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The remote implementation of all possible generalized quantum measurements on a single atomic qubit in a quantum network*

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To implement generalized quantum measurement (GQM) one has to extend the original Hilbert space. Generally speaking, the additional dimensions of the ancilla space increase as the number of the operators of the GQM n increases. This paper presents a scheme for deterministically implementing all possible n-operator GQMs on a single atomic qubit by using only one 2-dimensional ancillary atomic qubit repeatedly, which remarkably reduces the complexity of the realistic physical system. Here the qubit is encoded in the internal states of an atom trapped in an optical cavity and single-photon pulses are employed to provide the interaction between qubits. It shows that the scheme can be performed remotely, and thus it is suitable for implementing GQM in a quantum network. What is more, the number of the total ancilla dimensions in our scheme achieves the theoretic low bound.

Keywords: generalized quantum measurement, single atomic qubit, cavity-QED
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1. Introduction

The rapid development in quantum information theory and its applications results in renewed interest in the theory of generalized quantum measurement (GQM). In fact, GQM is intimately related to two fundamental tasks of quantum information processing (QIP). On the one hand, one can optimally extract information from quantum states via GQM, e.g., in quantum state discrimination or entanglement detection. On the other hand, one can obtain a desired state by GQM, e.g., in teleportation or entanglement distillation.[1] Hence, over the past few decades the implementation of GQM on various kinds of physical systems has been increasingly studied.[2−9]

The standard implementation of GQM is based on the Neumark's theorem,[2] which requires an ancilla space to extend the original Hilbert space. Generally speaking, the additional dimensions of the ancilla space increase as the number of the operators of the GQM n increases. However, for many physical systems, it is difficult or impossible to find enough extra dimensions for the extension, let alone perform operations on the extended system. This realistic challenge has motivated a resurgence of interest in the problem of economically implementing GQM with less ancilla resource on different physical systems.[3−7]

One possible candidate of physical systems for QIP is the cavity quantum electrodynamics (QED) system with a trapped atom.[10−16] Compared with the photonic qubit, we find that the atomic qubit is more suitable for quantum memory and large-scale quantum computation. Moreover, utilizing photons for quantum transmission between trapped atoms in spatially separated cavities is the main idea of the quantum network.[17] Hence the implementation of GQM on an atomic qubit is particularly attractive for researchers. Franke-Arnold *et al*[9] devised a scheme to implement all possible n-operator GQMs on a single atomic qubit, which used another n level of the atom as the ancilla space. Unfortunately, to find n stable levels of the atom and perform unitary transformations on the entire space is not easy, especially when n is large.

In this paper, we present an explicit scheme to deterministically implement all possible n-operator GQMs on a single atomic qubit via cavity QED. Our

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scheme has four outstanding advantages. (i) Only one atomic qubit is required as the ancilla qubit. By appropriately entangling two atomic qubits (the original qubit and the ancilla qubit) and sequentially measuring the ancilla qubit, we can implement all possible n-operator GQMs on the original qubit step by step. Thus our scheme is simpler and more realizable. (ii) The scheme can be performed remotely in a quantum network model. (iii) The roles of the original qubit and the ancilla qubit are exchangeable. (iv) The number of the total ancilla dimensions reaches the theoretic minimum.

2. Implementation of GQM (2-operator, n-operator and extension)

Mathematically a GQM corresponds to a set of positive-definite operators $\{E_i, 1 \leq i \leq n\}$, satisfying

$$\sum_{i=1}^{n} E_i = I, \tag{1}$$

where I represents the unit matrix. Each possible outcome i associates with an operator $E_i = M_i^\dagger M_i$, where M_i denotes the Kraus operator. If the quantum system is initially prepared in the state ρ^{in}, the post-measurement state becomes ρ_i^{out}, where

$$\rho_i^{\text{out}} = \frac{M_i \rho M_i^\dagger}{\text{tr}(M_i \rho M_i^\dagger)} \tag{2}$$

with the probability

$$p_i = \text{tr}(M_i \rho M_i^\dagger). \tag{3}$$

Accordingly, an arbitrary 2-operator GQM on two-dimensional Hilbert space can be expressed as

$$E + \tilde{E} = M^\dagger M + \tilde{M}^\dagger \tilde{M} = I\,,$$

where both M and $\tilde{M}$ are 2×2 matrixes. For arbitrary M and $\tilde{M}$, we obtain the uniquely determined polar decomposition (PD)

$$\begin{aligned} M &= S\sqrt{E}, \\ \tilde{M} &= \tilde{S}\sqrt{\tilde{E}}, \end{aligned} \tag{4}$$

where S and $\tilde{S}$ are unitary operators. Since

$$E\tilde{E} = (I - \tilde{E})\tilde{E} = \tilde{E}(I - \tilde{E}) = \tilde{E}E\,,$$

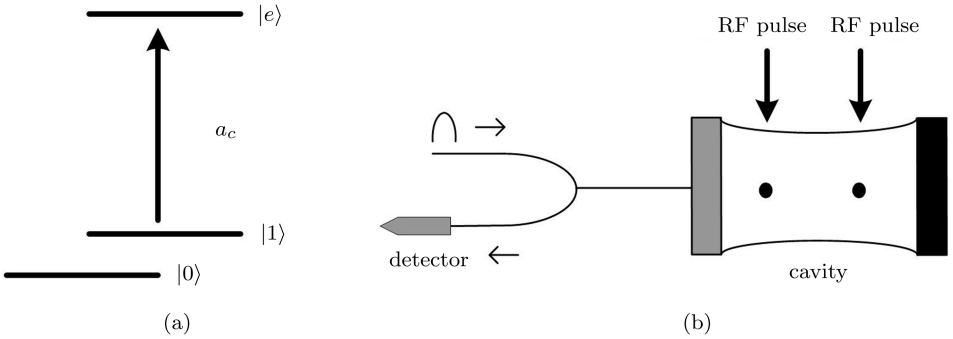
i.e., E commutes with $\tilde{E}$, E and $\tilde{E}$ are simultaneously diagonalizable by a certain unitary operator U_1. Thus, equation (4) can be rewritten as

$$\begin{aligned} M &= SU_1^\dagger \Lambda U_1 = V_1 \Lambda U_1, \\ \tilde{M} &= \tilde{S}U_1^\dagger \tilde{\Lambda} U_1 = \tilde{V}_1 \tilde{\Lambda} U_1, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \Lambda &= \Lambda^\dagger = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2})\,, \\ \tilde{\Lambda} &= \tilde{\Lambda}^\dagger = \text{diag}(\sqrt{\tilde{\lambda}_1}, \sqrt{\tilde{\lambda}_2})\,, \end{aligned}$$

$\{\lambda_1, \lambda_2\}$ and $\{\tilde{\lambda}_1, \tilde{\lambda}_2\}$ denote the non-negative eigenvalues of E and $\tilde{E}$, respectively. It should be noted that equation (5), i.e., the PD of the two Kraus operators of 2-operator GQM, is a fundamental equation for the discussion of the implementation of GQM below.



(a) (b)

Fig.1. (a) The energy level diagram of atoms and the coupling configuration. (b) The schematic setup to realize all possible GQMs of a single atomic qubit inside a cavity. The radio frequency (RF) pulses are used to produce arbitrary rotation of the single atomic qubit coded on $\{|0\rangle, |1\rangle\}$. A single photon pulse is reflected from the cavity to get a robust controlled phase flip (CPF) gate between the two atoms.

In the following, we present an explicit scheme for realizing all possible GQMs on a single atomic qubit. Firstly we concentrate on the implementation of all possible 2-operator GQMs. The physical system of our scheme is shown in Fig.1. Two identical three-level atoms are well located in a high-finesse single-mode cavity. Atomic qubits are encoded on the two stable ground states $|0\rangle$ and $|1\rangle$. Single atomic qubit rotation is provided by RF pulses on each of the atoms. The state $|e\rangle$ is a low excited state. The atomic transition $|1\rangle \rightarrow |e\rangle$ resonantly interacts with the cavity mode a_c, which can also be resonantly driven by the input photon. A basic element of our scheme is a controlled phase flip (CPF) gate between the two atoms generated by cavity-assisted photon scattering.[14,15] Suppose that the duration of the input photon pulse is sufficiently long and the atom-cavity coupling is much stronger than both the cavity decay and spontaneous emission of the atomic state. If both atoms are in the state $|0\rangle$, the input photon pulse will interact resonantly with the bare cavity mode, and the reflected pulse would yield a pulse shape almost unchanged but with an added π phase. If either of the two atoms is in the state $|1\rangle$, the effective frequency of the dressed cavity mode will be shifted due to the atom-cavity coupling. Thus the input photon will be reflected by an off-resonant cavity with both its shape and phase unchanged. Therefore, after a detection of the reflected photon, the net effect of the above processes is a CPF gate between the atoms described by $U^{a,b} = \exp(\mathrm{i}\pi|0\rangle_a\langle 0| \otimes |0\rangle_b\langle 0|)$.

Suppose that the initial state of a two-atom system is $|\psi_0\rangle_a \otimes |0\rangle_b$, where atom a and atom b denote the original qubit and the ancilla qubit, respectively. Arbitrary 2-operator GQM $E + \tilde{E} = M^\dagger M + \tilde{M}^\dagger \tilde{M} = I$ can be realized by a combination of $U^{a,b}$ and single-qubit rotation operations, according to the following steps.

(i) Perform single-qubit rotation U_1^a described in Eq.(5) on atom a via an RF pulse:

$$|\psi_0\rangle_a|0\rangle_b \xrightarrow{U_1^a} (\alpha|0\rangle_a + \beta|1\rangle_a)|0\rangle_b, \tag{6}$$

where α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$.

(ii) Perform the unitary operation $U(\theta) = H^b U^{a,b} R_y^b(\theta/2) U^{a,b} H^b R_y^b(\theta/2)$ on the system, where the superscripts a and b indicate which atom is operated on, H represents a Hadamard gate, and $R_y(\theta) = \exp(-\mathrm{i}\theta\sigma_y/2)$ ($\{\sigma_k\}, (k = x, y, z)$ represent Pauli operators). In this step, two single photon pulses are required in order to realize the CPF gate $U^{a,b}$ twice. The single atomic qubit rotations H^b and $R_y^b(\theta/2)$ are provided by RF pulses. The total effect of $U(\theta)$ can be regarded as a controlled rotation operator $C_{R_y(\theta)}^{(a-b)} = |0\rangle_a\langle 0| \otimes R_y(\theta)_b + |1\rangle_a\langle 1| \otimes I_b$. Consequently the state of the system evolves into

$$\alpha|0\rangle_a(\cos\theta/2|0\rangle_b + \sin\theta/2|1\rangle_b) + \beta|1\rangle_a|0\rangle_b. \tag{7}$$

(iii) In a similar manner, as given in step (ii), next we perform the unitary operation $U(\gamma) = \sigma_x^a H^b U^{a,b} R_y^b(\gamma/2) U^{a,b} H^b R_y^b(\gamma/2) \sigma_x^a$ on the system. The state of the system becomes

$$\begin{aligned} &\alpha|0\rangle_a(\cos\theta/2|0\rangle_b + \sin\theta/2|1\rangle_b) \\ &+ \beta|1\rangle_a(\cos\gamma/2|0\rangle_b + \sin\gamma/2|1\rangle_b) \\ =\ &(\alpha\cos\theta/2|0\rangle_a + \beta\cos\gamma/2|1\rangle_a)|0\rangle_b \\ &+ (\alpha\sin\theta/2|0\rangle_a + \beta\sin\gamma/2|1\rangle_a)|1\rangle_b. \end{aligned} \tag{8}$$

Noting Eq.(5), let $\sin\gamma/2 = \sqrt{\lambda_1}$ and $\sin\theta/2 = \sqrt{\lambda_2}$ hold simultaneously by tuning the values of θ and γ, we can rewrite the state described by Eq.(8) as

$$\tilde{\Lambda} U_1|\phi_0\rangle_a|0\rangle_b + \Lambda U_1|\phi_0\rangle_a|1\rangle_b. \tag{9}$$

(iv) Then we detect the state of atom b. If the result of the measurement is $|0\rangle_b$ (with the probability $\tilde{p} = {}_a\langle\phi_0|U_1^\dagger \tilde{\Lambda}^\dagger \tilde{\Lambda} U_1|\phi_0\rangle_a$), we perform the unitary operation $\tilde{V}_1$ described in Eq.(5) on atom a so that the final state is $\tilde{V}_1 \tilde{\Lambda} U_1|\phi_0\rangle_a = \tilde{M}|\phi_0\rangle_a$. If the result of the measurement on atom b is $|1\rangle_b$ (with the probability $p = {}_a\langle\phi_0|U_1^\dagger \Lambda^\dagger \Lambda U_1|\phi_0\rangle_a$), we perform the unitary operation V_1 described in Eq.(5) on atom a so that the final state is $V_1 \Lambda U_1|\phi_0\rangle_a = M|\phi_0\rangle_a$. According to Eqs.(2) and (3), we have already implemented an arbitrary 2-operator GQM of a single atomic qubit.

Next, we consider how to implement GQM with many operators by sequential 2-operator GQM procedures.[3–5,7] It is instructive to look at the implementation of all possible 3-operator GQMs first. For an arbitrary 3-operator GQM $N_1^\dagger N_1 + N_2^\dagger N_2 + N_3^\dagger N_3 = I$, we can implement it by using the 2-operator GQM procedure twice. At the beginning, we implement the 2-operator GQM $\{M_1 = N_1, \tilde{M}_1 = \sqrt{I - M_1^\dagger M_1}\}$. If the outcome corresponds to M_1, i.e., N_1, the implementation of the 3-operator GQM is finished. If the outcome corresponds to $\tilde{M}_1$, we continue to perform another 2-operator GQM with the operators $\{M_2, \tilde{M}_2\}$ satisfying $\{M_2\tilde{M}_1 = N_2, \tilde{M}_2\tilde{M}_1 = N_3\}$ on the system. One can easily verify that $M_2^\dagger M_2 + \tilde{M}_2^\dagger \tilde{M}_2 = I$. Thus an arbitrary 3-operator GQM has been implemented with the operators' corresponding

relations $M_1 \to N_1$, $M_2 \to N_2$ and $\tilde{M}_2 \to N_3$. Similarly, an arbitrary n-operator GQM $\sum_{i=1}^{n} N_i^\dagger N_i = I$ on atom a can be implemented by utilizing $n-1$ sequential 2-operator GQM procedures[3−5,7] (see Fig.2). For each 2-operator GQM procedure, the collective unitary operation U_C^i represents all the unitary operations in steps (i)–(iii) in the 2-operator GQM procedure. The operators of each 2-operator GQM can be determined by[3−5,7]

$$M_1 = N_1, \quad M_i \prod_{j=1}^{i-1} \tilde{M}_{i-j-1} = N_i \quad (2 \le i \le n-1),$$

$$\tilde{M}_i = \sqrt{I - M_i^\dagger M_i} \quad (1 \le i \le n-2),$$

$$\tilde{M}_{n-1} \prod_{j=1}^{i-1} \tilde{M}_{i-j-1} = N_n. \tag{10}$$

From the above description, one can see that atom b is used repeatedly as the ancilla in our scheme. Noting Eq.(9), once we get a result corresponding to $\tilde{M}_i$ and need to perform the next 2-operator POVM procedure sequentially, the state of atom b collapses to its initial state $|0\rangle_b$. Hence we can carry out the next 2-operator GQM procedure with atom b without initializing it every time.

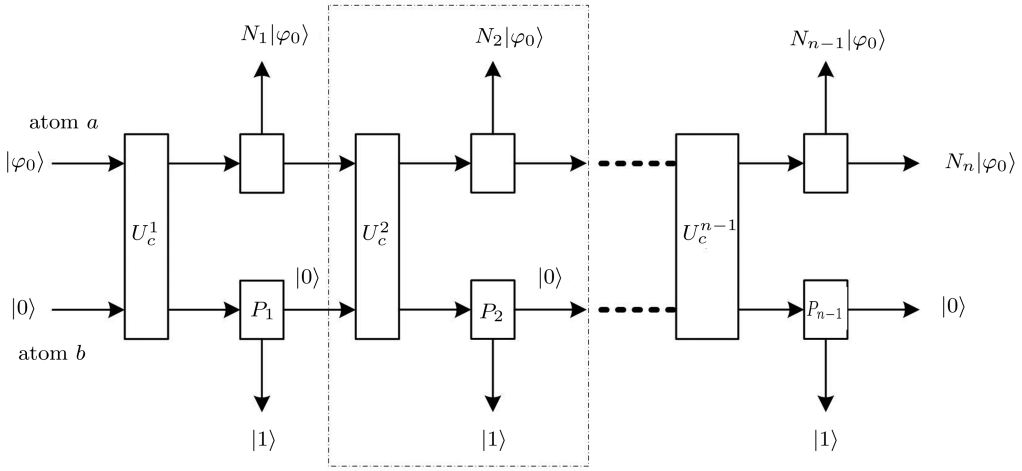


Fig.2. The circuit for implementing all possible n-operator GQMs on atom a by iteratively using an ancilla qubit atom b. The circuit includes $n-1$ sequential 2-operator GQM procedures, one of which is shown in the point-line pane. The P_i is a projective measurement on atom b with the computational bases $|0\rangle$ and $|1\rangle$. The U_c^j represents the total effect of all the unitary operations in steps (i)–(iii) described above, and by U_c^j the original atom a and the ancilla atom b are appropriately entangled in every 2-operator GQM procedure.

The above scheme can also be performed remotely, as illustrated in Fig.3. We consider two atoms (belonging to Alice and Bob respectively) in distant optical cavities, interacting with the local cavity fields. Alice and Bob can be regarded as two nodes of a large quantum network and Charlie may be a net administrator.[17] A similar physical setup has been proposed to carry out CPF gate in Ref.[15]. Since our scheme is a combination of single qubit rotations (here provided by the RF pulses), CPF gates and single qubit projective measurements, our scheme is available here without modification. One challenge of this scheme is the performance of operations on atoms at appropriate times with respect to the photon reflection. We can use classical communication to realize precise operation ordering. Note that Charlie can implement arbitrary GQM remotely on Alice's (Bob's) atom with the assistance of Bob (Alice). Therefore the roles of the two nodes are interchangeable and we do not need to decide which one is the ancilla before measurement. Suppose there are many nodes in a large quantum network. Our scheme allows the administrator to optionally choose a free node to be the ancilla for implementing GQM on another one without disturbing the others.

In the following, some discussions about our scheme are presented. Our scheme is different from the "binary measurement tree" in Ref.[7]. In Ref.[7] the measurement operator $\{M_i, \tilde{M}_i\}$ and the collective unitary rotation U_c^i of every 2-operator GQM procedure depend on the prevenient outcomes. Thus a classical feedback element

is needed in the setup of Ref.[7]. In our scheme the middle outcomes only decide whether we should continue the next procedure or not. Hence, by calculating all the operations from Eq.(10), our scheme would work in a repeat-until-success fashion without any classical feedback. Our scheme is also different from the schemes in Refs.[3]–[5] for the optical systems. Since the polarized photonic qubit is hard to be measured non-destructively, the schemes in Refs.[3]–[5] do not save the resource of the ancilla so that the optical system in Refs.[3]–[5] will be quite complex when n is large. However, in our scheme the one-atomic-qubit ancilla is repeatedly used, thus the complexity of the physical system for our scheme is significantly reduced.

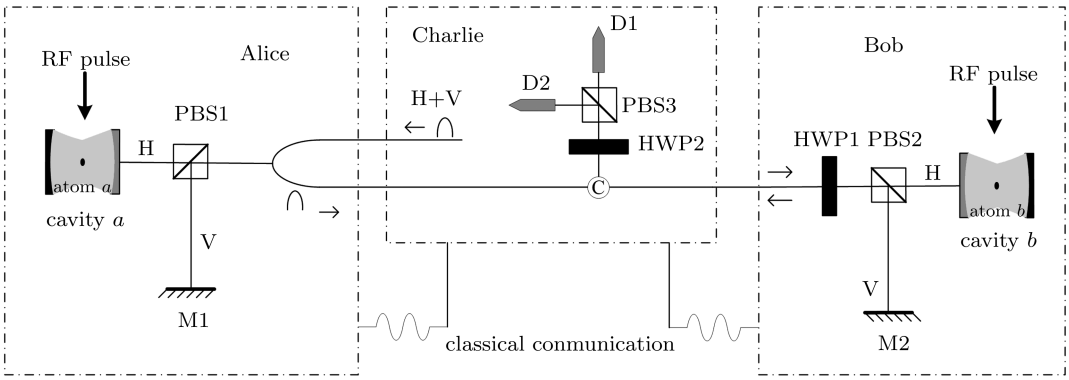


Fig.3. A schematic setup for implementing all possible GQMs on one atom catalyzed by another remote atom. The symbols PBS1, PBS2 and PBS3 represent polarization beam slitters. The symbols HWP1, HWP2 are half-wave plates. The symbols M1 and M2 symbolize reflecting mirrors. The symbols D1 and D2 are single-photon detectors, and the symbol C denotes a circulator. The RF pulses are used to produce single qubit rotation. The optical path from PBS1 to M1 (PBS2 to M2) and from PBS1 to cavity a (PBS2 to cavity b) are supposed to be equal.

Recently, Chen *et al*[6] investigated the theoretic low bound of ancilla dimensions needed for implementing GQMs. According to their result, to implement all possible n-operator GQMs on one qubit one needs to add an ancilla space with dimension $n_{\min} = 2n - 2$ at least. In every 2-operator GQM procedure of our scheme, the number of the added dimensions is 2. Note that by implementing the n-operator GQM on a single atomic qubit one needs to employ the 2-operator GQM procedure $n - 1$ times. If all the extra dimensions are counted, with the "sequentially occurring dimensions" counted separately, the total number of ancilla dimensions is $2(n - 1)$, which is equal to the theoretic minimum in Ref.[6]. In addition, $n_{\min}$ is the low bound of the number of ancilla dimensions that ensures the implementation of arbitrary GQM. By the meaning of adding the ancilla step-by-step in our scheme, since the GQM may be finished in the midst of the $n - 1$ 2-operator GQM procedure with certain probability, the practical number of the added ancilla dimensions may be less than $n_{\min}$. Note that if our scheme is finished in the i-th 2-operator GQM procedure (with the probability $\langle\phi_0| N_i^\dagger N_i |\phi_0\rangle$), the number of the added dimensions is $2i$. We calculate the average number of the added dimensions of our scheme, which is

$$\bar{n} = 2(n-1)\langle\phi_0| N_n^\dagger N_n |\phi_0\rangle + \sum_{i=1}^{n-1} 2i \langle\phi_0| N_i^\dagger N_i |\phi_0\rangle, \quad (11)$$

this is usually less than $n_{\min}$.

It is necessary to discuss the experimental feasibility of the proposed scheme. The basic building block $U^{a,b}$ of our scheme has been well studied previously by numerical simulation.[14,15] Let T and κ denote the duration of the input photon pulse and the cavity decay rate, respectively. Under the condition $\kappa T \gg 1$, $U^{a,b}$ is shown to be insensitive to atom-cavity coupling strength and Lamb–Dicke localization. Considering $\kappa T \sim 100$ and the atom-cavity coupling strength g is several times larger than κ and Γ (Γ is the atomic spontaneous emission rate), we find that the gate fidelity of $U^{a,b}$ is almost unity (about 99.5%) and the spontaneous emission probability is quite small.[15,18]

In fact, the dominant noise of our scheme comes from the photon loss. Photon loss would affect the subsequent interactions, leading to a failure and restart of the scheme. Actually in experiments, except for the spontaneous emission, there are several sources of photon loss, such as the absorptions of the cavity and the transmission line, the cavity scattering, the detector inefficiency and so on. However, with detectors at the end of our design, we can discard all those photon loss events. Thus, the photon loss events only affect the success probability, while doing no damage to the fidelity of GQM. As completely implementing an n-operator GQM we may need $4(n-1)$ single photon pulses, a stable and controllable single-photon source is necessary. Recently, a single-photon source which can continuously and deterministically generate 10000/s high-quality single photons has been reported.[19] With the rapid development of single-photon experimental technology, we hope that our scheme can be achieved in the near future.

3. Conclusion

To sum up, we have presented a scheme to implement arbitrary n-operator GQMs on a single atomic qubit via cavity QED, which is assisted by only one remote atomic ancilla qubit. The total number of the dimension of the ancilla system has achieved the theoretical minimum. Since the roles of the two atomic qubits are exchangeable in our proposal, our scheme can be well applied to the implementation of all possible GQMs on a single atomic qubit in a quantum network. We also point out that the idea of using the ancilla repeatedly to implement GQMs is available for any other non-destructively measurable physical ancilla system.

References

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

[2] Neumark M A and Akad I 1940 *Nauk SSSR Ser. Mat.* **4** 277

[3] Ahnert S E and Payne M C 2005 *Phys. Rev.* A **71** 012330

[4] Wang G M and Ying M S 2006 arXiv:quant-ph/0608235

[5] Ahnert S E and Payne M C 2006 *Phys. Rev.* A **73** 022333

[6] Chen P X Bergou J A Zhu S Y and Guo G C 2007 *Phys. Rev.* A **76** 060303

[7] Erika Andersson and Oi D K L 2008 *Phys. Rev.* A **77** 052104

[8] Zheng S B 2003 *Chin. Phys.* **12** 51

[9] Andersson F A S, Barnett E S M and Stenholm S 2001 *Phys. Rev.* A **63** 052301

[10] Turchette Q A, Hood C J, Lange W, Mabuchi H and Kimble H J 1995 *Phys. Rev. Lett.* **75** 4710

[11] Wei H, Deng Z J, Zhang X L and Feng M 2007 *Phys. Rev.* A **76** 054304

[12] Cirac J I, Zoller P, Kimble H J and Mabuchi H 1997 *Phys. Rev. Lett.* **78** 3221

[13] Pachos J and Walther H 2002 *Phys. Rev. Lett.* **89** 187903 Wei H, Fang R, Liu J B, Zhou F, Yang W L and Deng Z J 2008 *J. Phys.* B: *At. Mol. Opt. Phys.* **41** 085506

[14] Duan L M, Wang B and Kimble H J 2005 *Phys. Rev.* A **72** 032333

[15] Lin X M, Zhou Z W, Ye M Y, Xiao Y F and Guo G C 2006 *Phys. Rev.* A **73** 012323

[16] Wei H, Yang W L, Deng Z J and Feng M 2008 *Phys. Rev.* A **78** 014304

[17] Blinov B B, Moehring D L, Duan L M and Monroe C 2004 *Nature* (London) **428** 153

[18] Duan L M and Kimble H J 2004 *Phys. Rev. Lett.* **92** 127902

[19] Hijlkema M, Weber B, Specht H P, Webster S C, Kuhm A and Rempe G 2007 *Nature Phys.* **3** 253