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Pinning-controlled synchronization of complex networks with bounded or unbounded synchronized regions*

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This paper studies pinning-controlled synchronization of complex networks with bounded or unbounded synchronized regions. To study a state-feedback pinning-controlled network with N nodes, it first converts the controlled network to an extended network of $N+1$ nodes without controls. It is shown that the controlled synchronizability of the given network is determined by the real part of the smallest nonzero eigenvalue of the coupling matrix of its extended network when the synchronized region is unbounded; but it is determined by the ratio of the real parts of the largest and the smallest nonzero eigenvalues of the coupling matrix when the synchronized region is bounded. Both theoretical analysis and numerical simulation show that the portion of controlled nodes has no critical values when the synchronized region is unbounded, but it has a critical value when the synchronized region is bounded. In the former case, therefore, it is possible to control the network to achieve synchronization by pinning only one node. In the latter case, the network can achieve controlled synchronization only when the portion of controlled nodes is larger than the critical value.

Keywords: synchronization, pinning control, complex network, synchronized region
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1. Introduction

Both control and synchronization of complex networks are important subjects in the study of complex dynamics on networks. Since Pecora and Carroll[1] put forward the method of master stability functions (MSF), with which a network structure and its node dynamics can be taken apart, synchronization analysis of the network is consequently simplified，study of the effects of network structures on their synchronizability has become overwhelming.[2−11] In the earlier stages, attention was focused on symmetrical networks. Some studies (e.g. Refs.[4]–[6]) adopt an eigenratio criterion for the synchronizability and some others (e.g. Refs.[7]–[9]) use the smallest nonzero eigenvalue of the network coupling matrix instead. Lately, it was shown in Ref.[12] that the synchronized regions of networks can be classified into two cases: bounded and unbounded. When the synchronized region is unbounded, the network synchronizability is determined by the smallest nonzero eigenvalue; when the synchronized region is bounded, the network synchronizability is determined by the eigenratio. Moreover, some methods for improving the synchronizability of complex networks have been put forward (e.g. Refs.[13]–[17]). In particular, it was shown in Ref.[17] that the MSF method can also be applied to networks with a non-diagonalizable coupling matrix if such a network embeds at least one oriented spanning tree. This implies that the MSF method can also be extended to analyze the synchronizability of some asymmetrical networks.

Pinning control, on the other hand, is an effective control method applied to a small portion of nodes of a large-scale network, with which all the nodes of the network can be driven to a desired state such as a chaotic state or simply an equilibrium. When pinning control is realized, the entire network achieves controlled synchronization. It has been studied how to control a scale-free network to an equilibrium (e.g. in Ref.[18]) or to a general desired state (e.g., in Ref.[19]) by pinning. It is shown in Ref.[19] that the pinning control problem for a given network with N nodes can be converted to the synchronization problem of its extended network with $N+1$ nodes without con-

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trols, so that the MSF method can be directly applied to analyze the uncontrolled synchronizability of the extended network, which implies the equivalent controlled synchronizability of the original network.

This paper studies the interesting and challenging problem of pinning control of complex networks in more detail. Firstly, the analysis of pinning control of a network with N nodes is converted to the analysis of synchronization of its extended network with $N+1$ nodes. Then, the extended network is investigated, confirming that the real part of the smallest nonzero eigenvalue of the coupling matrix of the extended network can determine the controlled synchronizability of the given network if its synchronized region is unbounded. Also, the investigation reveals that the ratio of the real parts of the largest and the smallest nonzero eigenvalues of the coupling matrix of the extended network determines the controlled synchronizability of the original network if the synchronized region is bounded. Both theoretical analysis and numerical simulation show that, for networks with unbounded synchronized regions, the number of the controlled nodes has no critical values, therefore it is possible to control the network to achieve synchronization by pinning only one node, this supports the result of Ref.[20], and the controlled synchronizability can be improved monotonously with the increase of the control gain or the coupling strength; for networks with bounded synchronized regions, however, the portion of the controlled nodes has a critical value. For the latter case, investigation further shows that pinning-controlled synchronization can be achieved with an appropriately choosing control gain and a correspondingly determined coupling strength, when the portion of controlled nodes is larger than the critical value, but too large or too small a control gain or a coupling strength will desynchronize the controlled network.

2. The pinning-controlled network model

Consider the following network model:

$$\dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{x}_i) - \sigma \sum_{j=1}^{N} G_{ij} \boldsymbol{H}(\boldsymbol{x_j}),\ i = 1, \ldots, N, \quad (1)$$

where $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ denotes the dynamics of each individual node, $\boldsymbol{H}(\boldsymbol{x})$ is a vector-valued inner-coupling function, σ is the coupling strength, $\boldsymbol{G} = [G_{ij}]$ is a diffusive coupling matrix with $G_{ii} = -\sum_{j \neq i} G_{ij}$ (i.e., with a zero row-sum), which has information about the topology and weights of the network. In this setting, the coupling matrix $G = [G_{ij}]$ has one zero eigenvalue of multiplicity one and all the others are strictly positive.

With pinning controllers, the controlled network is described by

$$\dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{x}_i) - \sigma \sum_{j=1}^{N} G_{ij} \boldsymbol{H}(\boldsymbol{x}_j) + \sigma \delta_i u_i,$$
$$i = 1, 2, \ldots, N, \quad (2)$$

where the last item represents the pinning controllers taken from the index set $C = \{i_1, i_2, \ldots, i_n\}$ (e.g., $C = \{1, 2, \ldots, n\}$), where $n = \lfloor pN \rfloor$ for a small positive parameter $p \ll 1$ and $\lfloor a \rfloor$ is the integer part of real number a: if $i \in C$, then $\delta_i = 1$; otherwise, $\delta_i = 0$. Suppose that $\boldsymbol{s}(t)$ is a target state satisfying $\dot{\boldsymbol{s}}(t) = \boldsymbol{F}(\boldsymbol{s}(t))$, such as an equilibrium, and this state information is available for feedback. Then, by applying a simple linear state-feedback controller to some controlled node, that is, applying controller $u_i = k_i(\boldsymbol{H}(s) - \boldsymbol{H}(x_i))$ at the control node i, where k_i is the control gain, the controlled network becomes

$$\dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{x}_i) - \sigma \sum_{j=1}^{N} G_{ij} \boldsymbol{H}(\boldsymbol{x}_j) + \sigma \delta_i k_i (\boldsymbol{H}(\boldsymbol{s}) - \boldsymbol{H}(\boldsymbol{x}_i)),\quad i = 1, \ldots, N. \quad (3)$$

The objective of controlling this network is to achieve complete synchronization $||x_i(t) - x_j(t)|| \to 0$ as $t \to \infty$. Due to the diffusive coupling, after synchronization is achieved (even at a finite time), one has $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \ldots = \boldsymbol{x}_N(t) = \boldsymbol{s}(t)$ thereafter.

Notice that with the addition of controllers, the MSF method cannot be directly applied to analyze the synchronization of the controlled network Eq.(3). Nevertheless, according to the study of Ref.[19], such a synchronization problem of a pinning-controlled network with N nodes can be converted to an equivalent synchronization problem of its extended network of $N+1$ nodes without controls. More precisely, suppose that the network has an additional virtual node satisfying dynamics $\dot{\boldsymbol{s}}(t) = \boldsymbol{F}(\boldsymbol{s}(t))$, then, establish a connection from the virtual node to controlled node i, with k_i being viewed as the weight of the connection from the virtual node to node i, without connecting the virtual node to any uncontrolled nodes. In so doing, equation (3) is transformed to the following form:

$$\dot{\boldsymbol{y}}_i = \boldsymbol{F}(\boldsymbol{y}_i) - \sigma \sum_{j=1}^{N+1} \boldsymbol{M}_{ij}\boldsymbol{H}(\boldsymbol{y}_j) \quad i = 1, 2, \ldots, N+1, \tag{4}$$

where $\boldsymbol{y}_i = \boldsymbol{x}_i$, for $i = 1, 2, \ldots N$, $\boldsymbol{y}_{N+1} = \boldsymbol{s}$, and $\boldsymbol{M} = \{M_{ij}\}$ is an $(N+1)$-dimensional matrix defined by

$$\boldsymbol{M} = \begin{bmatrix} G_{11} + \delta_1 k_1 & G_{12} & \ldots & G_{1N} & -\delta_1 k_1 \\ G_{21} & G_{22} + \delta_2 k_2 & \ldots & G_{2N} & -\delta_2 k_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ G_{N1} & G_{N2} & \ldots & G_{NN} + \delta_N k_N & -\delta_N k_N \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{5}$$

in which $\delta_i = 1$, if node i is being controlled, but $\delta_i = 0$ otherwise. Consequently, pinning control analysis of system (3) is converted to synchronization analysis of system (4) with a pre-specified synchronous state $\boldsymbol{s}(t)$, so that the MSF method can be directly applied to the latter, which can finally be converted back to the former to complete the original task of synchronization analysis.

3. Two criteria for network synchronizability

To study the effects of the control gains and the portion of controlled nodes on the synchronizability of the extended network, criteria for determining the synchronizability have to be chosen. Today, there are two typical criteria for describing the synchronizability of a symmetric network with real eigen-spectra, which are (i) the smallest nonzero eigenvalue λ_2 of the coupling matrix and (ii) the eigenratio λ_N/λ_2 of the largest eigenvalue and the smallest nonzero eigenvalue of the coupling matrix. For an asymmetrical weighted network with a complex eigen-spectrum, the smallest nonzero eigenvalue and the eigenratio of the largest eigenvalue and the smallest nonzero eigenvalue are changed to be the real part of the smallest nonzero eigenvalue and the eigenratio of the real parts of the largest and the smallest nonzero eigenvalues, respectively. So, without loss of generality, the real parts of the eigenvalues of the coupling matrices are considered for the two criteria of network synchronizability throughout this paper.

Technically, when should the synchronizability be determined by criterion (i) and when by criterion (ii)? For the most general cases, there are no theoretical results on this issue available today, except for some numerical investigations[21] and some relevant analytic studies.[22−26] The special case with $\boldsymbol{H}$ being an identity matrix is relatively easy to analyze, therefore it is studied in this paper in more detail. For simplicity, the following analysis starts from an uncontrolled network with N nodes, which can be easily extended to controlled networks and then be converted back to uncontrolled ones, as discussed above.

Let the synchronous solution of Eq.(1) be $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \ldots = \boldsymbol{x}_N(t) = \bar{\boldsymbol{x}}$, and then linearize Eq.(1) near the synchronous solution, with $\boldsymbol{\xi}_k = \boldsymbol{x}_k - \bar{\boldsymbol{x}}$ denoting the k-th error state, $k = 1, 2, \ldots, N$. Following the derivation shown in Ref.[1], when the coupling matrix $\boldsymbol{G}$ is diagonalizable, one obtains the error dynamical system

$$\dot{\boldsymbol{\xi}}_k = [D\boldsymbol{F} - \sigma\lambda_k D\boldsymbol{H}]\xi_k, \quad k = 1, 2, \ldots N, \tag{6}$$

where λ_k are the eigenvalues of the coupling matrix $\boldsymbol{G}$. Since $\boldsymbol{G}$ is a zero-row-sum matrix, one has $\lambda_1 = 0$, as mentioned above, which corresponds to the synchronous mode, and the other λ_k correspond to the transverse modes, $k = 2, 3, \ldots, N$. Since $D\boldsymbol{F}$ and $D\boldsymbol{H}$ are same for all k in Eq.(6), the following so-called master stability equation is key to consider:

$$\dot{\boldsymbol{\xi}} = [D\boldsymbol{F} - (\alpha + \mathrm{i}\,\beta)D\boldsymbol{H}]\boldsymbol{\xi}. \tag{7}$$

For the case where $\boldsymbol{H}$ is an identity matrix, the following theoretical result can easily be derived.

When $\boldsymbol{H}$ is an identity matrix, let $l_{\max}$ denote the largest Lyapunov exponent of the node system $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$. Then when $\alpha > l_{\max}$, one has $\lim\limits_{t\to\infty} \boldsymbol{\xi} = 0$, so the network synchronized region is unbounded. Arrange the real parts of the eigenvalues of matrix $\boldsymbol{G}$ as $0 = \lambda_1 < \lambda_2^r \leq \ldots \leq \lambda_N^r$. Given any coupling strength σ satisfying the condition $\sigma\lambda_2^r > l_{\max}$, where λ_2^r is the real part of the smallest nonzero eigenvalue of the coupling matrix $\boldsymbol{G}$, then all points $\{\sigma\lambda_k; k = 2, 3, \ldots, N\}$

are located inside the synchronized region, therefore the network will achieve synchronization. Moreover, it can be seen that the larger the λ_2^r, the smaller the lower limit of σ which satisfies the synchronization condition, so the better the synchronizability of the network. Therefore, the synchronizability of the network is determined by the real part of the smallest nonzero eigenvalue of the coupling matrix in this case.

When $\boldsymbol{G}$ is a non-diagonalizable matrix, the MSF method can also be applied to analyze the synchronizability of the network (for detailed analysis, see Ref.[17]), thus the same conclusion still holds.

From an engineering point of view, when the network structure and the individual node dynamics are both given, one wishes that the vector-valued inner-coupling matrix $\boldsymbol{H}$ is as simple as possible, so that the control cost can be as low as possible. When $\boldsymbol{H}$ is a general diagonal matrix (not necessarily an identity matrix) with only 1 and 0 as its entries, if the synchronous solution is not an equilibrium, then it is difficult to determine the network synchronized region analytically. However, this can always be easily done via numerical simulation. More specifically, for a network with a general diagonalizable coupling matrix $\boldsymbol{H}$, if it has a bounded synchronized region, then according to Ref.[15], the range of the coupling strength σ to ensure network synchronization is given by $\sigma \in \left(\frac{\alpha_m}{\lambda_2^r}, \frac{\alpha_M}{\lambda_N^r}\right)$, where α_m and α_M are two parameters determined by the real-axis boundaries of the synchronized region, with $\alpha_m < \alpha_M$. When $\lambda_N^r/\lambda_2^r > \alpha_M/\alpha_m$, the range of σ for guaranteeing the network synchronization is empty, so synchronization cannot be achieved; when $\lambda_N^r/\lambda_2^r \leq \alpha_M/\alpha_m$, the bigger the λ_2^r and the smaller the λ_N^r, the bigger the region of the coupling strength σ for guaranteeing the network synchronization, thus the better the network synchronizability. In other words, when $\lambda_N^r/\lambda_2^r \leq \alpha_M/\alpha_m$, the smaller the eigenratio λ_N^r/λ_2^r, the better the network synchronizability. Consequently, when the synchronized region is bounded, the eigenratio λ_N^r/λ_2^r can be used to describe the network synchronizability.

4. Pinning control analysis

In this section, the pinning-controlled network Eq.(5) is further discussed. The question as to how the control gain affects the network synchronizability is analyzed both theoretically and numerically.

In Eq.(5), let

$$\boldsymbol{M} = \left[\begin{array}{ccccc} G_{11}+\delta_1 k_1 & G_{12} & \dots & G_{1N} & -\delta_1 k_1 \\ G_{21} & G_{22}+\delta_2 k_2 & \dots & G_{2N} & -\delta_2 k_2 \\ \vdots & & \ddots & & \vdots \\ G_{N1} & G_{N2} & \dots & G_{NN}+\delta_N k_N & -\delta_N k_N \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] := \left[\begin{array}{c|c} \boldsymbol{G'}_{N\times N} & \boldsymbol{K}_{N\times 1} \\ \hline \boldsymbol{O}_{1\times N} & 0 \end{array}\right], \tag{8}$$

where $\boldsymbol{G'}$ is

$$\boldsymbol{G'} = \left[\begin{array}{cccc} G_{11}+\delta_1 k_1 & G_{12} & \dots & G_{1N} \\ G_{21} & G_{22}+\delta_2 k_2 & \dots & G_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1} & G_{N2} & \dots & G_{NN}+\delta_N k_N \end{array}\right], \tag{9}$$

namely,

$$\boldsymbol{G'} = \boldsymbol{G} + \mathrm{diag}(\delta_1 k_1, \delta_2 k_2, \dots, \delta_N k_N). \tag{10}$$

Let λ and $\boldsymbol{x}$ denote an eigenvalue of matrix $\boldsymbol{M}$ and its corresponding right-eigenvector, respectively, then,

$$\boldsymbol{M}X = \lambda \boldsymbol{X}. \tag{11}$$

Decomposing the eigenvector as

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ x_{N+1} \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ x_{N+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}, \tag{12}$$

and substituting Eqs.(8) and (12) into Eq.(11), one obtains

$$\left[\begin{array}{c|c} \boldsymbol{G}' & \boldsymbol{K} \\ \hline \boldsymbol{O} & 0 \end{array}\right] \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}. \tag{13}$$

It then follows from Eq.(13) that

$$\begin{bmatrix} \boldsymbol{G}'\boldsymbol{X}_1 + \boldsymbol{K}\boldsymbol{X}_2 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}. \tag{14}$$

This implies that for any nonzero eigenvalue λ of $\boldsymbol{M}$, one always has $\boldsymbol{X}_2 = [x_{N+1}] = [0]$, so

$$\begin{bmatrix} \boldsymbol{G}'\boldsymbol{X}_1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{X}_1 \\ 0 \end{bmatrix}, \ \leftrightarrow \ \boldsymbol{G}'\boldsymbol{X}_1 = \lambda \boldsymbol{X}_1. \tag{15}$$

Equation (15) shows that the nonzero eigen-spectrum of $\boldsymbol{M}$ equals the eigen-spectrum of $\boldsymbol{G}'$.

In the following, for any matrix $\boldsymbol{B}$, let $\lambda_i^r(\boldsymbol{B})$ denote the real part of its i-th eigenvalue. The above analysis shows that $\lambda_2^r(\boldsymbol{M}) = \lambda_1^r(\boldsymbol{G}')$ and

$$R^{N+1} = \frac{\lambda_{N+1}^r(\boldsymbol{M})}{\lambda_2^r(\boldsymbol{M})} = \frac{\lambda_N^r(\boldsymbol{G}')}{\lambda_1^r(\boldsymbol{G}')}.$$

For simplicity, each controller is designed to have the same control gain below, i.e. $k_1 = k_2 = \ldots = k_N = k$. In order to analyze the effect of the control gain on the network synchronizability, first consider an extreme case where every node in the network is controlled, $\delta_1 = \delta_2 = \ldots = \delta_N = 1$, and leave the general case where only a portion of nodes are controlled to be further discussed later.

In this case, first, rewrite Eq.(10) as

$$\boldsymbol{G}' = \boldsymbol{G} + \mathrm{diag}(k, k, \ldots, k) = \boldsymbol{G} + k\boldsymbol{I}. \tag{16}$$

Then, suppose that the coupling matrix $\boldsymbol{G}$ is diagonalizable, so there exists a nonsingular matrix $\boldsymbol{L}$ such that $\boldsymbol{L}^{-1}\boldsymbol{G}\boldsymbol{L} = \boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix. Multiplying Eq.(16) by $\boldsymbol{L}^{-1}$ and $\boldsymbol{L}$ on both sides, respectively, one obtains

$$\boldsymbol{L}^{-1}\boldsymbol{G}'\boldsymbol{L} = \boldsymbol{L}^{-1}(\boldsymbol{G} + k\boldsymbol{I})\boldsymbol{L} = \boldsymbol{\Lambda} + k\boldsymbol{I}. \tag{17}$$

Since the matrix $\boldsymbol{L}^{-1}\boldsymbol{G}'\boldsymbol{L}$ is similar to the matrix $\boldsymbol{G}'$, they have the same eigen-spectrum, so

$$\lambda_2^r(\boldsymbol{M}) = \lambda_1^r(\boldsymbol{G}') = \lambda_1^r(\boldsymbol{G}) + k = k. \tag{18}$$

Consequently,

$$\lim_{k\to 0} \lambda_2^r(\boldsymbol{M}) = 0, \quad \text{and} \quad \lim_{k\to\infty} \lambda_2^r(\boldsymbol{M}) = \infty. \tag{19}$$

The eigenratio R^{N+1} of matrix $\boldsymbol{M}$ is

$$\begin{aligned} R^{N+1} &= \frac{\lambda_{N+1}^r(\boldsymbol{M})}{\lambda_2^r(\boldsymbol{M})} = \frac{\lambda_N^r(\boldsymbol{G}')}{\lambda_1^r(\boldsymbol{G}')} = \frac{\lambda_N^r(\boldsymbol{G}) + k}{\lambda_1^r(\boldsymbol{G}) + k} \\ &= 1 + \frac{\lambda_N^r(\boldsymbol{G})}{k}. \end{aligned} \tag{20}$$

It follows from Eq.(20) that $\lim\limits_{k\to 0} R^{N+1} = \infty$ and $\lim\limits_{k\to\infty} R^{N+1} = 1$. Thus, when every node in the network is being controlled, the synchronizability of the extended network is improved monotonously with the increase of the control gain k. When the control gain k varies from zero to infinity, $\lambda_2^r(\boldsymbol{M})$ varies from zero to infinity and the eigenratio R^{N+1} varies from infinity to 1. This means that the network synchronizability can be improved by increasing the control gain k no matter if the network synchronized region is bounded or unbounded.

Note that even if the matrix $\boldsymbol{G}$ is not diagonalizable, the same conclusion can be obtained by applying Gerschgorin's circle theorem.

Next, consider the case when only a portion of nodes is controlled. In this case, network synchronization can still be achieved through couplings among nodes. Suppose the coupling matrix $\boldsymbol{G}$ of the original network is irreducible, when the control gain satisfies $k > 0$, one has $\sum\limits_{i\in C} G'_{ij} > 0$ for the controlled node i, and $\sum\limits_{l\notin C} G'_{lj} = 0$ for the uncontrolled node l , so the matrix $\boldsymbol{G}'$ is irreducibly diagonally dominant, all its eigenvalues are positive.

Firstly, consider the two limits of $\lambda_2^r(\boldsymbol{M})$ and R^{N+1} as $k \to 0$ and $k \to \infty$.

(i) When the control gain $k \to 0$, the two limits are

$$\lim_{k\to 0} \lambda_2^r(\boldsymbol{M}) = \lim_{k\to 0} \lambda_1^r(\boldsymbol{G}')$$

and

$$\lim_{k\to 0} R^{N+1} = \lim_{k\to 0} \frac{\lambda_N^r(\boldsymbol{G}')}{\lambda_1^r(\boldsymbol{G}')}.$$

As $k \to 0$, since $\boldsymbol{G}' \cong \boldsymbol{G}$, one has $\lim\limits_{k\to 0} \lambda_1(\boldsymbol{G}') = \lambda_1(\boldsymbol{G}) = 0$, and $\lim\limits_{k\to 0} \lambda_N(\boldsymbol{G}') = \lambda_N(\boldsymbol{G})$, where $\lambda_N(\boldsymbol{G})$ is a constant for any given $\boldsymbol{G}$. Consequently, $\lim\limits_{k\to 0} \lambda_2^r(\boldsymbol{M}) = \lim\limits_{k\to 0} \lambda_1^r(\boldsymbol{G}') = 0$ and $\lim\limits_{k\to 0} R^{N+1} = \infty$.

(ii) When the control gain $k \to \infty$, the two limits are $\lim\limits_{k\to\infty} \lambda_2^r(\boldsymbol{M}) = \lim\limits_{k\to\infty} \lambda_1^r(\boldsymbol{G}')$ and $\lim\limits_{k\to\infty} R^{N+1} =$

$\lim\limits_{k\to\infty}\dfrac{\lambda_N^r(\boldsymbol{G}')}{\lambda_1^r(\boldsymbol{G}')}$. Since only a portion of nodes is controlled, according to Gerschgorin's circle theorem, when the topology and the weights of the network are given, i.e., when $\boldsymbol{G}$ is given, every Gerschgorin's circular radius of $\boldsymbol{G}'$ is fixed. The control gain k only changes those centers of Gerschgorin's circles which are determined by the rows corresponding to the nodes where the controllers are located. It then follows that $\lim\limits_{k\to\infty}\lambda_N^r(\boldsymbol{G}')\to\infty$ and $0<\lim\limits_{k\to\infty}\lambda_1^r(\boldsymbol{G}')\le c1$, where $c1$ is the maximum radius of Gerschgorin's circular radii which are determined by the rows corresponding to the uncontrolled nodes which do not contain k. Consequently, one has

$$\lim_{k\to\infty}\lambda_2^r(\boldsymbol{M})=\lim_{k\to\infty}\lambda_1^r(\boldsymbol{G}')\le c1$$

and

$$\lim_{k\to\infty}R^{N+1}=\lim_{k\to\infty}\frac{\lambda_N^r(\boldsymbol{G}')}{\lambda_1^r(\boldsymbol{G}')}=\infty.$$

When the control gain k is a constant, it is clear that $\lambda_2^r(\boldsymbol{M})$ and $R^{N+1}=\dfrac{\lambda_{N+1}^r(\boldsymbol{M})}{\lambda_2^r(\boldsymbol{M})}$ are both constants. When the control gain k takes a small value near zero, the increase of k mainly affects the eigenvalue λ_1^r of $\boldsymbol{G}'$, which increases $\lambda_1^r(\boldsymbol{G}')$ more significantly than increasing $\lambda_N^r(\boldsymbol{G}')$, so $\lambda_2^r(\boldsymbol{M})=\lambda_1^r(\boldsymbol{G}')$ increases and

$$R^{N+1}=\frac{\lambda_{N+1}^r(\boldsymbol{M})}{\lambda_2^r(\boldsymbol{M})}=\frac{\lambda_N^r(\boldsymbol{G}')}{\lambda_1^r(\boldsymbol{G}')}$$

decreases. When the control gain k is much larger than the biggest radius of Gerschgorin's circles of $\boldsymbol{G}$, with the increase of k, $\lambda_N^r(\boldsymbol{G}')$ increases quickly, however, $\lambda_1^r(\boldsymbol{G}')$ increases very slowly, therefore, $\lambda_2^r(\boldsymbol{M})=\lambda_1^r(\boldsymbol{G}')$ increases slowly and eventually approaches a constant, but

$$R^{N+1}=\frac{\lambda_{N+1}^r(\boldsymbol{M})}{\lambda_2^r(\boldsymbol{M})}=\frac{\lambda_N^r(\boldsymbol{G}')}{\lambda_1^r(\boldsymbol{G}')}$$

still increases quickly.

For a given portion p of controlled nodes, the above analysis shows that when the control gain k increases from zero to infinity, the eigenvalue λ_2^r of the extended network increases monotonously and eventually approaches a constant; the eigenratio R^{N+1} of the extended network firstly decreases and then increases, which shows the existence of a minimum. For a given value of p, if when $k=k_g$, the eigenratio R^{N+1} takes the smallest value, then the range of the coupling strength σ which ensures synchronization of the extended network is the largest, so the extended network achieves the best synchronizability with $k=k_g$. It should be noted that for a given scheme of choosing controlled nodes, k_g varies with the portion of controlled nodes p.

When the network synchronized region is unbounded, the real part of the smallest nonzero eigenvalue, $\lambda_2^r(\boldsymbol{M})$, can be used to describe the synchronizability of the extended network. The larger the value of k, the larger the value of $\lambda_2^r(\boldsymbol{M})$, thus the better the synchronizability of the extended network. In other words, the larger the value of k, the better the pinning-controllability of the original network. Actually, when the control gain $k\neq 0$, one has $\lambda_2^r(\boldsymbol{M})\neq 0$. In this case, no matter how small the value of $\lambda_2^r(\boldsymbol{M})$ is, if the coupling strength σ is big enough to satisfy $\sigma\lambda_2^r(\boldsymbol{M})>l_{\max}$, then the extended network can always achieve synchronization. This means that even only one node in the original network is pinned, if the coupling strength is big enough, network synchronization can still be achieved, this supports the result of Ref.[20]. In this case, the portion of the controlled nodes has no critical values.

For a given control gain, when the portion of controlled nodes p increases, $\lambda_2^r(\boldsymbol{M})$ also increases, so the lower-limit of the coupling strength for realizing the synchronization of the extended network decreases, which shows that the synchronizability of the extended network increases.

When $\boldsymbol{H}$ is diagonal but is not an identity matrix, the network synchronized region should be determined by the MSF method. In general, when the network synchronized region is bounded, such as coupled Lorenz systems or coupled Rosller systems, there exists a constant α_M/α_m,[4,15] when the eigenratio satisfies $R^{N+1}<\alpha_M/\alpha_m$ the extended network will achieve synchronization with an appropriately choosing coupling strength σ.

For a given scheme of choosing the controlled nodes and a given portion p of controlled nodes, there exists a curve of R^{N+1} versus k where the minimum of R^{N+1} corresponds to the control gain $k=k_g$. If, for some p, there exists $R^{N+1}\big|_{k=kg}=\alpha_M/\alpha_m$, then this p is the critical value p_c of the portion of controlled nodes, when $p>p_c$, the network can be controlled by pinning. This critical value p_c is determined by the node dynamics, the pre-specified state $\boldsymbol{s}(t)$, the vector-valued inner-coupling function $\boldsymbol{H}$, the network topology and weights.

The above analysis is not confined to a particular network topology. For a homogenous network, since every node has roughly the same number of con-

nections, choosing which nodes to pin has very small impact on the control performance. However, for a heterogeneous network, since a few nodes have very large numbers of connections but most other nodes have very few connections, choosing which nodes to pin generally has a great impact on the control results. Previous studies[18,19] have shown that the selective pinning scheme, which gives a high priority to controlling the nodes with large numbers of connections, is more efficient than the random pinning scheme which simply controls a portion of randomly-chosen nodes. So, in the following simulation, the selective control scheme is applied, with which the controlled nodes are chosen according to their descending numbers of connections.

Figure 1 shows the relations between λ_2^r and k, and between $R^{N+1} = \lambda_{N+1}^r/\lambda_2^r$ and k, both in a symmetrical unweighted Barabasi–Albert (BA) scale-free network model and in an asymmetrical weighted BA scale-free network model. It can be seen from Fig.1 that, for a given p, λ_2^r increases monotonously and eventually approaches a constant value with increasing k; $R^{N+1} = \lambda_{N+1}^r/\lambda_2^r$ firstly decreases and then increases, which shows the existence of a minimum. For a given p, if the minimum of the eigenratio R^{N+1} is larger than the constant α_m/α_m, pinning control cannot realize network synchronization. The relation between the synchronizability and the control gain shown in Fig.1 is consistent with the above theoretical analysis.

In the simulation, a symmetrical unweighted BA scale-free network[27] with the number of nodes $N = 500$ and the smallest degree $m = 5$ is used, with results shown in Figs.1(a) and 1(b); and an asymmetrical weighted BA scale-free network[14] with $N = 500$, $m = 5$ is used, with results shown in Figs.1(c) and 1(d). Here, the coupling matrix of the weighted network is

$$G_{ij} = -A_{ij}\frac{\Theta_{ij}}{\sum_{j\in\Lambda_i}\Theta_{ij}},$$

$\boldsymbol{A} = [A_{ij}]$ is the adjacency matrix, Λ_i is the set of neighbors of node i,

$$\Theta_{ij} = \frac{1-\theta}{2} \quad \left(\Theta_{ij} = \frac{1+\theta}{2}\right)$$

for $i > j$ $(i < j)$, in which the parameter $-1 < \theta < 1$ governs the asymmetry of the coupling matrix in the network and $\theta = -0.9$ is used in simulation.

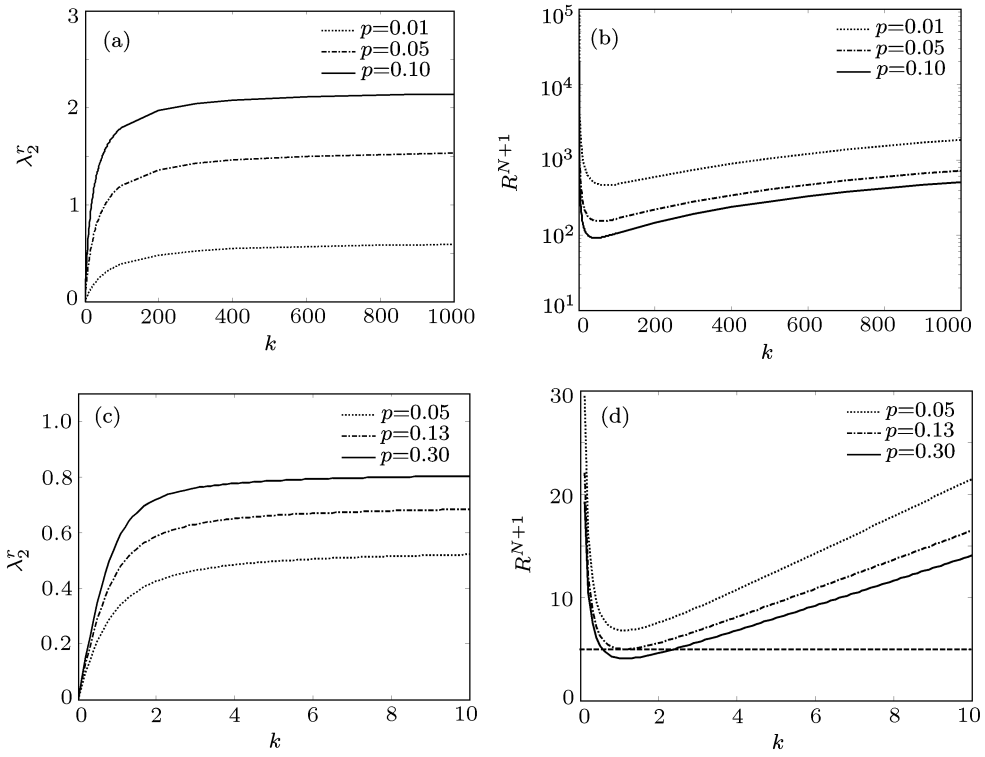


Fig.1. The relation of network synchronizability and control gain k: (a), (c) λ_2^r vs. k; (b), (d) $R^{N+1} = \lambda_{N+1}^r/\lambda_2^r$ vs. k. In (a) and (b), the symmetrical unweighted BA scale-free network is used; in (c) and (d), the asymmetrical weighted BA scale-free network with $\theta = -0.9$ is used. In (d), the line running parallel with the x-axis corresponds to $\alpha_M/\alpha_m = 5.17$.

5. Pinning control of a network of Lorenz systems

In this section, pinning control of a network of Lorenz systems is simulated for illustration. Consider a network of chaotic Lorenz systems, described by

$$\begin{aligned} \dot{x} &= 10(y-x), \\ \dot{y} &= 23x - y - xz, \\ \dot{z} &= xy - z. \end{aligned} \tag{21}$$

The desired synchronous state $\boldsymbol{s}(t)$ is a chaotic solution of Eq.(21). Every node of the asymmetrical weighted BA scale-free network described in the above section is a chaotic Lorenz system in this study. In the simulation, the parameters of the network are $N = 500$, $m = 5$, $\theta = -0.9$. When the vector-valued inner-coupled function $\boldsymbol{H} = \mathrm{diag}(1,\ 1,\ 1)$, the network synchronized region is unbounded, therefore the controlled synchronizability of the network is determined by λ_2^r of the coupling matrix of its extended network.

The synchronized region calculated by the MSF method is shown in Fig.2(a), in which the network can achieve controlled synchronization with $\sigma\lambda_2^r \geq l_{\max} = 0.43$. So, even if only one node is controlled with a nonzero control gain k, one has $\lambda_2^r \neq 0$, hence the extended network can achieve controlled synchronization with a large enough coupling strength σ.

In the following, only one node with the largest number of connections is controlled, that is, $p = 0.002$ and $n = pN = 1$. When the control gain $k = 5$, the average value of the smallest nonzero eigenvalue is $\bar{\lambda}_2^r = 0.0452$, obtained with the average of 20 simulation runs. When $\sigma \geq 9.5133$, the inequality $\sigma\lambda_2^r \geq 0.43$ is satisfied; that is, the network can achieve controlled synchronization with $k = 5$ and $\sigma \geq 9.5133$. Figure 2(b) shows that the nonzero eigenvalues of the coupling matrix $\boldsymbol{M}$ are all located in the synchronized region with $k = 5, \sigma = 17.25$, which means that pinning control indeed realizes network synchronization. This simulation verifies the above theoretical analysis.

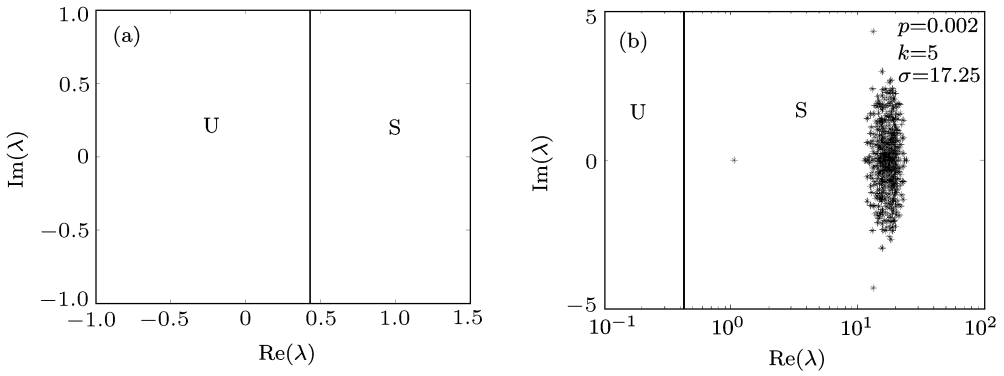


Fig.2. The simulated network with an unbounded synchronized region: (a) $\boldsymbol{H} = I_m$; (b) the relation of points $\{\sigma\lambda_l;\ l = 2, 3, \ldots, N+1\}$ and the network synchronized region with only the largest node being controlled by pining.

Next, the network with $\boldsymbol{H} = \mathrm{diag}(0,\ 0,\ 1)$ is simulated. The desired synchronous state is still a chaotic orbit of the Lorenz system. In this case, the network synchronized region determined by the MSF method is bounded, as shown in Fig.3(a). Based on the analysis of Section 3, the synchronizability of the extended network should be described by the eigenratio. According to the synchronized region shown in Fig.3(a), it can be seen that $\alpha_M/\alpha_m = 6.56/1.27 \approx 5.17$, hence only if the eigenratio R^{N+1} satisfies the inequality $R^{N+1} \leq 5.17$, pinning the original network can achieve network synchronization. The critical value of the portion of controlled nodes is $p_c = 0.13$, which is obtained from Fig.1(d), where the line running parallel with the x-axis corresponds to $\alpha_M/\alpha_m = 5.17$. Only when $p > p_c$, can pinning-controlled synchronization be achieved. Figure 3(b) shows that the nonzero eigenvalues of the coupling matrix $\boldsymbol{M}$ are all located inside the synchronized region with $p = 0.4$, $k = 1.0$, $\sigma = 2.5$, which means that the pinning-controlled synchronization is achieved.

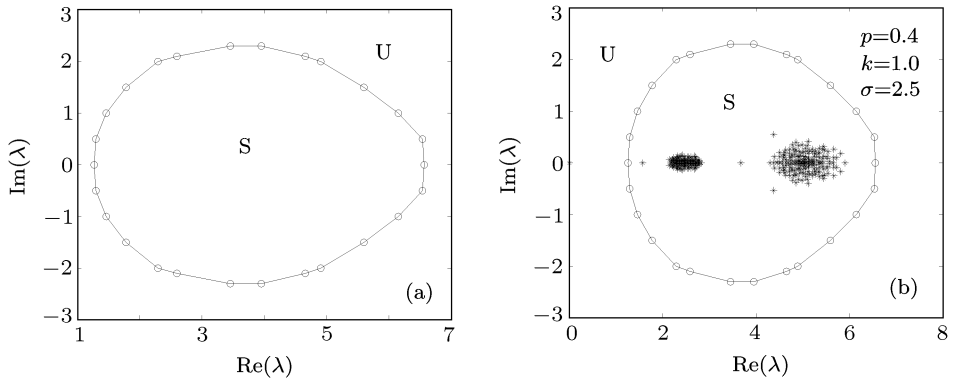


Fig.3. The simulated network with a bounded synchronized region: (a) $\boldsymbol{H} = \mathrm{diag}(0, 0, 1)$; (b) the relation of points $\{\sigma\lambda_l;\ l = 1,\ 2,\ 3, \ldots, N + 1\}$ and the network synchronized region with $p = 0.4$, $k = 1.0$ and $\sigma = 2.5$.

When the portion p of the controlled nodes and the control gain k are kept unchanged, e.g., $p = 0.4$, $k = 1.0$, the controlled synchronization of the network cannot be achieved by changing the coupling strength σ, as shown in Fig.4(a) (increasing σ) and Fig.4(b) (decreasing σ), respectively. Similarly, if the portion p of controlled nodes and the coupling strength σ are both kept unchanged, e.g., $p = 0.4$, $\sigma = 2.5$, the controlled synchronization cannot be realized by changing the control gain k, as shown in Fig.4(c) (increasing k) and Fig.4(d) (decreasing k), respectively. If the coupling strength σ or the control gain k increases, the desynchronization of the extended network can arise due to the overflow of the largest eigenvalue of the coupling matrix from the synchronized region, which are shown in Figs.4(a) and 4(c). If the coupling strength σ or the control gain k decreases, the desynchronization of the extended network can also arise due to the overflow of the smallest nonzero eigenvalue of the coupling matrix from the synchronized region, which are shown in Figs.4(b) and 4(d).

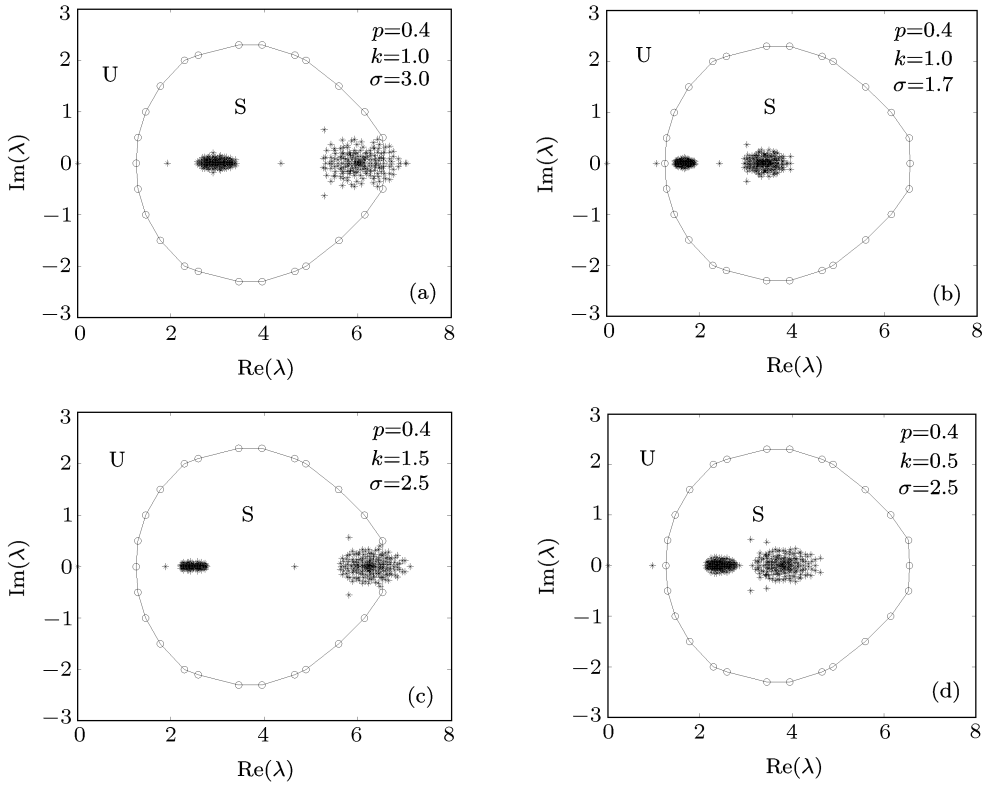


Fig.4. Four cases of desynchronization: (a) increasing the coupling strength σ; (b) decreasing the coupling strength σ; (c) increasing the control gain k; (d) decreasing the control gain k.

The results of the simulation study, shown in Figs.3 and 4 as well as Fig.1(d), have shown that when the network synchronized region is bounded, the portion p of controlled nodes has a critical value p_c, therefore only when $p > p_c$, the network will achieve pinning-controlled synchronization with an appropriately choosing control gain k and a correspondingly determined coupling strength σ. If the control gain k or the coupling strength σ is too large or too small, the extended network will be desynchronized, namely, pinning-controlled synchronization of the original network cannot be achieved.

6. Conclusions

This paper studies pinning-controlled synchronization of complex networks with bounded or unbounded synchronized regions. It has shown, both theoretically and numerically, that if every node in a network has full-state connections, e.g., $\boldsymbol{H} = \mathrm{diag}(1,\ 1,\ 1)$ in the 3-dimensional setting, the network synchronized region will be unbounded, so pinning control of the network can be easily realized. In this case, theoretically controlled synchronization can be achieved by pinning only one node in the network. However, when every node has only partial-state connections, e.g., $\boldsymbol{H} = \mathrm{diag}(0,\ 0,\ 1)$, if the network synchronized region is bounded, pinning-controlled synchronization of the network becomes difficult. In this case, only when the number of controlled nodes exceeds a threshold, controlled synchronization can be achieved with an appropriately choosing control gain and a correspondingly determined coupling strength.

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