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Approximate direct reduction method: infinite series reductions to the perturbed mKdV equation*

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The approximate direct reduction method is applied to the perturbed mKdV equation with weak fourth order dispersion and weak dissipation. The similarity reduction solutions of different orders conform to formal coherence, accounting for infinite series reduction solutions to the original equation and general formulas of similarity reduction equations. Painlevé II type equations, hyperbolic secant and Jacobi elliptic function solutions are obtained for zero-order similarity reduction equations. Higher order similarity reduction equations are linear variable coefficient ordinary differential equations.

Keywords: perturbed mKdV equation, approximate direct reduction method, series reduction solutions

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1. Introduction

As one of the most significant tools in nonlinear science, perturbation theory is extensively used to solve many problems with disturbance quantities, which fail many analytical methods in achieving prospective results. Perturbation techniques facilitate the investigation of these problems by decomposing them into some subproblems that can be solved analytically.

Lie group theory provides effective analytical tools such as the classical method[1−3] and nonclassical method[4] for similarity reduction of partial differential equations. In Ref.[5], the classical method was generalized to get exact solutions of the nonlinear Schrödinger equation, implying the possibility of making new optical fibre amplifiers. Compared with the classical method, the direct method proposed by Clarkson and Kruskal[6] is capable of generating more similarity reductions for many partial differential equations. The remarkable characteristic of the direct method is that it involves no group theory. The direct method was extended to find the similarity reduction of partial differential equations.[7,8] A comparison between the nonclassical method and the direct method can be found in Refs.[9]–[11].

In Ref.[12], the approximate direct method was proposed and applied to the singularly perturbed Boussinesq equation, generating more approximate similarity reduction than the approximate symmetry method which was developed by Fushchich and Shtelen.[13] The approximate direct method is virtually the successive use of perturbation techniques and the direct method, i.e., reducing the original equation to an infinite number of equations which can be manipulated by the direct method.

The celebrated modifed Korteweg-de Vries (mKdV) equation appears in many branches of nonlinear science. Due to the existence of perturbation, its specific forms generally vary from the standard form. One form of perturbation is

$$u_t + 6u^2u_x + u_{xxx} = \epsilon(u_{xx} + u_{xxxx}), \qquad (1)$$

which arises in a number of places, such as models of shallow water on tilted planes.[14] The soliton perturbation property of the mKdV equation has been analysed.[15−17]

In this paper, we consider two special forms of the

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above equation with weak fourth order dispersion

$$u_t + 6u^2u_x + u_{xxx} = \epsilon u_{xxxx},\ \epsilon \ll 1 \tag{2}$$

and weak dissipation

$$u_t + 6u^2u_x + u_{xxx} = \epsilon u_{xx},\ \epsilon \ll 1, \tag{3}$$

which are separately solved by the approximate direct reduction method in the subsequent two sections. Section 4 is the conclusion and discussion.

2. Approximate direct method to Eq.(2)

For Eq.(2), according to perturbation theory, the solution can be expressed as a series containing a small parameter

$$u = \sum_{j=0}^{\infty} \epsilon^j u_j \tag{4}$$

with u_j functions of x and t. Substituting Eq.(4) into Eq.(2) and vanishing the coefficients of all different powers of ϵ, we obtain a system of partial differential equations

$$u_{jt} + 6\sum_{k=0}^{j}\sum_{l=0}^{k} u_l u_{k-l} u_{j-k,x} + u_{jxxx} - u_{j-1,xxxx} = 0. \tag{5}$$

The similarity solutions for the above equations are of the form

$$u_j = U_j(x, t, P_j(z(x,t))),\ (j = 0, 1, \ldots), \tag{6}$$

where U_j, P_j and z are functions with respect to the indicated variables and all $P_j(z)$ satisfy a system of ordinary differential equations, which can be obtained by substituting Eq.(6) into Eq.(5). After the substitution, it is easily seen that the coefficients for $P_{j,zzz}$ and $P_{j,zz}P_{j,z}$ are $U_{j,P_j}z_x^3$ and $3U_{j,P_jP_j}z_x^3$ respectively. We reserve uppercase Greek letters for undetermined functions of z hereafter. The ratios of the coefficients are functions of z, namely,

$$3U_{j,P_jP_j}z_x^3 = U_{j,P_j}z_x^3\Gamma_j(z),\ (j = 0, 1, \ldots) \tag{7}$$

with the solution $U_j = F_j(x,t) + G_j(x,t)\mathrm{e}^{\frac{1}{3}\Gamma_j(z)P_j}$. Hence, it is also sufficient to seek similarity reduction of Eq.(5) in the special form

$$u_j = \alpha_j(x,t) + \beta_j(x,t)P_j(z(x,t)),\ (j = 0, 1, \ldots), \tag{8}$$

instead of the most general form Eq.(6).

Remark Three freedoms in the determination of $\alpha_j(x,t)$, $\beta_j(x,t)$ and $z(x,t)$ should be noted:

(i) if $\alpha_j(x,t)$ has the form $\alpha_j(x,t) = \alpha_j'(x,t) + \beta_j\Omega(z)$, then one can take $\Omega(z) = 0$;

(ii) if $\beta_j(x,t)$ has the form $\beta_j(x,t) = \beta_j'(x,t)\Omega(z)$, then one can take $\Omega(z) =$ constant;

(iii) if $z(x,t)$ is determined by $\Omega(z) = z_0(x,t)$, where $\Omega(z)$ is any invertible function, then one can take $\Omega(z) = z$.

Setting $j = 0$ and substituting Eq.(8) into Eq.(5), we can see that the coefficients for P_{0zzz}, $P_{0z}P_0^2$, $P_{0z}P_0$ and P_{0zz} are $\beta_0 z_x^3$, $6\beta_0^3 z_x$, $12\alpha_0\beta_0^2 z_x$ and $3\beta_{0x}z_x^2 + 3\beta_0 z_x z_{xx}$, respectively. To make sure Eq.(5) ($j = 0$) is an ordinary differential equation, we require that

$$6\beta_0^3 z_x = \beta_0 z_x^3 \Psi_0(z), \tag{9}$$

$$12\alpha_0\beta_0^2 z_x = \beta_0 z_x^3 \Phi_0(z), \tag{10}$$

$$3\beta_{0x}z_x^2 + 3\beta_0 z_x z_{xx} = \beta_0 z_x^3 \Omega_0(z), \tag{11}$$

and thus, from Eq.(9) and **Remark** (ii), we have

$$\beta_0 = z_x. \tag{12}$$

From Eqs.(10), (12) and **Remark** (i), we have

$$\alpha_0 = 0. \tag{13}$$

From Eqs.(11), (12) and **Remark** (iii), we have

$$z = \theta(t)x + \sigma(t). \tag{14}$$

Equation (5) ($j = 0$) is then simplified to

$$\theta^4 P_{0zzz} + 6\theta^4 P_0^2 P_{0z} + (\theta_t z - \theta_t\sigma + \theta\sigma_t)P_{0z} + \theta_t P_0 = 0. \tag{15}$$

From the coefficients of P_{0zzz}, zP_{0z} and P_{0z} for the above equation which needs to be an ordinary differential equation, it is easily seen that

$$\theta_t = A\theta^4,\ -\theta_t\sigma + \theta\sigma_t = B\theta^4 \tag{16}$$

with A and B being arbitrary constants.

Case 1 When $A \neq 0$, Eq.(16) has the solution

$$\theta = -(3A(t - t_0))^{-\frac{1}{3}},\ \sigma = -\frac{B}{A} + s_0(t - t_0)^{-\frac{1}{3}}, \tag{17}$$

where s_0 and t_0 are arbitrary constants.

Inserting Eq.(8) into the general form (5), we obtain the coefficients of $P_{j,zzz}$, $P_{j-1,zzzz}$ and $P_{0z}P_0$, which are $\beta_j z_x^3$, $-\beta_{j-1}z_x^4$ and $12\alpha_j\beta_0^2 z_x$, respectively. These coefficients are subjected to

$$\beta_{j-1}z_x^4 = -\beta_j z_x^3 \Psi_j(z),\ (j = 1, 2, \ldots), \tag{18}$$

$$12\alpha_j\beta_0^2 z_x = \beta_j z_x^3 \Phi_j(z),\ (j = 1, 2, \ldots). \tag{19}$$

From Eqs.(12), (18) and **Remark** (ii), we obtain

$$\beta_j = z_x^{j+1},\ (j = 0, 1, \ldots). \tag{20}$$

From Eqs.(12), (19), (20) and **Remark** (i), we obtain

$$\alpha_j = 0,\ (j = 0, 1, \ldots). \tag{21}$$

Equations (4), (8), (12)–(14), (17), (20) and (21) determine the perturbation series solution to Eq.(2)

$$u = \sum_{j=0}^{\infty} (-1)^{j+1} (3A(t-t_0))^{-\frac{1}{3}(j+1)} \epsilon^j P_j(z) \tag{22}$$

with the similarity variable $z = -x(3A(t-t_0))^{-\frac{1}{3}} + s_0(t-t_0)^{-\frac{1}{3}} - \dfrac{B}{A}$. The related similarity reduction equations are

$$\begin{aligned} &P_{j,zzz} + (j+1)AP_j + (Az+B)P_{j,z} \\ &+6\sum_{k=0}^{j}\sum_{l=0}^{k} P_l P_{k-l} P_{j-k,z} - P_{j-1,zzzz} \\ &= 0,\ (j = 0, 1, \ldots) \end{aligned} \tag{23}$$

with $P_{-1} = 0$. When $j = 0$, Eq.(23) degenerates to a Painlevé II type equation. Equation (23) is in fact a third order linear variable coefficient ordinary differential equation of P_j with the assumption that P_k for $k < j$ are determined in advance, since we can rewrite Eq.(23) as

$$\begin{aligned} &P_{j,zzz} + [(j+1)A + 12(1-\delta_{j,0})P_0 P_{0,z}]P_j \\ &+(Az + B + 6P_0^2)P_{j,z} = f_j,\ (j = 0, 1, \ldots), \end{aligned} \tag{24}$$

where

$$\begin{aligned} f_j \ =\ & P_{j-1,zzzz} - 6P_0 \sum_{k=1}^{j-1} P_k P_{j-k,z} \\ & -6\sum_{l=1}^{j-1}\sum_{k=l}^{j} P_l P_{k-l} P_{j-k,z}, \end{aligned} \tag{25}$$

and the notation $\delta_{i,j}$ satisfying $\delta_{0,0} = 1$, $\delta_{k,0} = 0,\ (k \neq 0)$ is adopted in the following text.

Case 2 When $A = 0$, Eq.(16) has the solution

$$\theta = t_0,\ \sigma = Bt_0^3 t + s_0, \tag{26}$$

where s_0 and t_0 are arbitrary constants. Equations (14) and (26) imply an equivalent travelling wave form $z = x + ct$ with c an arbitrary constant, so that we obtain the perturbation series solution to Eq.(2)

$$u = \sum_{j=0}^{\infty} \epsilon^j P_j(z), \tag{27}$$

where all $P_j(z)$ satisfy

$$\begin{aligned} &cP_j + 2\sum_{k=0}^{j}\sum_{l=0}^{k} P_{j-k}P_{k-l}P_l + P_{j,zz} \\ &-P_{j-1,zzz} + a_j = 0,\ (j = 0, 1, \ldots), \end{aligned} \tag{28}$$

where $P_{-1} = 0$, and a_j are arbitrary integral constants. Taking $j = 0$ and $a_0 = 0$, we find that Eq.(28) has the hyperbolic secant function solution

$$P_0 = \sqrt{-c}\ \mathrm{sech}(\sqrt{-c}z + C_0),\ (c < 0), \tag{29}$$

and the Jacobi elliptic function solution

$$\begin{aligned} &P_0 \\ &= \frac{1}{2}\sqrt{2(C_0^2 - c)}\ \mathrm{cn}\left[C_1 + C_0 z \frac{1}{2C_0}\sqrt{2(C_0^2 - c)}\right], \end{aligned} \tag{30}$$

where C_0 and C_1 are arbitrary constants.

An equivalent form of Eq.(28) is

$$cP_j + P_{j,zz} + 2(3 - 2\delta_{j,0})P_0^2 P_j = g_j,\ (j = 0, 1, \ldots), \tag{31}$$

where

$$\begin{aligned} g_j \ =\ & P_{j-1,zzz} - 2\sum_{k=1}^{j-1}\sum_{l=0}^{k} P_{j-k}P_{k-l}P_l \\ & -2P_0\sum_{l=1}^{j-1} P_{j-l}P_l - a_j,\ (j = 0, 1, \ldots). \end{aligned} \tag{32}$$

Equation (31) is obviously a second order linear variable coefficient ordinary differential equation of P_j with the solution

$$P_j = P_{0,z}\left[c_j + \int P_{0,z}^{-2}\left(b_j + \int g_j P_{0,z}\mathrm{d}z\right)\mathrm{d}z\right], \tag{33}$$

where b_j and c_j are arbitrary integral constants.

3. Approximate direct method to Eq.(3)

For Eq.(3), in a similar manner, substituting Eq.(4) and vanishing the coefficients of all different powers of ϵ, we have

$$\begin{aligned} &u_{jt} + 6\sum_{k=0}^{j}\sum_{l=0}^{k} u_l u_{k-l} u_{j-k,x} + u_{jxxx} \\ &-u_{j-1,xx} = 0,\ (j = 0, 1, \ldots) \end{aligned} \tag{34}$$

with $u_{-1} = 0$.

As in Section 2, we insert Eq.(6) into Eq.(34). The coefficients for $P_{j,zzz}$ and $P_{j,zz}P_{j,z}$ are $U_{j,P_j} z_x^3$

and $3U_{j,P_jP_j}z_x^3$ respectively. For Eq.(34) to be an ordinary differential equation after the similarity reduction, we need

$$3U_{j,P_jP_j}z_x^3 = U_{j,P_j}z_x^3\Gamma_j(z),\ (j=0,1,\ldots) \tag{35}$$

with the solution $U_j = F_j(x,t) + G_j(x,t)\mathrm{e}^{\frac{1}{3}\Gamma_j(z)P_j}$. Hence, Eq.(8) is also available for similarity reduction of Eq.(34) rather than the general form Eq.(6). Since Eq.(34) is exactly the same as Eq.(5) in the case of $j=0$, Eqs.(12)–(16) can also be obtained after the substitution of Eq.(8) into Eq.(34) for $j=0$.

Case 1 When $A\neq 0$, Eq.(16) has the solution (17). For $j>0$, after substitution of Eq.(8) into Eq.(34), the coefficients of $P_{j,zzz}$, $P_{j-1,zz}$ and $P_{0z}P_0$ are $\beta_j z_x^3$, $-\beta_{j-1}z_x^2$ and $12\beta_0^2\alpha_j z_x$, respectively. Therefore,

$$-\beta_{j-1}z_x^2 = \beta_j z_x^3\Psi_j(z),\ (j=1,2,\ldots) \tag{36}$$

$$12\beta_0^2\alpha_j z_x = \beta_j z_x^3\Phi_j(z),\ (j=1,2,\ldots) \tag{37}$$

From Eqs.(12), (36) and **Remark** (ii), we obtain

$$\beta_j = z_x^{1-j},\ (j=0,1,\ldots). \tag{38}$$

From Eqs.(37), (38) and **Remark** (i), we obtain

$$\alpha_j = 0,\ (j=0,1,\ldots). \tag{39}$$

Accordingly, Eqs.(4), (8), (12)–(14), (17), (38) and (39) present the perturbation series solution to Eq.(3)

$$u = \sum_{j=0}^{\infty}(-1)^{j-1}(3A(t-t_0))^{\frac{1}{3}(j-1)}\epsilon^j P_j(z) \tag{40}$$

with the similarity variable $z = -x(3A(t-t_0))^{-\frac{1}{3}} + s_0(t-t_0)^{-\frac{1}{3}} - \dfrac{B}{A}$. The similarity reduction equations are

$$\begin{aligned} &P_{j,zzz} - (j-1)AP_j + (Az+B)P_{j,z} \\ &+6\sum_{k=0}^{j}\sum_{l=0}^{k}P_lP_{k-l}P_{j-k,z} \\ &-P_{j-1,zz} = 0,\ (j=0,1,\ldots) \end{aligned} \tag{41}$$

with $P_{-1}=0$. When $j=0$, Eq.(41) is equivalent to a Painlevé II type equation. We can rearrange the terms in Eq.(41) as

$$\begin{aligned} &P_{j,zzz} + [(1-j)A + 12(1-\delta_{j,0})P_0P_{0,z}]P_j \\ &+(Az+B+6P_0^2)P_{j,z} = P_{j-1,zz} \\ &-6P_0\sum_{k=1}^{j-1}P_kP_{j-k,z} \\ &-6\sum_{l=1}^{j-1}\sum_{k=l}^{j}P_lP_{k-l}P_{j-k,z},\ (j=0,1,\ldots), \end{aligned} \tag{42}$$

which is a third order linear variable coefficient ordinary differential equation of P_j.

Case 2 When $A=0$, we can also obtain Eq.(26) from Eq.(16) in Section 2, which makes Eq.(14) a travelling wave form. Thus, we obtain the perturbation series solution (27) to Eq.(3) and all $P_j(z)$ satisfy

$$\begin{aligned} &cP_j + 2\sum_{k=0}^{j}\sum_{l=0}^{k}P_{j-k}P_{k-l}P_l + P_{j,zz} \\ &-P_{j-1,z} + a_j' = 0,\ (j=0,1,\ldots), \end{aligned} \tag{43}$$

where $P_{-1}=0$, and a_j' are arbitrary integral constants. We can also see that this equation has the solution (29) and (30) for $j=0$.

Equation (43) is equivalent to

$$cP_j + P_{j,zz} + 2(3-2\delta_{j,0})P_0^2P_j = h_j,\ (j=0,1,\ldots), \tag{44}$$

where

$$\begin{aligned} h_j \;=\; &P_{j-1,z} - 2\sum_{k=1}^{j-1}\sum_{l=0}^{k}P_{j-k}P_{k-l}P_l \\ &-2P_0\sum_{l=1}^{j-1}P_{j-l}P_l - a_j',\ (j=0,1,\ldots). \end{aligned} \tag{45}$$

Equation (44) is easily seen to be a second order linear variable coefficient ordinary differential equation of P_j with the solution

$$P_j = P_{0,z}\left[c_j' + \int P_{0,z}^{-2}\left(b_j' + \int h_jP_{0,z}\mathrm{d}z\right)\mathrm{d}z\right], \tag{46}$$

where b_j' and c_j' are arbitrary integral constants.

Remark The convergence of infinite series solution (22) is superior to that of Eq.(40), because the general terms of Eq.(22) become infinitesimal for sufficiently large time t, while the general terms of Eq.(40) may be infinite when the time t is large enough.

4. Conclusion and discussion

In summary, the approximate similarity reduction of the perturbed mKdV equation was studied in terms of the approximate direct method. We summarized the general formulas for the similarity reduction solutions and similarity reduction equations of different orders. Series reduction solutions were derived.

Zero-order similarity reduction equations are equivalent to Painlevé II type equations, or have hyperbolic secant and Jacobi elliptic function solutions

for both weak fourth order dispersion and weak dissipation. j-order similarity reduction equations are linear variable coefficient ordinary differential equations of P_j which depend on particular solutions of the previous similarity reduction equations from zero-order to $(j-1)$-order.

The approximate direct reduction method is applicable to other perturbed nonlinear partial differential equations with small parameters and it is thought-provoking to explore a general principle for perturbed nonlinear partial differential equations with similar results.

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