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Local distinguishability by projective measurement*

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This paper proves that a set of orthogonal pure states are indistinguishable by restricted local projective measurement and classical communication if the sum of their Schmidt ranks is larger than the dimension of their joint Hilbert space. This result is useful in determining the local distinguishability of quantum states and is stronger in some respects than that of Hayashi *et al* [Phys. Rev. Lett. **96**, 040501]. In addition, it presents a new method to determine the local distinguishability of orthogonal states by projecting measurement operators into their subspaces.

Keywords: local distinguishability, projective measurement, Schmidt rank
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Local distinguishability of a set of orthogonal multipartite quantum states is an interesting topic in quantum information because of its intimate yet subtle relation with nonlocality and quantum entanglement.[1] For example, Bennett *et al* showed that there are nine orthogonal product states in a 3×3 system which cannot be distinguished by local operation and classical communication (LOCC).[1] Obviously, a set of multipartite orthogonal states can always be distinguished by nonlocal projective measurements. However, when the available operations are limited to LOCC, it is not easy to determine whether a set of multipartite orthogonal pure states can be locally distinguished.[1,2]

In the past few years, there have been many interesting results on local distinguishability of orthogonal states.[1−17] In spite of these, there has not been a general criterion in determining the local distinguishability of orthogonal states. Some authors suggested that if a set of orthogonal quantum states can be distinguished by LOCC protocols, the sum of Schmidt ranks[18] for the set of states should be equal to or less than the dimension of their Hilbert space.[3] Schmidt rank is defined as follows: an N-partite pure state can be written in the form

$$|\psi\rangle = \sum_{i=1}^{r} a_i |\psi_1^{(i)}\rangle \otimes |\psi_2^{(i)}\rangle \otimes \ldots \otimes |\psi_N^{(i)}\rangle, \qquad (1)$$

and Schmidt rank is the minimal number of product terms r in such a decomposition of $|\psi\rangle$. This criterion, if valid, would be very useful in studying local distinguishability of quantum states. For example, in a 2×2 system, if four states can be locally distinguished, the four states must be product states. If three states can be locally distinguished, at most one of them could be entangled. Unfortunately, this conjecture is proved to be wrong by the clever counterexample given by Cohen.[4]

In this paper, we prove that local distinguishability of orthogonal states can indeed be determined by the sum of the Schmidt ranks of these states, but by restricting available operations to restricted local projective measurement and classical communication (RLPCC).

Before we proceed to prove our results, we review how an LOCC protocol works in distinguishing orthogonal states.

In order to distinguish states in $\varepsilon = \{|\psi_i\rangle\}$, Alice and Bob perform successive rounds of local measurement and classical communication. Suppose Alice is the first to perform a POVM $\{A_{j_1}^\dagger A_{j_1}\}_{j_1}$. If the outcome is j_1, $\{|\psi_i\rangle\}$ becomes $\{A_{j_1}|\psi_i\rangle\}_i$, up to normalization. After communicating the result of Alice's measurement to Bob, Bob carries out a measurement $\{B_{k_1}^\dagger(j_1)B_{k_1}(j_1)\}_{k_1}$, obtains outcome k_1 and gets the

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state

$$\{A_{j_1} \otimes B_{k_1}(j_1)|\psi_i\rangle\}_i. \tag{2}$$

After N_0 rounds of measurement and communication, there are many possible outcomes corresponding to measurement operators acting on Alice and Bob's Hilbert space, each of which is a product of the N_0 operators. We denote these operators as $\{A_i \otimes B_i\}$. They satisfy

$$\sum_i A_i^\dagger \otimes B_i^\dagger A_i \otimes B_i = \sum_i M_i = I, \tag{3}$$

where $M_i = A_i^\dagger A_i \otimes B_i^\dagger B_i$.

For a local projective measurement and classical communication (LPCC) protocol, similar to LOCC protocols described above, we get the state $P_{j_1}^A \otimes P_{k_1}^B(j_1)|\psi\rangle$ after the first round, where $\{P_{j_1}^A\}$ and $\{P_{k_1}^B(j_1)\}_{k_1}$ are Alice's and Bob's projective measurement separately (for any projective measurement $\{P_i\}$, $P_i = P_i^2$ and $P_i = P_i^\dagger$ for all i). In the second round, Alice performs another projective measurement $\{P_{j_2}^A(j_1,k_1)\}_{j_2}$ and the outcome is j_2. In general, the operator $P_{j_1}^A P_{j_2}^A(j_1,k_1)$ is not a projective operator. But here, we restrict LPCC protocols to satisfy

$$[P_{j_1}^A, P_{j_2}^A(j_1,k_1)] = 0 \tag{4}$$

for all j_1, k_1 and j_2. Equation (4) ensures that the set $\{P_{j_1}^A P_{j_2}^A(j_1,k_1)\}$ is still a set of projective operators, which holds the following relations:

$$[P_{j_1}^A P_{j_2}^A(j_1,k_1)]^2 = P_{j_1}^A P_{j_2}^A(j_1,k_1), \tag{5}$$

and

$$P_{j_1}^A P_{j_2}^A(j_1,k_1) = [P_{j_1}^A P_{j_2}^A(j_1,k_1)]^\dagger. \tag{6}$$

In the following every steps, we give similar restrictions as Eq.(4). Thus, after N_0 rounds of measurement and communication, there are many possible outcomes, each of which is a product of the N_0 operators. We denote these operators as $\{P_i^A \otimes P_i^B\}$ and they satisfy that $\sum_i (P_i^A)^\dagger \otimes (P_i^B)^\dagger P_i^A \otimes P_i^B = \sum_i P_i^A \otimes P_i^B = \sum_i P_i = I$. We define this LPCC protocol as the RLPCC protocol.

Theorem 1 Let $\varepsilon = \{|\psi_i\rangle | i = 1, \ldots, r\}$ be a set of bipartite orthogonal states. If states in ε can be perfectly identified by RLPCC, then the sum of their Schmidt ranks is equal to or less than N, the dimension of their joint Hilbert space.

Proof If the states in ε can be distinguished by some RLPCC protocol, then the LPCC protocol can be described by a complete set of local projective operations $\{P_i\}$. Thus, for every $|\psi_i\rangle$, there exists at least one measurement operator P_i satisfying

$$P_i|\psi_i\rangle \neq 0 \quad \text{and} \quad P_i|\psi_j\rangle = 0 \tag{7}$$

for all $j(\neq i)$.

Corresponding to the set $\varepsilon = \{|\psi_i\rangle | i = 1, \ldots, r\}$, we divide $\{P_i\}$ into r sets: $\sigma_1, \sigma_2, \ldots, \sigma_r$, where $\sigma_i = \{P_i^{(h)} | h = 1, \ldots, m_i\}$ is composed of all operators identifying $|\psi_i\rangle$. Then,

$$\sum_{i,h} P_i^{(h)} = I \quad \text{and} \quad P_i^{(h)} P_j^{(k)} = P_i^{(h)} P_j^{(k)} \delta_{ij} \delta_{hk}. \tag{8}$$

Since $P_i^{(h)}$ is a projective operator, it can be written in the following form:

$$\begin{aligned} P_i^{(h)} = {} & |\varphi_i^{1(h)}\rangle\langle\varphi_i^{1(h)}| + |\varphi_i^{2(h)}\rangle\langle\varphi_i^{2(h)}| + \ldots \\ & + |\varphi_i^{n_i(h)}\rangle\langle\varphi_i^{n_i(h)}|, \end{aligned} \tag{9}$$

where $\{|\varphi_i^{1(h)}\rangle, \ldots, |\varphi_i^{n_i(h)}\rangle\}$ is the set of eigenbases of $P_i^{(h)}$ and every one of them is a product state, $n_i(h)$ is the rank of $P_i^{(h)}$.

From the above argument, we know

$$P_j^{(h)}|\psi_i\rangle = 0, \tag{10}$$

for all $j(\neq i)$. Thus for every $|\psi_i\rangle$ we have

$$\begin{aligned} |\psi_i\rangle &= \{I - [P_i^{(1)} + \ldots + P_i^{(m_i)}] \\ &\quad + [P_i^{(1)} + \ldots + P_i^{(m_i)}]\}|\psi_i\rangle \\ &= [P_i^{(1)} + \ldots + P_i^{(m_i)}]|\psi_i\rangle \\ &= \sum_{k=1}^{x_i} a_k^i |\varphi_i^k\rangle, \end{aligned} \tag{11}$$

where number k from 1 to x_i is a rearrangement of $\{1(1), \ldots, n_i(1), \ldots, 1(m_i), \ldots, n_i(m_i)\}$; $a_k^i = \langle\varphi_i^k|\psi_i\rangle$; $x_i = n_i(1) + \ldots + n_i(m_i)$. From Eq.(11), we conclude that $S(|\psi_i\rangle)$, the Schmidt rank of the state $|\psi_i\rangle$, is equal to or less than x_i.

Therefore, S, the sum of Schmidt ranks of the set, is equal to or less than N ($N = \sum_i x_i$). This completes the proof.

It is interesting to compare our result with that in Ref.[5]. In Ref.[5], it was shown that a necessary condition for deterministic LOCC discrimination of a set of states $\{\rho_i\}$ is that $\sum_i d(\rho_i) \leq N$, where $d(\rho_i)$ is a quantity characterizing the distance between ρ_i and the set of separable states; N is the dimension of the Hilbert space. Since $d(\rho_i)$ is very complicated to compute, a quantity $r(\rho_i) := |P_i|(1 + R_G(P_i/|P_i|)) \leq d(\rho_i)$ is often used, and $\sum_i r(\rho_i) \leq N$. Here, P_i is the support of state ρ_i, $|P_i| := \mathrm{tr}\{P_i\}$, $R_G(P_i/|P_i|)$ is the robustness of entanglement of state $P_i/|P_i|$.

The conclusion in Ref.[5] is more general than ours in the sense that LOCC is allowed and mixed states $\{\rho_i\}$ are also considered. However, our result is more effective in distinguishing pure states by using RLPCC. For any pure bipartite state $|\psi_i\rangle = \sum_{j=1}^{S(|\psi_i\rangle)} a_j^i|jj\rangle$, $R_G(|\psi_i\rangle) = \Big(\sum_j a_j^i\Big)^2 - 1$, so we have $r(|\psi_i\rangle) = \Big(\sum_j a_j^i\Big)^2 \leq S(|\psi_i\rangle)$.[1)] Consequently,

$$\sum_i r(|\psi_i\rangle) \leq \sum_i S(|\psi_i\rangle) \leq N. \tag{12}$$

Therefore, our result is a stronger bound.

In certain systems such as multi-qubit systems,[6] if a set of orthogonal states can be distinguished by an LOCC protocol, they can also be discriminated by some RLPCC protocol. Particularly, if a full set of orthogonal product basis states of a quantum system can be distinguished by some LOCC protocol, we can always find an RLPCC protocol to identify them.[7] In the following, we prove this conclusion and derive a corollary from it which provides an effective method in distinguishing quantum states.

Theorem 2 If a full set of orthogonal product basis states $\{\phi_i\rangle_{AB} = |\theta_i\rangle_A|\delta_i\rangle_B\}_{i=1}^{mn}$ of an $m \times n$ quantum system can be distinguished by some LOCC protocol, they can also be distinguished by an RLPCC protocol.

Proof We can prove this theorem by showing that, in each step of an LOCC protocol, the local operation can be replaced with a restricted local projective operation to achieve the same goal.

Without loss of generality, we assume that we perform POVM $\{M_i^A\}\big(\sum_{i=1}^h M_i^A = I\big)$ to the particle A with result k. Then

$$\begin{aligned}&(M_k^A \otimes I^B)|\phi_i\rangle_{AB} = 0, \quad i = 1, \ldots, l,\\ &(M_k^A \otimes I^B)|\phi_i\rangle_{AB} \neq 0, \quad i = l+1, \ldots, mn.\end{aligned} \tag{13}$$

If for an $s \in \{l+1, l+2, \ldots, mn-1, mn\}$,

$$(M_k^A \otimes I^B)|\phi_s\rangle_{AB} \neq |\phi_s\rangle_{AB}, \tag{14}$$

up to normalization, then

$$\begin{aligned}|\phi_s\rangle_{AB} &= [(I^A - M_k^A) \otimes I^B]|\phi_s\rangle_{AB}\\ &\quad + (M_k^A \otimes I^B)|\phi_s\rangle_{AB}\\ &= |\varphi_1^s\rangle + |\varphi_2^s\rangle,\end{aligned} \tag{15}$$

where we define $|\varphi_1^s\rangle = [(I^A - M_k^A) \otimes I^B]|\phi_s\rangle_{AB}$ and $|\varphi_2^s\rangle = M_k^A \otimes I^B)|\phi_s\rangle_{AB}$. Suppose $|\varphi_1^s\rangle$ and $|\varphi_2^s\rangle$ are linearly dependent, then there exist some nonzero coefficients a and b such that

$$a|\varphi_1^s\rangle + b|\varphi_2^s\rangle = 0. \tag{16}$$

From Eqs.(15) and (16), we get

$$\begin{aligned}|\phi_s\rangle_{AB} &= [(a-b)/a]|\varphi_2^s\rangle\\ &= [(a-b)/a](M_k^A \otimes I^B)|\phi_s\rangle_{AB},\end{aligned} \tag{17}$$

which contradicts Eq.(14). Therefore, $|\varphi_1^s\rangle$ and $|\varphi_2^s\rangle$ must be linearly independent. Then, by Eq.(13) we get

$$\begin{aligned}\langle\phi_\mu|\varphi_2^s\rangle_{AB} &= \langle\phi_\mu|(M_k^A \otimes I^B)|\phi_s\rangle_{AB}\\ &= 0, \quad \mu = 1, \ldots, l.\end{aligned} \tag{18}$$

Since any effective measurement for local distinguishability must keep those residual states orthogonal, we obtain

$$\begin{aligned}\langle\phi_\nu|\varphi_2^s\rangle_{AB} &= \langle\phi_\nu|(M_k^A \otimes I^B)|\phi_s\rangle_{AB}\\ &= 0, \quad \nu = l+1, \ldots, s-1, s+1, \ldots, mn.\end{aligned} \tag{19}$$

As $|\varphi_1^s\rangle = |\phi_s\rangle_{AB} - |\varphi_2^s\rangle$ and $\langle\phi_\sigma|\phi_s\rangle_{AB} = 0, \sigma = 1, \ldots, s-1, s+1, \ldots, mn$, we get $\langle\phi_\sigma|\varphi_1^s\rangle = 0$. In this way, we can construct $mn+1$ linearly independent states $\{|\varphi_1^s\rangle, |\varphi_2^s\rangle, |\phi_1\rangle, \ldots, |\phi_{s-1}\rangle, |\phi_{s+1}\rangle, \ldots, |\phi_{mn}\rangle\}$. This is inconsistent with the fact that the dimension of the Hilbert space is mn. Therefore, for all $s \in \{l+1, l+2, \ldots, mn-1, mn\}$, $(M_k^A \otimes I^B)|\phi_s\rangle_{AB} = |\phi_s\rangle_{AB}$, up to normalization.

So, for every state $|\theta_i\rangle_A$, there is always a set of POVM operators $\{M_j^A\}_{j=i(1)}^{i(I)}$ which satisfy $M_j^A|\theta_i\rangle_A = C_{ji}|\theta_i\rangle_A$ (C_{ji} is a real nonzero coefficient). Thus, every state $|\theta_i\rangle_A$ corresponds to a set of labels, which is defined as $\xi_i = \{i(1), i(2), \ldots, i(I)\}$. We divide all the states $\{|\theta_i\rangle_A\}_{i=1}^{mn}$ into corresponding groups with the same labels. For two different groups of states, there must exist at least one subscript (suppose that it is r) which cannot simultaneously appear in the two corresponding sets of labels. For any two states $|\theta_u\rangle_A$ and $|\theta_w\rangle_A$ in two different groups, suppose r labels the group of $|\theta_w\rangle_A$, that is, $M_r|\theta_w\rangle = C_{rw}|\theta_w\rangle$ and $M_r|\theta_u\rangle_A = 0$, we have

[1)] It is known that $\Big(\sum_{i=1}^k a_i^2\Big)\Big(\sum_{i=1}^k b_i^2\Big) \geq \Big(\sum_{i=1}^k a_i b_i\Big)^2$. When $\sum_{i=1}^k a_i^2 = 1$ and $b_i = 1$ for all i, we get $k \geq \Big(\sum_{i=1}^k a_i\Big)^2$.

$$\begin{aligned}\langle\theta_u|\theta_w\rangle_A &= \langle\theta_u|\frac{C_{rw}}{C_{rw}}|\theta_w\rangle_A \\ &= \langle\theta_u|\frac{M_r}{C_{rw}}|\theta_w\rangle_A \\ &= 0. \end{aligned} \tag{20}$$

This shows that the subspaces spanned by the two different groups of states are orthogonal to each other. Using this result, we can construct a set of orthogonal projectors $\{P_i^A\}$ on the orthogonal subspaces. As $\{P_i^A\}$ spans the whole space of the particle A, the set of projectors $\{P_i^A\}$ are complete, i.e. $\sum_i P_i^A = I_A$. Thus, we can replace the original measurement $\{M_i^A\}$ with projective measurement $\{P_i^A\}$. This completes the proof.

Corollary If a full set of orthogonal product basis states $\{\phi_i\rangle_{AB} = |\theta_i\rangle_A|\delta_i\rangle_B\}_{i=1}^{mn}$of an $m \times n$ quantum system can be distinguished by some LOCC protocol $\{M_i\}_{i=1}^t$, then any two different M_i's are either parallel ("parallel" here means linearly dependent) or orthogonal.

Proof In the proof of Theorem 2, we know that the state in $\{\phi_i\rangle_{AB}\}$ is unchanged in every measurement in the protocol $\{M_i\}_{i=1}^t$ identifying it. Thus, suppose state $\phi_i\rangle_{AB}$ is identified by M_i, we have $M_i \propto |\phi_i\rangle_{AB}\langle\phi_i|$, $M_i|\phi_i\rangle_{AB} \neq 0$ and $M_i|\phi_j\rangle_{AB} = 0, j = 1, \ldots, i-1, i+1, \ldots, mn$. Generally, there exists at least one M_i identifying state $|\phi_i\rangle_{AB}$, which implies $t \geq mn$, so any two different M_i's are either parallel or orthogonal. We complete the proof.

Corollary provides an effective method to determine the distinguishability of states.

Suppose that some LOCC protocol $\{M_i\}$ can locally distinguish the set of orthogonal states $\{\psi_i\rangle\}$. We project the operators $\{M_i\}$ into the subspace spanned by states $\{\psi_i\rangle\}$ and define $M_i' = PM_iP$. If we can prove that there exist two operators in $\{M_i'\}$ which are neither parallel nor orthogonal, then we can conclude that the supposed LOCC protocol $\{M_i\}$ does not exist. More generally, if we can prove that for any LOCC protocol $\{M_i\}$, there always exist two operators in $\{M_i'\}$ which are neither parallel nor orthogonal, the states in $\{\psi_i\rangle\}$ cannot be discriminated by LOCC protocols.

In conclusion, we proved that if a set of orthogonal states can be distinguished by RLPCC, then the sum of their Schmidt ranks is not greater than the dimension of their Hilbert space. This criterion is stronger than previous results in Ref.[5] in certain situations. For orthogonal product basis states, we showed that distinguishability by LOCC implies distinguishability by RLPCC, where we emphasize that states stay unchanged in every local measurement. This allows us to derive a corollary which offers an effective way to determine local indistinguishability.

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