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Entanglement dynamics of a Heisenberg chain with Dzyaloshinskii–Moriya interaction*

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This paper investigates the entanglement dynamics of the system, composed of two qubits A and B with Heisenberg XX spin interactation. There is a third controller qubit C, which only has Dzyaloshinskii–Moriya (DM) spin-orbit interaction with the qubit B. It is found that depending on the initial state of the controller qubit C and DM interaction, the entanglement of the system displays amplification and sudden birth effects. These effects indicate that one can control the entanglement of the system, which may be helpful for quantum information processing.

Keywords: entanglement sudden birth, entanglement amplification, DM interaction
PACC: 0365, 4250

1. Introduction

Entanglement not only plays a basic role in quantum theory,[1,2] but also is a key resource in quantum information processing (QIP),[3] for example, quantum teleportation,[4,5] quantum dense coding,[6] and quantum cryptography.[7] Because of its importance, many proposals have been suggested to generate the entanglement states.[8−10]

The solid-state system, especially the spin chain, is the natural candidate for a quantum information processer, as it is believed to have the best scalability and the exchange interaction between spins enables it to establish the two-qubit entanglement. Bose[11] firstly noticed that the spin chain can be used as a channel for short distance quantum communications. One kind of static entanglement, thermal entanglement of the spin chain has been extensively studied.[12−14] Recently, in addition to spin-spin interaction, the effect of spin-orbit DM interaction[15,16] on QIP has excited wider interest,[17−20] as it provides one new parameter to control the entanglement. The anisotropic and antisymmetric DM interaction was introduced about fifty years ago, to explain the weak ferromagnetism of antiferromagnetic crystals. The DM interaction can lead to higher mutual information of the two-qubit Heisenberg chain, especially for the ferromagnetic case.[18] Quantum teleportation by the thermal entanglement of a two-qubit Heisenberg chain is investigated in the presence of DM interaction.[19,20] More recently, the influence of the x-component of the DM interaction on entanglement has been reported.[21,22]

Entanglement dynamics is another important field. One interesting phenomenon, so-called entanglement sudden death (ESD) was found by Yu *et al.*[23] The special property of ESD is that entanglement disappears non-smoothly for a finite time, out of the expectation that entanglement disappears smoothly. Moreover, Almeida *et al* have observed ESD in an optical system.[24] For the spin chain, there are ESD effects in the two-qubits mixed state with the Ising spin chain,[25] and two qubits coupled by a spin channel with an XY-type interaction.[26] Contrary to the currently extensively discussed sudden death of entanglement, another intriguing phenomenon, entanglement sudden birth (ESB) is also found.[27] As its name suggests, ESB means that entanglement can also be created non-smoothly.

In this paper, similar to Ref.[28], we consider the entanglement dynamics of the system composed

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of two qubits A and B. The difference from Ref.[28] is that there is Heisenberg XX interaction between them. Similarly, the third controller qubit C is introduced,[28] which only has DM interaction with the qubit B. We will show that DM interaction and the initial state of controller qubit C are the efficient controlling parameters for the entanglement of the system qubits A and B. Depending on these two parameters, there are ESB and entanglement amplification effects for our two-qubits system. Our results show that statements like "entanglement should not increase under local operations and classical communication" are not generally true outside the set of local operations considered in the original proof.[29] Moreover, these effects provide a resource for the creation of entanglement between two qubits on demand.

2. Measure of entanglement

For the bipartite state, there are some equivalent measures of the degree of entanglement, such as concurrence, entanglement entropy, negativity, tangle. In this paper, we adopt the Wootters concurrence.[30] For two qubits, the concurrence can be calculated by

$$C(\rho) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}), \quad (1)$$

where λ_i $(i = 1, 2, 3, 4)$ are the eigenvalues in decreasing order of the matrix

$$R = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y). \quad (2)$$

Here ρ^* is the complex conjugation of ρ in the standard basis and σ_y is the Pauli matrix

$$\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \quad (3)$$

The concurrence is monotone for the bipartite entanglement, $C = 0$ corresponds to the separable state, and $C = 1$ to the maximal entanglement one. For density matrix

$$\rho = \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ w^* & 0 & 0 & d \end{pmatrix}, \quad (4)$$

the concurrence can be easily obtained,[31]

$$C(\rho) = 2\ \max\{0, |z| - \sqrt{ad}, |w| - \sqrt{bc}\}. \quad (5)$$

3. Entanglement dynamics of system qubits

We focus on the entanglement dynamics of the system consisting of two qubits A and B. The interaction between them is the Heisenberg XX spin model. The third controller qubit C is introduced, which only has DM interaction with the qubit B. The Hamiltonian of the whole system is

$$H = H_{\mathrm{sys}} + H_{\mathrm{DM}}, \quad (6)$$

where the system interaction between qubits A and B is

$$H_{\mathrm{sys}} = \frac{1}{2}\ w\ \sigma_{\mathrm{A}}^x\ \sigma_{\mathrm{B}}^x + \frac{1}{2}\ w\ \sigma_{\mathrm{A}}^y\ \sigma_{\mathrm{B}}^y, \quad (7)$$

and that between qubit B and C is

$$H_{\mathrm{DM}} = \boldsymbol{D} \cdot (\boldsymbol{\sigma}_B \times \boldsymbol{\sigma}_C). \quad (8)$$

Here w is the coupling parameter between qubits of the system, σ^x and σ^y are Pauli matrices. Choosing $\boldsymbol{D} = D\boldsymbol{z}$, we can reduce DM interaction H_{DM} to[19]

$$H_{\mathrm{DM}} = D(\sigma_{\mathrm{B}}^x\ \sigma_{\mathrm{C}}^y - \sigma_{\mathrm{B}}^y\ \sigma_{\mathrm{C}}^x). \quad (9)$$

This paper only considers the antiferromagnetic coupling $w > 0$ case, and we are working in units where D and w are dimensionless.

Without loss of generality, one can define state $|g\rangle$ ($|e\rangle$) as the ground (excited) state of a qubit. It is convenient to choose the initial state of the whole system as

$$\rho_0 = \rho^{\mathrm{AB}}(0) \otimes \rho^{\mathrm{C}}(0) \quad (10)$$

with

$$\rho^{\mathrm{AB}}(0) = r|\varphi^{\mathrm{AB}}\rangle\langle\varphi^{\mathrm{AB}}| + \frac{1}{4}(1-r)I_{4\times4},$$
$$\rho^{\mathrm{C}}(0) = |\varphi^{\mathrm{C}}\rangle\langle\varphi^{\mathrm{C}}|. \quad (11)$$

Where $I_{4\times4}$ is (4×4) the identity matrix of the state of the system's qubits

$$|\varphi^{\mathrm{AB}}\rangle = \cos(\alpha)|eg\rangle + \sin(\alpha)|ge\rangle, \quad (12)$$

in which the first index representing the state of qubit A and the second for that of B. When $\alpha = \pi/4$, $|\varphi^{\mathrm{AB}}\rangle$ reduces to the Bell state. The state of the controller qubit is

$$|\varphi^{\mathrm{C}}\rangle = \cos(\gamma)|e\rangle + \sin(\gamma)|g\rangle. \quad (13)$$

The density matrix of the system and the controller at arbitrary time t is

$$\rho(t) = U(t)\rho_0 U(t)^\dagger \quad (14)$$

with the evolution operator $U(t) = \exp[-\mathrm{i}\,Ht]$. Tracing over the state of the controller qubit C, we get the reduced density matrix of qubits A and B

$$\rho^{\mathrm{AB}} = \mathrm{Tr}_{\mathrm{C}}[\rho(t)]. \tag{15}$$

We firstly study one simple case $r = 0$, $w = 2$, and $\gamma = \pi/2$, which is possible to get the analytical result. After some straightforward calculations, we obtain the elements of the unitary operator $U(t)$,

$$\begin{aligned}
&U_{11} = U_{88} = 1,\\
&U_{22} = \frac{1 + (-1 + \Delta)\,\cos(2\,t\,\Omega)}{\Delta} = U_{77},\\
&U_{23} = \frac{D\,\sin(2\,t\,\Omega)}{\Omega} = -U_{32},\\
&U_{25} = \frac{-2\,\mathrm{i}\;D\,\sin(t\,\Omega)^2}{\Delta} = [U_{52}]^*,\\
&U_{33} = \cos(2\,t\,\Omega) = U_{66},\\
&U_{35} = \frac{\sin(2\,t\,\Omega)}{\sqrt{-\Delta}} = U_{53},\\
&U_{44} = \frac{-1 + \Delta + \cos(2\,t\,\Omega)}{\Delta} = U_{55},\\
&U_{46} = \frac{\sin(2\,t\,\Omega)}{\sqrt{-\Delta}} = U_{64},\\
&U_{47} = \frac{-2\,\mathrm{i}\;D\,\sin(t\,\Omega)^2}{\Delta} = [U_{74}]^*,\\
&U_{67} = \frac{D\sin(2\,t\,\Omega)}{\Omega} = -U_{76}
\end{aligned} \tag{16}$$

with $\Delta = 1 + D^2 = \Omega^2$, the other elements equal zero. Note that in this case there is no initial entanglement between the system qubits A and B. According to Eqs.(14) and (15), the density matrix of the system qubits ρ^{AB} is obtained at arbitrary time t, when DM interaction $D = 0$, which is the same to the initial state

$$\rho_1(t)^{\mathrm{AB}} = \frac{1}{4} I_{4\times 4} = \rho_1(t=0)^{\mathrm{AB}}. \tag{17}$$

Therefore, $C[\rho_1(t)^{\mathrm{AB}}] = 0$.

However, if $D = 1$, ρ^{AB} at arbitrary time becomes to

$$\rho_2(t)^{\mathrm{AB}} = \begin{pmatrix} \rho_{11}^{\mathrm{AB}} & 0 & 0 & 0\\ 0 & \rho_{22}^{\mathrm{AB}} & \rho_{23}^{\mathrm{AB}} & 0\\ 0 & \rho_{32}^{\mathrm{AB}} & \rho_{33}^{\mathrm{AB}} & 0\\ 0 & 0 & 0 & \rho_{44}^{\mathrm{AB}} \end{pmatrix} \tag{18}$$

with

$$\begin{aligned}
\rho_{11}^{\mathrm{AB}} &= \frac{2 + \cos(2\,\gamma)\,\mu\,\sin(\tau)^2}{8}|_{\gamma=\frac{\pi}{2}} = \cos(\tau)^4/4,\\
\rho_{22}^{\mathrm{AB}} &= \frac{2 - \cos(2\,\gamma)\,\nu\,\sin(\tau)^2}{8}|_{\gamma=\frac{\pi}{2}} = [2 + \nu\,\sin(\tau)^2]/8,\\
\rho_{33}^{\mathrm{AB}} &= \frac{2 + \cos(2\,\gamma)\,\nu\,\sin(\tau)^2}{8}|_{\gamma=\frac{\pi}{2}} = [2 - \nu\,\sin(\tau)^2]/8,\\
\rho_{44}^{\mathrm{AB}} &= \frac{2 - \cos(2\,\gamma)\,\mu\,\sin(\tau)^2}{8}|_{\gamma=\frac{\pi}{2}} = [2 + \mu\,\sin(\tau)^2]/8,\\
\rho_{23}^{\mathrm{AB}} &= \frac{-\,\mathrm{i}\cos(2\,\gamma)\;\cos(\tau)\;\sin(\tau)^3}{\sqrt{2}}|_{\gamma=\frac{\pi}{2}}\\
&= [\rho_{32}^{\mathrm{AB}}]^*|_{\gamma=\frac{\pi}{2}} = \mathrm{i}\cos(\tau)\;\sin(\tau)^3/\sqrt{2}.
\end{aligned} \tag{19}$$

Here $\tau = \sqrt{2}t$, $\mu = 3 + \cos(2\,\tau)$, and $\nu = 1 + 3\,\cos(2\,\tau)$. So, the concurrence of the system is

$$C[\rho_2^{\mathrm{AB}}] = 2\max\{0,\ |\rho_{23}^{\mathrm{AB}}| - \sqrt{\rho_{11}^{\mathrm{AB}}\rho_{44}^{\mathrm{AB}}}\}. \tag{20}$$

This demonstrates that when qubit C is in the state $|g\rangle$ and there is no DM interaction between qubit B and C, the concurrence of system will be almost zero [Eq.(17)]. However, if DM interaction $D = 1$, the entanglement of the system can be created [Eq.(20)] even its initial state is the separable state ($r = 0$). This simple case shows that DM interaction is one of the effective parameters to control the entanglement of the system.

For the general parameters γ and D, the full solution is quite complicated and its form is not particularly instructive, hence it will not be reproduced here. We perform a numerical simulation to gain knowledge about the effect of DM interaction and the initial state of the controller qubit on entanglement dynamics of the system. Our main results are summarized in Figs.1–3. Although we only concentrate on the initial state $|\varphi^{\mathrm{AB}}\rangle = \cos(\alpha)|eg\rangle + \sin(\alpha)|ge\rangle$, the similar phenomena are also found for the state $|\phi^{\mathrm{AB}}\rangle = \cos(\alpha)|ee\rangle + \sin(\alpha)|gg\rangle$, which are not shown here.

Figures 1(a), 1(b) and 1(c) are the evolutions of entanglement of the system for the initial separable state ($r = 0$). When there is no DM interaction and the controller qubit C is in ground state $|\varphi^{\mathrm{C}}\rangle = |g\rangle$ [Fig.1(a)], the entanglement of system always equals zero, while when DM interaction $D = 1$, the entanglement of the system can be created [Fig.1(b)]. This is the so-called ESB.[27] Interestingly ESB also depends on the initial state of the controller qubit. When $\gamma = \pi/4$, there is no ESB, even DM interaction $D = 1$ [Fig.1(c)]. In order to show the effect of the controller qubit C, figure 1(d) plots the maximum value of the concurrence $C_{\max}$ of system qubits versus γ. It is clear for some values of γ, $C_{\max}$ equals zero, i.e., there is no ESB. Figure 1(d) also displays when the state of controller bit C is $|g\rangle$ or $|e\rangle$, $C_{\max}$ is maximum. This case implies that the initial state of the qubit C is another parameter to control the entanglement of the system qubits A and B.

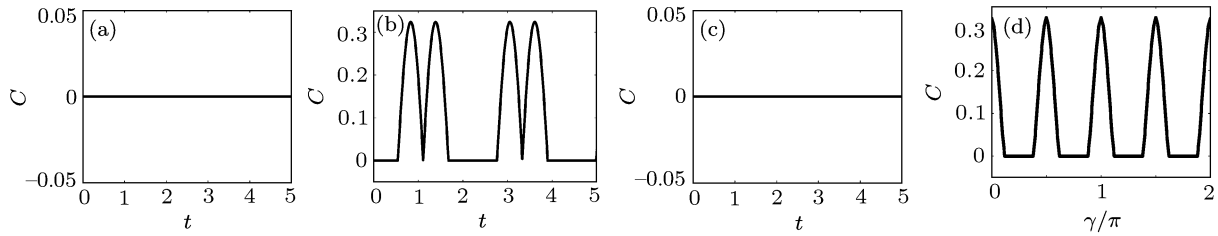


Fig.1. Figures (a), (b) and (c) are the evolutions of the concurrence of ρ^{AB} for the initial state of system $|\varphi^{\mathrm{AB}}\rangle$ with $\alpha = \pi/6$, corresponding to $(D,\ \gamma) = (0,\ \pi/2)$, $(1,\ \pi/2)$ and $(1,\ \pi/4)$, respectively. The other parameters are $w = 2$ and $r = 0$. Figure (d) is the variation of the maximum value of concurrence $C_{\max}$ with respect to γ/π. The other parameters are $w = 2$, $r = 0$ and $D = 1$.

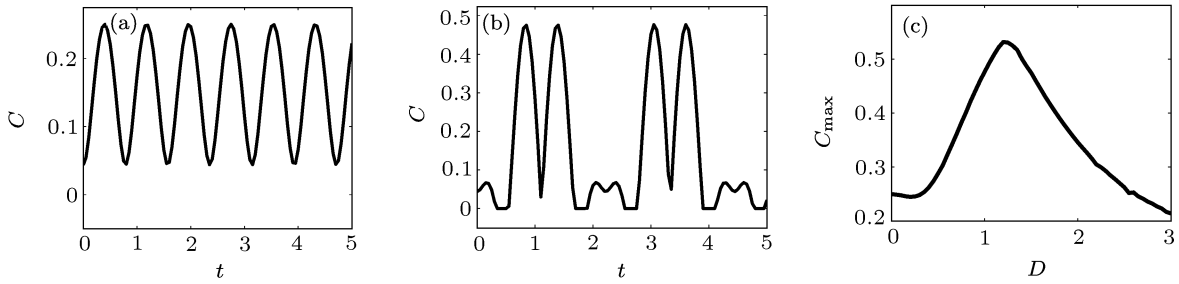


Fig.2. Figures (a) and (b) are the evolutions of the concurrence of ρ^{AB}, corresponding to $D = 0$ and $D = 1$, respectively. Figure (c) is the variation of the maximum of concurrence $C_{\max}$ with respect to D. The other parameters are $w = 2$, $r = 0.5$, $\alpha = \pi/10$ and $\gamma = \pi/2$.

Figure 2 shows that DM interaction can amplify the entanglement of the system when it is in the initial entanglement mixed state. When $D = 0$, the maximum value of system concurrence is about 0.25 [Fig.2(a)], while that is about 0.5 if $D = 1$ [Fig. 2(b)]. It increases twice as there is DM interaction. $C_{\max}$ versus DM interaction D is plotted in Fig.2(c), which has one small dip and one big hump. It means that for the small value of D, DM interaction makes $C_{\max}$ decrease, and there is the maximum amplification effect in the neighborhood of $D = 1.1$.

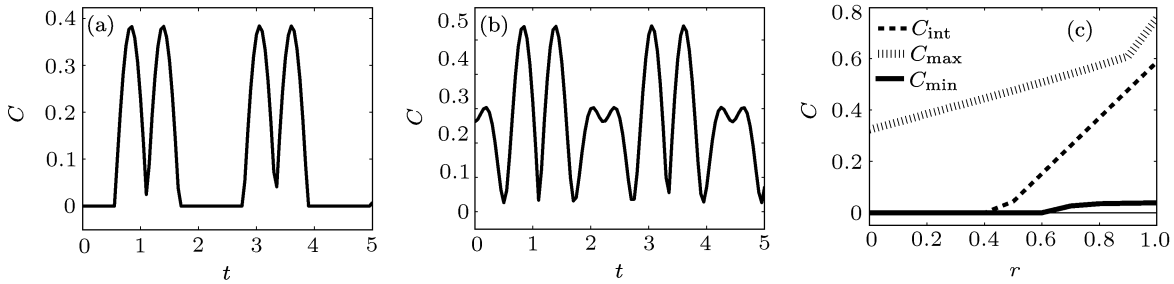


Fig.3. Figures (a) and (b) are the evolutions of concurrence with the different initial state of system, corresponding to $r = 0.2$ and $r = 0.7$, respectively. Figure (c) is the variation of the initial concurrence C_{int}, the minimum and maximum ones $C_{\min}$ and $C_{\max}$ versus r. The other parameters are the same as those in Fig.2 except that D is fixed to 1.

The initial state of the system in Fig.3(a) is the separable state $[C(t = 0) = 0]$. The DM interaction can induce the entanglement for system qubits at the later time. If the initial entanglement of the system is big enough, the system always keeps in entanglement state [Fig.3(b)]. Note only when r is bigger than $r_{\mathrm{m}} \simeq 0.4$ [Fig.3(c)], is the initial state of the system the entangled state. If one needs the system to always stay in the entangled state, that is, the minimum value of concurrence is above zero, figure 3(c) shows that the initial entanglement should be larger than $r_{\mathrm{m2}} \simeq 0.6 > r_m$. This point can be understood as follows. When the initial concurrence of the system is small, such as $r = 0.45$, by the interaction between qubits A, B and C, this

initial entanglement may transfer to other subsystems, (A, C) or (B, C). This makes the entanglement between qubits A and B vanish for some time. However, if the initial entanglement is big enough (for example, $r = 0.8$), the system qubits can keep its entanglement even if there is a transferring process. Figure 3(c) also exhibits $C_{\max} = k_1 r$ when $0 < r < 0.9$, while $C_{\max} = k_2 r$ if $r > 0.9$, and $k_1 \neq k_2$. This non-smooth change of the rates k_1 and k_2 needs further work.

4. Conclusion

In this paper, with entanglement degree of concurrence, we study the entanglement dynamics of the system qubits A and B with Heisenberg XX spin interaction. The third controller qubit C is introduced, which only has the Dzyaloshinskii–Moriya spin-orbit interaction with the qubit B. For one simple case, we get the analytical expression of the system state and find that DM interaction can change the initial separable state of the system to the entanglement mixed state. For the general parameters, numerical experiments show that there are effects of entanglement amplification and ESB, which can be controlled by the initial state of the controller qubit and DM interaction. This study sheds some light on the control of quantum entanglement by manipulation of DM interaction. It is expected that these effects may be helpful for QIP based on quantum entanglement.

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