k=1}^{+\infty} 2(\int_0^1 (g \sin(k\pi y) \, \mathrm{d}y) \sin(k\pi y) \end{bmatrix}. \tag{27}$$

Note that the two components of the vector-valued function on the right hand side of Eq.(27) are the corresponding Fourier sine series expansion associated with f and g in $L^2[0,1]$, respectively. Therefore, Eq.(25) is valid.

Similarly, we immediately obtain the following consequence.

Theorem 4.3. The infinite dimensional Hamiltonian operator $-A^*$ has a normed symplectic orthogonal eigenfunction system in the symplectic space $\widehat{W}$. In particular, apart from a multiple of a constant, a complete symplectic orthogonal eigenfunction system of $-A^*$ is given by $\{\widehat{U}_k \,|\, k \in \Lambda\}$, where

$$\widehat{U}_k = \begin{bmatrix} -k\pi \sin(k\pi y) \\ \sin(k\pi y) \end{bmatrix} \tag{28}$$

is the eigenfunction associated with the eigenvalue $\hat{\mu}_k = k\pi$.

4.2. Analytical solution

In Section 3.1, we have written the plate bending problem (Eq.(6) and boundary condition (7)) as the Hamiltonian form (13). Let

$$Z = \begin{bmatrix} \omega \\ \theta \end{bmatrix}, \quad \widehat{Z} = \begin{bmatrix} q \\ m \end{bmatrix},$$

$$F = \begin{bmatrix} 0 \\ -m/D \end{bmatrix}, \quad \text{and} \quad \widehat{F} = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

then, the Hamiltonian form Eq.(13) can be equivalently written as

$$\frac{\partial}{\partial x} Z = AZ + F, \tag{29}$$

$$\frac{\partial}{\partial x} \widehat{Z} = -A^* \widehat{Z} + \widehat{F}, \tag{30}$$

with $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ defined in expressions (19) and (20), respectively.

Note that the completeness of the symplectic orthogonal eigenfunction system $\left\{\widehat{U}_k \,|\, k \in \Lambda\right\}$ of $-A^*$ in Theorem 4.3, then we will apply separation of variables to the problem (30). First, consider the corresponding homogeneous case

$$\frac{\partial}{\partial x} \widehat{Z} = -A^* \widehat{Z}. \tag{31}$$

We look for the solutions of the form

$$\widehat{Z}(x, y) = \widehat{T}(x)\widehat{U}(y). \tag{32}$$

Then, inserting this expression for $\widehat{Z}$ into Eq.(31) yields

$$\frac{\widehat{T}'(x)}{\widehat{T}(x)} = \frac{-A^*\widehat{U}(y)}{\widehat{U}(y)}. \tag{33}$$

Both sides are therefore constant, say equal to $\hat{\mu}$. So, the problem (31) reduces to the following

$$\widehat{T}'(x) = \hat{\mu}\widehat{T}(x), \tag{34}$$

$$-A^*\widehat{U}(y) = \hat{\mu}\widehat{U}(y). \tag{35}$$

It should be noticed that separating variables for the Hamiltonian system (31) will obtain the eigenvalue problem of the Hamiltonian operator $-A^*$. Theorem 4.3 shows that $\hat{\mu} = k\pi\,(k \in \Lambda)$ and $\widehat{U}(y)$ is a multiple of the vector-valued function $\widehat{U}_k$, and hence $\widehat{T}(x)$ is a multiple of $\mathrm{e}^{k\pi x}$. By superposing these solutions, we have the solution of the form

$$\widehat{Z}(x, y) = \sum_{k=1}^{+\infty} \widehat{T}_k(x)\widehat{U}_k(y) + \widehat{T}_{-k}(x)\widehat{U}_{-k}(y) = \sum_{k=1}^{+\infty} \begin{bmatrix} -(c_k \mathrm{e}^{k\pi x} + c_{-k}\mathrm{e}^{-k\pi x}) k\pi \sin(k\pi y) \\ (c_k \mathrm{e}^{k\pi x} - c_{-k}\mathrm{e}^{-k\pi x}) \sin(k\pi y) \end{bmatrix} \tag{36}$$

with $c_{\pm k}$ being unknown constants to be determined.

By the completeness of the eigenfunction system of $-A^*$ in the symplectic space $\widehat{W}$, we then see that the inhomogeneous item $\widehat{F}$ in Eq.(30) has the following symplectic Fourier expansion

$$\widehat{F} = \sum_{k=1}^{+\infty} \widehat{F}_k(x)\widehat{U}_k(y) + \widehat{F}_{-k}(x)\widehat{U}_{-k}(y), \tag{37}$$

where

$$\widehat{F}_k(x) = \frac{\int_0^1 \widehat{U}_{-k}^T J \widehat{F}\, \mathrm{d}y}{\int_0^1 \widehat{U}_{-k}^T J \widehat{U}_k\, \mathrm{d}y} = -\frac{1}{k\pi} \int_0^1 f(x, \xi) \sin(k\pi\xi)\, \mathrm{d}\xi = \widehat{F}_{-k}(x), \quad k \in \Lambda. \tag{38}$$

Thus, for the inhomogeneous problem, we can still find the solution of the form (36). Substituting Eqs.(36) and (37) into problem (30), we have

$$\widehat{T}_k'(x) = \hat{\mu}_k \widehat{T}_k(x) + \widehat{F}_k(x), \quad k \in \Lambda. \tag{39}$$

Straightforward calculations show that

$$\widehat{T}_k(x) = \hat{d}_k \mathrm{e}^{k\pi x} + \int_0^x \widehat{F}_k(\eta) \mathrm{e}^{k\pi(x-\eta)}\, \mathrm{d}\eta, \quad k \in \Lambda, \tag{40}$$

where $\hat{d}_k, k \in \Lambda$ are unknown constants to be determined. Therefore, the solution of the inhomogeneous problem is given by

$$\begin{aligned}\widehat{Z}(x,y) &= \sum_{k=1}^{+\infty} \widehat{T}_k(x)\widehat{U}_k(y) + \widehat{T}_{-k}(x)\widehat{U}_{-k}(y) \\ &= \sum_{k=1}^{+\infty} \begin{bmatrix} -\big(\hat{d}_k \mathrm{e}^{k\pi x} + \hat{d}_{-k}\mathrm{e}^{-k\pi x} + \int_0^x \widehat{F}_k(\eta)(\mathrm{e}^{k\pi(x-\eta)} + \mathrm{e}^{-k\pi(x-\eta)})\,\mathrm{d}\eta\big) k\pi \sin(k\pi y) \\ \big(\hat{d}_k \mathrm{e}^{k\pi x} - \hat{d}_{-k}\mathrm{e}^{-k\pi x} + \int_0^x \widehat{F}_k(\eta)(\mathrm{e}^{k\pi(x-\eta)} - \mathrm{e}^{-k\pi(x-\eta)})\,\mathrm{d}\eta\big) \sin(k\pi y) \end{bmatrix} \\ &= \sum_{k=1}^{+\infty} \begin{bmatrix} -\big(\hat{d}_k \mathrm{e}^{k\pi x} + \hat{d}_{-k}\mathrm{e}^{-k\pi x} - \dfrac{1}{k\pi}\displaystyle\int_0^x \mathrm{d}\eta \int_0^1 (\mathrm{e}^{k\pi(x-\eta)} + \mathrm{e}^{-k\pi(x-\eta)}) f(\eta,\xi)\sin(k\pi\xi)\,\mathrm{d}\xi\big) k\pi \sin(k\pi y) \\ \big(\hat{d}_k \mathrm{e}^{k\pi x} - \hat{d}_{-k}\mathrm{e}^{-k\pi x} - \dfrac{1}{k\pi}\displaystyle\int_0^x \mathrm{d}\eta \int_0^1 (\mathrm{e}^{k\pi(x-\eta)} - \mathrm{e}^{-k\pi(x-\eta)}) f(\eta,\xi)\sin(k\pi\xi)\,\mathrm{d}\xi\big) \sin(k\pi y) \end{bmatrix}. \end{aligned} \tag{41}$$

To solve the problem, we substitute m given by the second component in expression (41) into Eq.(29). Then, we obtain a concrete inhomogeneous Hamiltonian system. Similarly, from Theorems 4.1 and 4.2, we see that the inhomogeneous item F in Eq.(29) has the following symplectic Fourier expansion

$$F = \sum_{k=1}^{+\infty} F_k(x)U_k(y) + F_{-k}(x)U_{-k}(y), \tag{42}$$

where

$$\begin{aligned} F_k(x) &= \frac{\int_0^1 U_{-k}^T JF\,\mathrm{d}y}{\int_0^1 U_{-k}^T JU_k\,\mathrm{d}y} = -\frac{1}{k\pi D}\int_0^1 m(x,y)\sin(k\pi y)\,\mathrm{d}y = -\frac{1}{2k\pi D}\bigg(\hat{d}_k\mathrm{e}^{k\pi x} - \hat{d}_{-k}\mathrm{e}^{-k\pi x} \\ &\quad - \frac{1}{k\pi}\int_0^x \mathrm{d}\eta\int_0^1 (\mathrm{e}^{k\pi(x-\eta)} - \mathrm{e}^{-k\pi(x-\eta)}) f(\eta,\xi)\sin(k\pi\xi)\,\mathrm{d}\xi\bigg) \\ &= F_{-k}(x) \end{aligned} \tag{43}$$

for $k \in \Lambda$; and direct calculations for $k \in \Lambda$ yield

$$T_k(x) = d_k \mathrm{e}^{k\pi x} + \int_0^x F_k(\tau)\mathrm{e}^{k\pi(x-\tau)}\,\mathrm{d}\tau \tag{44}$$

with unknown constants d_k to be determined. Therefore, the analytical solutions of Eq.(29) are given by

$$\begin{aligned} Z(x,y) &= \sum_{k=1}^{+\infty} T_k(x)U_k(y) + T_{-k}(x)U_{-k}(y) \\ &= \sum_{k=1}^{+\infty} \begin{bmatrix} \big(d_k\mathrm{e}^{k\pi x} - d_{-k}\mathrm{e}^{-k\pi x} + \int_0^x F_k(\tau)(\mathrm{e}^{k\pi(x-\tau)} - \mathrm{e}^{-k\pi(x-\tau)})\,\mathrm{d}\tau\big)\sin(k\pi y) \\ \big(d_k\mathrm{e}^{k\pi x} + d_{-k}\mathrm{e}^{-k\pi x} + \int_0^x F_k(\tau)(\mathrm{e}^{k\pi(x-\tau)} + \mathrm{e}^{-k\pi(x-\tau)})\,\mathrm{d}\tau\big) k\pi\sin(k\pi y) \end{bmatrix} \\ &= \sum_{k=1}^{+\infty} \begin{bmatrix} \left(d_k\mathrm{e}^{k\pi x} - d_{-k}\mathrm{e}^{-k\pi x} + \dfrac{\hat{d}_k}{2k\pi D}\left(\dfrac{\sinh(k\pi x)}{k\pi} - \mathrm{e}^{k\pi x}x\right)\right. \\ \left(d_k\mathrm{e}^{k\pi x} + d_{-k}\mathrm{e}^{-k\pi x} - \dfrac{\hat{d}_k}{2k\pi D}\left(\mathrm{e}^{k\pi x}x + \dfrac{\sinh(k\pi x)}{k\pi}\right)\right. \\ \quad + \dfrac{\hat{d}_{-k}}{2k\pi D}\left(\dfrac{\sinh(k\pi x)}{k\pi} - \mathrm{e}^{-k\pi x}x\right) + \dfrac{1}{(k\pi)^2 D}\displaystyle\int_0^x \mathrm{d}\tau\int_0^\tau \mathrm{d}\eta\int_0^1 (\cosh(k\pi(x-\eta)) \\ \quad + \dfrac{\hat{d}_{-k}}{2k\pi D}\left(\mathrm{e}^{-k\pi x}x + \dfrac{\sinh(k\pi x)}{k\pi}\right) + \dfrac{1}{(k\pi)^2 D}\displaystyle\int_0^x \mathrm{d}\tau\int_0^\tau \mathrm{d}\eta\int_0^1 (\sinh(k\pi(x-\eta)) \\ \quad \left. - \cosh(k\pi(x+\eta-2\tau)))f(\eta,\xi)\sin(k\pi\xi)\,\mathrm{d}\xi\right)\sin(k\pi y) \\ \quad \left. - \sinh(k\pi(x+\eta-2\tau)))f(\eta,\xi)\sin(k\pi\xi)\,\mathrm{d}\xi\right) k\pi\sin(k\pi y) \end{bmatrix}. \end{aligned} \tag{45}$$

Remark The preceding $Z(x,y)$ is in fact a general solution of Eq.(29) with $d_{\pm k}$ and $\hat{d}_{\pm k}$ being unknown constants to be determined through the other two given edges, and its first component is the deflection function, i.e.

$$\begin{aligned} w(x,y) = \big(& d_k \mathrm{e}^{k\pi x} - d_{-k}\mathrm{e}^{-k\pi x} + \frac{\hat{d}_k}{2k\pi D}\left(\frac{\sinh(k\pi x)}{k\pi} - \mathrm{e}^{k\pi x}x\right) \\ & + \frac{\hat{d}_{-k}}{2k\pi D}\left(\frac{\sinh(k\pi x)}{k\pi} - \mathrm{e}^{-k\pi x}x\right) + \frac{1}{(k\pi)^2 D}\int_0^x \mathrm{d}\tau \int_0^\tau \mathrm{d}\eta \int_0^1 (\cosh(k\pi(x-\eta)) \\ & - \cosh(k\pi(x+\eta-2\tau)))f(\eta,\xi)\sin(k\pi\xi)\,\mathrm{d}\xi\big)\sin(k\pi y). \end{aligned} \tag{46}$$

Direct calculations show that $w(x,y)$ given in expression (46) is a general solution of the plate bending problem (Eq.(6) and boundary condition (7)). Also, using Levy's trigonometric series method, we have the following general solution

$$\begin{aligned} w(x,y) = \sum_{k=1}^{+\infty} & (a_k \cosh(k\pi x) + b_k \sinh(k\pi x) \\ & + c_k x \cosh(k\pi x) + d_k x \sinh(k\pi x) \\ & + f_k^*(x))\sin(k\pi y), \end{aligned} \tag{47}$$

where the function $f_k^*(x)$ and unknown constants a_k, b_k, c_k, d_k are determined by lateral load $f(x,y)$ and the other two edges. By the independence of $\mathrm{e}^{k\pi x}$ and $\mathrm{e}^{-k\pi x}$, we see that these two solutions can transform into each other. This shows that from the aspect of the symplectic eigenfunction expansion theorem, Levy's method is reasonable.

5. Conclusions

Generally speaking, there are three kinds of methods for solving partial differential equations, i.e., analytical, semi-analytical and numerical methods.[13,14] The analytical method is the basis of the other two kinds of methods. To overcome the restrictions of self-adjointness, extend the Hilbert–Schmidt theorem, and develop the analytical method of solving problems in applied mechanics, numerous authors have studied the separation of variables based on Hamiltonian systems. However, the mathematical foundation, i.e. the feasibility of this kind of method, has not been established. In this paper, for rectangular plates with two opposite edges simply supported, we obtain the following results: the infinite-dimensional Hamiltonian operator (10) associated with the domain (12) is found to have no normed symplectic orthogonal eigenfunction system, let alone completeness of its eigenfunction systems; using the block operators in the Hamiltonian operator H, the appropriate Hamiltonian theoretical aspects are proposed; and then analytical solutions of rectangular plates with two opposite edges simply supported and the other edges supported in any manner are given. We remark that the symplectic eigenfunction expansion method is universal; in particular, it can be applied to other problems in mathematical physics.

References

[1] Zhong W X 1991 *Computational Structural Mechanics and Applications* **8** 229 (in Chinese)

[2] Zhong W X 1995 *A New Systematic Methodology for Theory of Elasticity* (Dalian: Dalian University of Technology Press) (in Chinese)

[3] Xu X S, Wang G P and Sun F M 2008 *Appl. Math. Mech.* **29** 705

[4] Yao W A and Sui Y F 2004 *Appl. Math. Mech.* **25** 178

[5] Yao Z, Zhang H W, Wang J B and Zhong W X 2008 *Chin. J. Solid Mechanics* **29** 13

[6] Huang J J, Alatancang and Chen Aruna 2008 *Acta Mathematicae Applicatae Sinica* **31** 457 (in Chinese)

[7] Hou G L and Alatancang 2008 *Chin. Phys.* B **17** 2753

[8] Zhang H Q, Alatancang and Zhong W X 1997 *Appl. Math. Mech.* **18** 237

[9] Zhang H Q and Alatancang 1995 *J. Dalian University of Technology* **35** 754 (in Chinese)

[10] Huang J J, Alatancang and Fan X Y 2008 *Sci. China Ser. A-Mathematics* **38** 71 (in Chinese)

[11] Vainberg M M 1964 *Variational Methods for the Study of Nonlinear Operators* (San Francisco: Holden-Day)

[12] Huang J J and Alatancang 2008 *Commun. Theor. Phys.* **49** 562

[13] Chen L and Cheng Y M 2008 *Acta Phys. Sin.* **57** 1 (in Chinese)

[14] Lin W T and Mo J Q 2008 *Acta Phys. Sin.* **57** 1291 (in Chinese)