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A rigorous criterion to identify the validity of the Born approximation*

Li Jian-Bing(李健兵)†, Wang Xue-Song(王雪松), and Wang Tao(王 涛)

College of Electronic Science and Engineering, National University of Defense Technology, Changsha 410073, *China*

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This paper intends to identify the validity of the orn approximation by a new universal criterion, which is ultimately reduced to the calculation of an operator norm. With the purpose of enabling the criterion to be applicable to general scattering problems, a method is proposed to estimate the norm of the operator concerned. Compared with the conventional criterion, this method excels in its ability to acquire a quantificational upper bound of the relative error by Born approximation as well as to extend its valid frequency to a wider range. Two canonical scattering examples are given as evidence for the validity of the criterion.

Keywords: electromagnetic scattering, Born approximation, operator theory, relative error
PACC: 0350D, 0380, 1180F, 4110H

1. Introduction

Scattering problems have been investigated for a very long time, and a lot of significant effort in this field has been made for the past few decades.

A general scattering situation can be depicted in Fig.1. In light of the Maxwell equations, the scattering field can be expressed as:[1]

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_{\mathrm{i}}(\boldsymbol{r}) - k^2 \iiint_{V'} \Delta\varepsilon_r(\boldsymbol{r}')\boldsymbol{E}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}', \tag{1}$$

where k is the wave number of the incident electromagnetic wave; $\boldsymbol{r}$ is the radius vector from the origin to the field point; $\boldsymbol{r}'$ is the radius vector from the origin to the source point; $\Delta\varepsilon_r$ is the difference between the relative dielectric constant of the scatterer and that of the ambient medium; $\boldsymbol{E}$ is the total electric field; $\boldsymbol{E}_{\mathrm{i}}$ is the electric field of the incident wave; G is the Green function: $G(\boldsymbol{r},\boldsymbol{r}') = \exp(\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)/4\pi|\boldsymbol{r}-\boldsymbol{r}'|$.

A scattering study is intended to work out the total electric field $\boldsymbol{E}$ from Eq.(1), which is a 3-dimensional (3D) Fredholm integral equation of the second kind. Generally, this equation is very hard to resolve especially in the case that the scatterer is electrically large, i.e., the dimension of V' is much larger than the wave length. In effect, it remains essential in computational electromagnetics to resolve the integral equation with both accuracy and speed.[2]

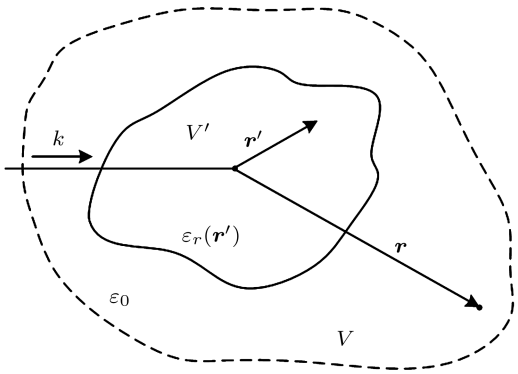


Fig.1. A general scattering situation.

So far, quite a few approximation methods have been proposed to ease the problem, among which the Born approximation is one of the most classic. [3] It has the merit of easy implementation, but it is also believed inapplicable to the incident wave of high frequency. Thereafter, the identification of its validity has been an issue of major concern for a long time in the study of (inverse) scattering.

The existing validity criteria include $\Delta\varepsilon_r kL \ll 1$ and the like,[4−6] in which L is the scale of the scatterer. Another criterion, $\Delta\varepsilon_r(kL)^2 \ll 1$, was studied in Ref.[7]; and Shariff and Wray[8] presented a conclusive one: $\Delta\varepsilon_r\{1, kL, (kL)^2\} \ll 1$. All of them are the results derived from certain numerical experiments where the scatterers are supposed homogeneous, i.e.,

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†Corresponding author. E-mail: ljbnudt@tom.com

$\Delta\varepsilon_r(\boldsymbol{r}') = \text{constant}$ holds for all r in the scatterer. In this sense, the criteria are neither rigorous nor universal. So, it is of absolute necessity to find a rigorous criterion to identify the validity of Born approximation.

2. A unified criterion to identify the validity of the Born approximation

2.1. Relative error of the Born approximation

For a general medium scatterer, we assume an operator, $\boldsymbol{T}$, as

$$\boldsymbol{T}\boldsymbol{E} = -k^2 \iiint_{V'} \Delta\varepsilon_r(\boldsymbol{r}')\boldsymbol{E}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'. \quad (2)$$

Thus equation (1) is rewritten as

$$\boldsymbol{E} = \boldsymbol{E}_\mathrm{i} + \boldsymbol{T}\boldsymbol{E}, \quad (3)$$

which is a 3D Fredholm integral equation of the second kind.

The Born approximation is usually adopted as a classical approximation method for Eq.(3). It replaces the total electric field ($\boldsymbol{E}$) on the right side of Eq.(3) with the incident electric field ($\boldsymbol{E}_\mathrm{i}$), which significantly simplifies the calculation of the scattering field but inevitably causes error as well. The criterion proposed in this section attempts to provide an answer to how much influence the accompanied error will exert on the exactness of the scattering field calculation.

First of all, some definitions and variables need be specified as follows:

1) The vector norm reads

$$||\boldsymbol{E}|| = \max_{r\in V} |\boldsymbol{E}|, \quad (4)$$

where $|\cdot|$ denotes the vector length.[9]

2) The operator norm is given by

$$||\boldsymbol{T}|| = \max_{E\neq 0} \frac{||\boldsymbol{T}\boldsymbol{E}||}{\boldsymbol{E}}. \quad (5)$$

3) A ratio is forwarded by this article to measure the influence of the Born approximation on the calculation of the scattering field, which is named relative error by the Born approximation and defined as

$$q = \frac{||\boldsymbol{E}_\mathrm{s}^\mathrm{Error}||}{||\boldsymbol{E}_\mathrm{s}^\mathrm{Born}||}, \quad (6)$$

where $\boldsymbol{E}_\mathrm{s}^\mathrm{Born}$ is the scattering field acquired via the Born approximation, and $\boldsymbol{E}_\mathrm{s}^\mathrm{Error} = \boldsymbol{E} - \boldsymbol{E}_\mathrm{s}^\mathrm{Born}$ is the scattering field error caused by the Born approximation.

Theorem 1 For a general medium scatterer, the relative error of its scattering electric field caused by the Born approximation is no larger than $||\boldsymbol{T}||/(1-||\boldsymbol{T}||)$, i.e.,

$$q \leq q_\mathrm{upper} = \frac{||\boldsymbol{T}||}{1-||\boldsymbol{T}||}. \quad (7)$$

Proof In light of energy conservation, a passive scatterer never generates scattering wave of amplitude higher than that of the incident wave. Hence $||\boldsymbol{T}\boldsymbol{E}|| < ||\boldsymbol{E}||$, and then there is $\|\boldsymbol{T}\| < 1$. If $\|\boldsymbol{T}\| < 1$, the resolution of Eq.(3) can be expressed as a Neumann series[1]

$$\boldsymbol{E} = (\boldsymbol{I}-\boldsymbol{T})^{-1}\boldsymbol{E}_\mathrm{i} = \sum_{j=0}^{\infty} \boldsymbol{T}^j \boldsymbol{E}_\mathrm{i}. \quad (8)$$

Since the Born approximation employs the incident field instead of the total electric field on the right side of Eq.(3), we derive

$$\boldsymbol{E}^\mathrm{Born} = \boldsymbol{E}_\mathrm{i} + \boldsymbol{T}\boldsymbol{E}_\mathrm{i}. \quad (9)$$

Obviously, approximation (9) is the first two items of the series (8).

Scattering field is generally the major concern in a scattering analysis. Thus by introducing the relation $\boldsymbol{E} = \boldsymbol{E}_\mathrm{i} + \boldsymbol{E}_\mathrm{s}$ to Eq.(9), we obtain the scattering field of the Born approximation:

$$\boldsymbol{E}_\mathrm{s}^\mathrm{Born} = \boldsymbol{E}^\mathrm{Born} - \boldsymbol{E}_\mathrm{i} = \boldsymbol{T}\boldsymbol{E}_\mathrm{i}. \quad (10)$$

Likewise, the theoretically ideal scattering field can be deduced from Eq.(8):

$$\boldsymbol{E}_\mathrm{s} = \sum_{j=0}^{\infty} \boldsymbol{T}^j \boldsymbol{E}_\mathrm{i} - \boldsymbol{E}_\mathrm{i} = \sum_{j=1}^{\infty} \boldsymbol{T}^j \boldsymbol{E}_\mathrm{i}. \quad (11)$$

It is observed that equation (10) is the first term of Eq.(11); so a higher ordered Born approximation can be adopted to yield a more accurate scattering field. For example, the first two terms of Eq.(11) can be used to represent the second-order Born approximation, and so on and so forth.

The error of the scattering field caused by Born approximation is the result of Eq.(11) minus Eq.(10):

$$\boldsymbol{E}_\mathrm{s}^\mathrm{Error} = \boldsymbol{E}_\mathrm{s} - \boldsymbol{E}_\mathrm{s}^\mathrm{Born} = \sum_{j=2}^{\infty} \boldsymbol{T}^j \boldsymbol{E}_\mathrm{i}, \quad (12)$$

which can be re-written as

$$\boldsymbol{E}_\mathrm{s}^\mathrm{Error} = \sum_{j=1}^{\infty} \boldsymbol{T}^j (\boldsymbol{T}\boldsymbol{E}_\mathrm{i}) = \sum_{j=1}^{\infty} \boldsymbol{T}^j \boldsymbol{E}_\mathrm{s}^\mathrm{Born}. \quad (13)$$

According to the operator theory,[9] there are

$$\left\|\sum_{j=1}^{\infty}\boldsymbol{T}^j\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}\right\| \leq \sum_{j=1}^{\infty}\left\|\boldsymbol{T}^j\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}\right\|$$

and

$$\begin{aligned}\left\|\boldsymbol{T}^j\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}\right\| &= \left\|\boldsymbol{T}\left(\boldsymbol{T}^{j-1}\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}\right)\right\| \\ &\leq \|\boldsymbol{T}\|\left\|\boldsymbol{T}^{j-1}\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}\right\| \\ &\leq \cdots \\ &\leq \|\boldsymbol{T}\|^j\left\|\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}\right\|.\end{aligned}$$

Thus the norm of Eq.(13) is estimated as

$$\begin{aligned}\left\|\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Error}}\right\| &= \left\|\sum_{j=1}^{\infty}\boldsymbol{T}^j\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}\right\| \\ &\leq \sum_{j=1}^{\infty}\left\|\boldsymbol{T}^j\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}\right\| \\ &\leq \sum_{j=1}^{\infty}\|\boldsymbol{T}\|^j\left\|\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}\right\| \\ &\leq \frac{\|\boldsymbol{T}\|\left(1-\|\boldsymbol{T}\|^{\infty}\right)}{1-\|\boldsymbol{T}\|}\left\|\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}\right\|. \qquad (14)\end{aligned}$$

$\|\boldsymbol{T}\| < 1$ signifies $\lim\limits_{j\to\infty}\|\boldsymbol{T}\|^j = 0$, then $\left\|\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Error}}\right\|$ is eventually estimated as

$$\left\|\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Error}}\right\| \leq \frac{\|\boldsymbol{T}\|}{1-\|\boldsymbol{T}\|}\left\|\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}\right\|. \qquad (15)$$

Therefore, if equation (6) is replaced by Eq.(15), the relative error of the Born approximation is approximated as

$$q \leq q_{\mathrm{upper}} = \frac{||\boldsymbol{T}||}{1-||\boldsymbol{T}||},$$

where q_{upper} is named the upper bound of the relative error.

Q. E. D.

Theorem 1 is a universal theorem to estimate the relative error of the Born approximation, which is ultimately reduced to the calculation of the operator norm $||\boldsymbol{T}||$. Based on this theorem, it is observed that a larger $||\boldsymbol{T}||$ corresponds to a larger q_{upper} because $f(x) = x/(1-x)$ is an increasing function in the domain $[0, 1)$.

2.2. Validity of the Born approximation

Criterion 1 For a general medium scatterer, if the Born approximation is considered valid when the relative error q is not larger than a given threshold $q_{\mathrm{th}} \in (0, 1)$, then

$$||\boldsymbol{T}|| \leq \frac{q_{\mathrm{th}}}{1+q_{\mathrm{th}}} \qquad (16)$$

serves as a sufficient condition for the validity of Born approximation.

Proof Because of $q_{\mathrm{th}} \in (0, 1)$, $||\boldsymbol{T}|| \leq q_{\mathrm{th}}/(1+q_{\mathrm{th}})$ is equivalent to

$$q_{\mathrm{th}} \geq \frac{||\boldsymbol{T}||}{1-||\boldsymbol{T}||}. \qquad (17)$$

To sum up, a relation can be derived from the inequalities (16) and (17) if $||\boldsymbol{T}|| \leq q_{\mathrm{th}}/(1+q_{\mathrm{th}})$:

$$q \leq q_{\mathrm{th}}.$$

In this way, the Born approximation certainly provides a result of adequate accuracy.

Q. E. D.

Criterion 1 serves as a universal criterion to identify the validity of the Born approximation, because $\boldsymbol{T}$ is an operator related to f, $\Delta\varepsilon_r$, and $\boldsymbol{r}$, its exact norm $||\boldsymbol{T}||$ is difficult to obtain. Thus it is of necessity to give a proper estimation of $||\boldsymbol{T}||$ in order to make Criterion 1 applicable to real scattering problems.

3. Estimation of the operator norm and its application to the universal criterion

3.1. Estimation of the operator norm $||\boldsymbol{T}||$

Lemma 1 For a general medium scatterer, its operator norm $||\boldsymbol{T}||$ satisfies

$$||\boldsymbol{T}|| \leq \frac{\pi f^2}{c^2}||I||, \qquad (18)$$

where f is the incident wave frequency, $c = 3\times10^8$ m/s is the light velocity, and $||I|| = \max_{r\in V}\iiint_{V'}(|\Delta\varepsilon_r(r')|/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3r'$ is an integral norm.

Proof In light of the definition of $\boldsymbol{T}$ in Eq.(2), it is easy to reach the following estimation:

$$\begin{aligned}||\boldsymbol{T}\boldsymbol{E}|| &= \max_{r\in V}\left|k^2\iiint_{V'}\Delta\varepsilon_r(\boldsymbol{r}')\boldsymbol{E}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'\right| \\ &\leq k^2\max_{r\in V}\iiint_{V'}|\Delta\varepsilon_r(\boldsymbol{r}')\boldsymbol{E}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')|\,\mathrm{d}^3\boldsymbol{r}' \\ &\leq k^2\|\boldsymbol{E}\|\max_{\boldsymbol{r}\in V}\iiint_{V'}|\Delta\varepsilon_r(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')|\,\mathrm{d}^3\boldsymbol{r}'. \qquad (19)\end{aligned}$$

Given $G(\boldsymbol{r},\boldsymbol{r}')=\exp(\mathrm{i}k|r-r'|)/4\pi|\boldsymbol{r}-\boldsymbol{r}'|$, $|\exp(\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)|=1$, and $k=2\pi f/c$, equation (19) can be rewritten as

$$||\boldsymbol{T}\boldsymbol{E}||\leq\frac{\pi f^2}{c^2}||\boldsymbol{E}||||I||, \tag{20}$$

where $||I||=\max_{r\in V}\iiint_{V'}(|\Delta\varepsilon_r|(r')/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3\boldsymbol{r}'$ is the integral norm.

Estimation (20) holds for all $\boldsymbol{E}\in\mathbb{C}^3$, so it can be rewritten as the following inequality if $\boldsymbol{E}\neq 0$:

$$\frac{||\boldsymbol{T}\boldsymbol{E}||}{||\boldsymbol{E}||}\leq\frac{\pi f^2}{c^2}||I||, \tag{21}$$

which further indicates

$$\max_{E\neq 0}\frac{||\boldsymbol{T}\boldsymbol{E}||}{||\boldsymbol{E}||}\leq\frac{\pi f^2}{c^2}||I||. \tag{22}$$

Therefore, the operator norm $||\boldsymbol{T}||$, defined by Eq.(6), satisfies

$$||\boldsymbol{T}||\leq\frac{\pi f^2}{c^2}||I||.$$

Q. E. D.

Lemma 1 provides an approach to estimate the operator norm, which is ultimately reduced to the acquisition of the integral norm $||I||$. However, it is rather conservative to some extent in spite of the strict estimation (19), and further improvement of the estimation of the operator norm concerned should be made in the future.

3.2. A deduced validity criterion of the Born approximation

As is known, $\|\boldsymbol{T}\|<1$ is a sufficient condition for the convergence of the Neumann series (8). In order to ensure the convergence of the Neumann series, this article focuses the discussion on the situation in which the following inequality is satisfied:

$$\frac{\pi f^2}{c^2}||I||<1. \tag{23}$$

According to inequality (23) and Lemma 1, there must be $\|\boldsymbol{T}\|<1$. Condition (23) means that the frequency concerned should satisfy

$$f<\frac{c}{\sqrt{\pi||I||}}. \tag{24}$$

Obviously, inequality (24) is also a sufficient condition for the convergence of the Neumann series. Literally, the frequency concerned can extend to a wider range if more rigorous estimation of the operator norm is made.

By applying Lemma 1 to Criterion 1, we obtain the following corollary to estimate the upper bound of the relative error caused by the Born approximation and its validity.

Corollary 1 For a general medium scatterer, if $f<c/\sqrt{\pi||I||}$, then the relative error caused by the Born approximation will satisfy

$$q\leq q_{\max}=\frac{\pi f^2\,\|I\|}{c^2-\pi f^2\,\|I\|}, \tag{25}$$

where $||I||=\max_{r\in V}\iiint_{V'}(|\Delta\varepsilon_r(r')|/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3\boldsymbol{r}'$ is the integral norm.

Proof $f<c/\sqrt{\pi||I||}$ signifies $\|\boldsymbol{T}\|<1$. Because $f(x)=x/(1-x)$ is an increasing function in the domain $[0,1)$, the following inequality can be derived from Lemma 1:

$$\frac{||\boldsymbol{T}||}{1-||\boldsymbol{T}||}\leq q_{\max}=\frac{\pi f^2||I||}{c^2-\pi f^2||I||}. \tag{26}$$

Then the combination of Eqs.(7) and (26) leads to

$$q\leq q_{\max}=\frac{\pi f^2\,\|I\|}{c^2-\pi f^2\,\|I\|}.$$

Q. E. D.

Based on Corollary 1 the following criterion is easy to reach.

Criterion 2 For a general medium scatterer, if the Born approximation is considered valid when the relative error q is not larger than a given threshold $q_{\mathrm{th}}\in(0,1)$, then

$$f\leq f_{\max}=\sqrt{\frac{q_{\mathrm{th}}}{1+q_{\mathrm{th}}}}\cdot\frac{c}{\sqrt{\pi\,\|I\|}} \tag{27}$$

serves as a sufficient condition for the validity of the Born approximation. In Eq.(27), $||I||=\max_{r\in V}\iiint_{V'}(|\Delta\varepsilon_r(r')|/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3\boldsymbol{r}'$ is the integral norm.

Proof $q_{\mathrm{th}}>0$ signifies $\sqrt{q_{\mathrm{th}}/(1+q_{\mathrm{th}})}<1$, then $f<c/\sqrt{\pi||I||}$ is obtained from Eq.(27). In this sense, $f\leq f_{\max}=\sqrt{q_{\mathrm{th}}/(1+q_{\mathrm{th}})}\cdot c/\sqrt{\pi\,\|I\|}$ is equivalent to

$$q_{\mathrm{th}}\geq\frac{\pi f^2||I||}{c^2-\pi f^2||I||}. \tag{28}$$

Based on Eqs.(25) and (28), $q\leq q_{\max}\leq q_{\mathrm{th}}$; thus the Born approximation is valid on this frequency.

Q. E. D.

In radar theory, 3 dB (i.e., $q_{\mathrm{th}}=0.5$) is commonly used as a threshold to identify the reasonableness of

a relative error. If $q_{\rm th} = 0.5$, then inequality (27) becomes

$$f \leq f_{\max} = \frac{c}{\sqrt{3\pi \, \|I\|}}, \tag{29}$$

which is a corollary from Criterion 2.

Clearly, Criterion 2 is an application of the estimation (18) to Criterion 1, and the difference in criteria results from the difference in the estimation of the operator norm concerned. More effort should be made to enhance the rigor of the estimation.

3.3. Calculation of the integral norm $||I||$

To obtain $||I||$ is to obtain $\boldsymbol{r}_0 \in V$ with the purpose to satisfy $\iiint_{V'}(|\Delta\varepsilon_r(r')|/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3\boldsymbol{r}' = \max_{r\in V}\iiint_{V'}(|\Delta\varepsilon_r(r')|/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3\boldsymbol{r}'$, which can be generally solved by the optimization method. Since more computation cost may be required for a larger searching region (i.e., the ambient medium V), it is advisable to limit $\boldsymbol{r}$ properly within a smaller searching region in order to lessen the computational difficulty.

As for the problem, this article divides scatterers into two categories: convex scatterers and non-convex scatterers.

Proposition 1 For a general medium scatterer (V') located in the ambient medium (V), if V' is a convex volume, then $||I|| = \max_{r\in V}\iiint_{V'}(|\Delta\varepsilon_r(r')|/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3\boldsymbol{r}' = \max_{r\in V'}\iiint_{V'}(|\Delta\varepsilon_r(r')|/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3\boldsymbol{r}'$, i.e., $\boldsymbol{r}$ can be limited within V' in the acquisition of the integral norm; if V' is a non-convex volume, then $||\boldsymbol{I}|| = \max_{r\in V_1}\iiint_{V'}(|\Delta\varepsilon_r(r')|/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3\boldsymbol{r}'$, where V_1 is a convex volume containing the scatterer concerned.

Proof for Proposition 1 is presented in the appendix.

Proposition 1 can limit $\boldsymbol{r}$ in V' or V_1 with respect to the scatterer shape. The computational difficulty is expected to be lessened significantly given that both V' and V_1 are much smaller than V in most cases.

A simple scheme to estimate the integral norm is as follows: mesh the volume V' or (the mesh can be much sparser than that in general computational electromagnetics methods, such as method of moment) and figure out the integral values I for the nodes. Then the maximum value of them is found, and it is assumed as an estimation of the integral norm $||I|| = \max_{r\in V'}\iiint_{V'}(|\Delta\varepsilon_r(r')|/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3\boldsymbol{r}'$. Although this process may be insufficiently precise, it is applicable in most real engineering problems. Moreover, the above searching method can further be used around the maximum value to obtain a more accurate estimation of the integral norm.

Since the impact of the oscillatory factor $\exp(\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)$ is excluded in the integral $I = \iiint_{V'}(|\Delta\varepsilon_r(r')|/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3\boldsymbol{r}'$, it is easy to work out the integral value for a node. Furthermore, the number of integrals to be calculated is not large for the mesh being sparse, so the total computational time is almost negligible in comparison with that of solving a 3D Fredholm integral equation. Hence, the less computational time strengthens the advantage of adopting the Born approximation method to solve the scattering problems.

4. Numerical experiments

In this section, the criterion proposed is verified by two canonical scattering models.

The scattering characteristics of the dielectric sphere have been elaborately studied via Mie theory for a long time,[10−12] so we adopt two spheres, a homogeneous dielectric sphere and a multi-layered dielectric sphere, to testify the validity of our criterion.

4.1. In the case of a homogeneous dielectric sphere

Table 1 lists the parameters of the homogeneous dielectric sphere concerned.

Table 1. Parameters of the homogeneous dielectric sphere.

parameters	values
sphere radius R	1 m
detection range R_t	500 m
dielectric constant difference $\Delta\varepsilon_r$	10^{-3}
amplitude of incident wave $\|E_{\rm i}\|$	1 V/m
frequency of incident wave f	10 MHz–3 GHz, equal steps in logarithmic scale

Thereafter its integral norm is derived as follows:

$$\begin{aligned} ||I|| &= \max_{r\in V} \iiint_{V'} \frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}' \\ &= \iiint_{V'} \frac{|10^{-3}|}{|\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}' \\ &= 2\pi\times 10^{-3}. \end{aligned}$$

According to Table 1, it is easy to obtain the scattering electric fields via the Mie theory and the Born approximation respectively. Figure 2 shows the relation between their amplitudes and the frequency. As is shown, the two curves almost coincide; this indicates the capability of the Born approximation to provide precise results for weak scatterers.

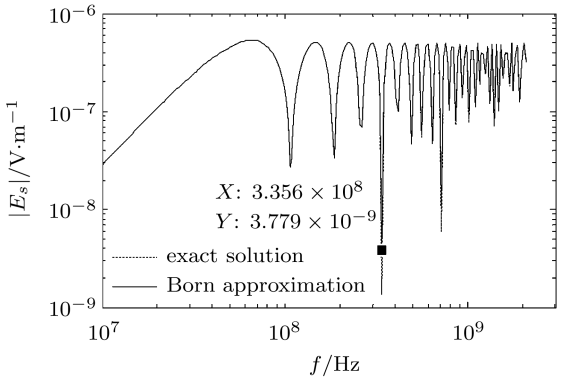


Fig.2. Amplitudes of the scattering electric field for the homogeneous sphere.

Figure 3 shows the error of the scattering field for a detailed contrast. It can be observed that the error is relatively small and it increases as a whole with the increase of frequency.

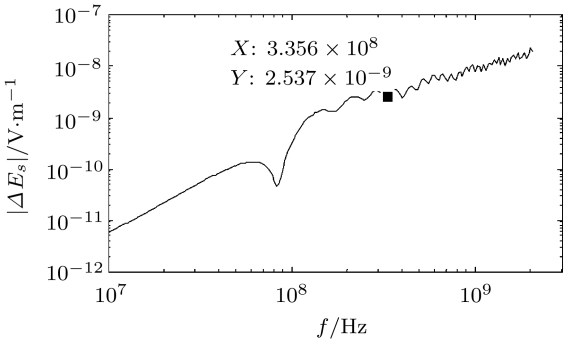


Fig.3. Scattering electric field errors caused by the Born approximation.

Figure 4 shows the actual relative error, the upper bound of the relative error ($q_{\max}$), and the threshold (q_{th}). In the log-log coordinate system, a linear increase is observed of $q_{\max}$ in low frequency band with the increase of the frequency; while it turns nonlinear in the high frequency band. The explanation for the phenomenon is as follows: when $f \ll c/\sqrt{\pi||I||}$, there will be $q_{\max} \approx \pi f^2||I||/c^2$ according to Corollary 1, thus $\log_{10} q_{\max} \approx 2\log_{10} f + \log_{10}\left(\pi\,\|I\|/c^2\right)$ approximates to a linear relationship. However, when f is comparable to $c/\sqrt{\pi||I||}$, the relationship between $q_{\max}$ and f becomes nonlinear. The crooked part of the curve in the top right corner of Fig.4 well accounts for the situation.

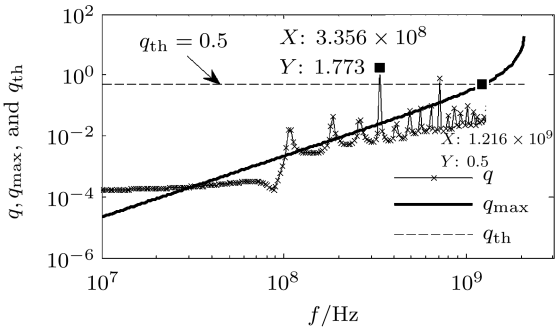


Fig.4. Relative error of Born approximation.

In addition, figure 4 indicates that the relative errors corresponding to each sample point are below the curve $q_{\max}$, with those few peak values and those sample points in the low frequency as exceptions. Its explanation goes as follows:

1) The exact solution is computed via the Mie theory which expands the solution as a series with infinite terms. The series must be truncated in real computation and it consequently brings about error to the final result. In general, 3 to 4 significant digits are provided by the truncated series, so the relative error in the low frequency remains flat with a magnitude of about 10^{-4} (see Fig.4).

2) Because $q = ||\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Error}}||/||\boldsymbol{E}_{\mathrm{s}}^{\mathrm{Born}}||$, the relative error tends to be considerably large if the scattering field obtained by the Born approximation is very small. Therefore, the presence of valley values of the scattering field results in peak values of relative error even if the error curve is smooth (see Fig.4).

Hence, figure 4 can serve as an evidence for the validity of Criterion 2. It shows that the upper bound of the valid frequency deduced from our criterion is 1.2 GHz, which is higher than that (0.48 GHz) provided by Shariff's criterion. In other words, our criterion has extended the valid frequency to a wider range.

4.2. In the case of a multi-layered dielectric sphere

Figure 5 illustrates a multi-layered dielectric sphere.

Table 2 lists the parameters of the multi-layered sphere.

The amplitudes of the scattering field obtained via the Mie theory and Born approximation respectively are presented in Fig.6, where the two curves almost coincide in the log-log coordinate system.

Table 2. Parameters of multi-layered dielectric sphere

parameters	values
outmost radius of the sphere R	1 m
detection range R_t	500 m
dielectric constant difference $\Delta\varepsilon_r$	$10^{-3}/(1+50r^2)$
amplitude of incident wave $\|E_\mathrm{i}\|$	1 V/m
frequency of incident wave f	10 MHz–8 GHz, equal steps in logarithmic scale

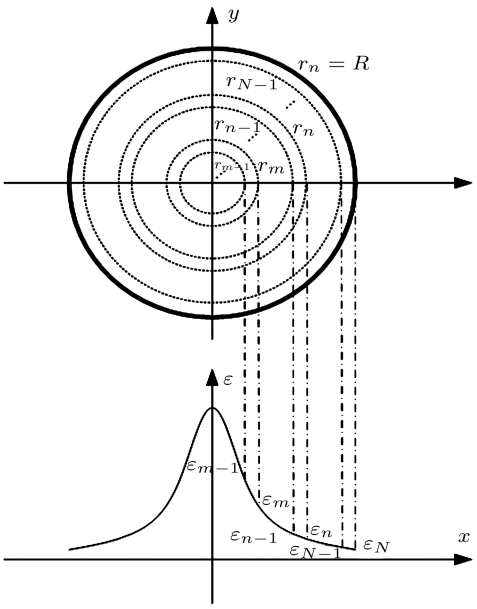


Fig.5. Dielectric constant distribution of the multi-layered dielectric sphere.

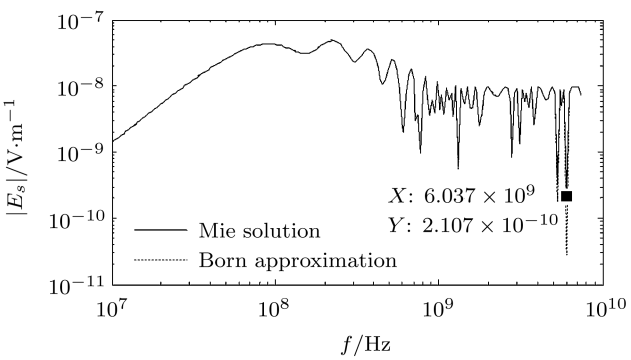


Fig.6. Amplitudes of the scattering electric field for the multi-layered sphere.

The scattering field error is presented in Fig.7; the upper bound of the relative error and the real relative error are shown in Fig.8. All of them are evidence for the validity of Criterion 2. Moreover, the valid frequency by our method amounts to 4.5 GHz, which can be deemed as great progress compared with 0.48 GHz by Sharff's criterion.

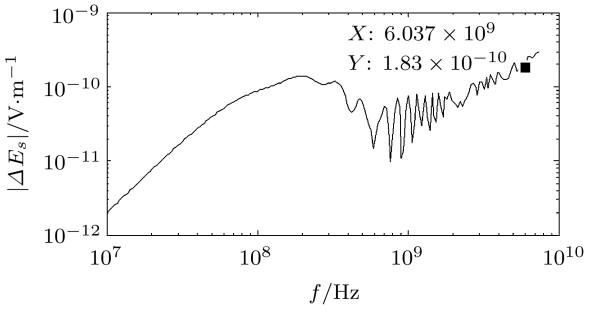


Fig.7. Scattering electric field errors caused by Born approximation.

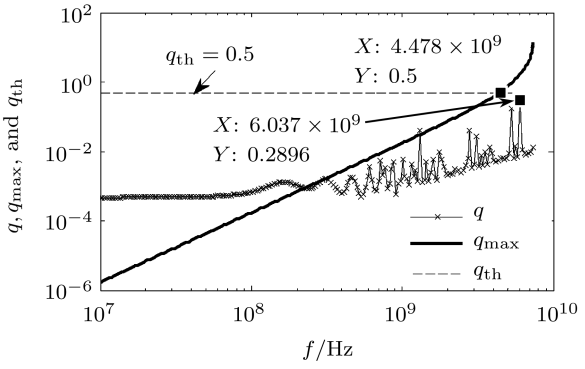


Fig.8. Relative error of Born approximation.

5. Conclusion and further work

This article is intended to establish a rigorous criterion to identify the validity of the Born approximation. Aiming at this, firstly it proposes a universal criterion via operator theory, which is ultimately reduced to the estimation of the norm of an operator $||\boldsymbol{T}||$. A method is then proposed to estimate the norm, based on which a new criterion is developed to identify the valid frequency band for Born approximation. Apart from its rigorousness, the new criterion also extends the valid frequency to a wider range. Numerical experiments have been conducted to verify its validity.

However, it should be noted that this criterion is still conservative in some senses. To begin with, the employment of the operator norm theory to make estimation is conservative per se (e.g., the derivation (14)); and the estimation of the operator norm is conservative too. For instance, the highly oscillatory factor $\exp(\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)$ is assumed 1 in the derivation from formula (19) to (20), which in effect neglects the counteracting effect inside the oscillator. That is to say, the valid frequency for our criterion can be extended to a wider range in application.

Studies on the evaluation of the counteracting effect of $\exp(\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)$ on the operator norm as well as estimation of a more accurate operator norm $||\boldsymbol{T}||$ to better calculate the validity range of the Born approximation are on our working schedule for the near future.

Appendix A: Proof for Proposition 1

A lemma is forwarded preceding our proof for Proposition 1.

Lemma 2 If V' is a convex volume, then for any point $\boldsymbol{r}\in V'$ there exists at least one $\boldsymbol{r}''\in\partial V'$ which enables the relation $|\boldsymbol{r}''-\boldsymbol{r}'|<|\boldsymbol{r}-\boldsymbol{r}'|$ to hold for all $\boldsymbol{r}'\in V'$, where $\partial V'$ denotes the boundary of the volume.

Proof Figure 9 depicts the situation of Lemma 2, where A is a point outside the volume ($\overline{OA}=\boldsymbol{r}$) and Q is a sphere with the center at A. The sphere extends to a larger volume as its radius gradually increases from 0. Obviously there must be a sphere that is externally tangent to the given convex volume. Assume that the tangent point is B and the tangent plane is T, then the point A and the convex volume V' lie on the different side of T because a convex volume only lies on one side of a tangent plane of it.[13]

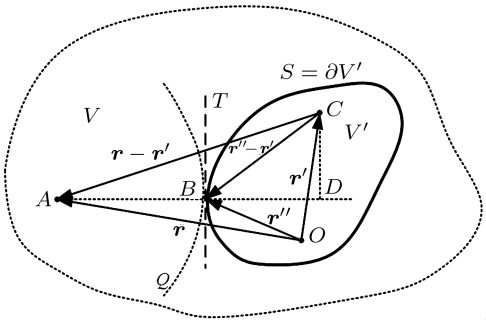


Fig.9. Volume with a convex boundary.

For any point C inside the scatterer, we assume $\overline{CD}$ be orthogonal to $\overline{AB}$. Because A and D are located on two sides of T, there must be $\overline{AD}>\overline{BD}$; thus $|\boldsymbol{r}''-\boldsymbol{r}'|<|\boldsymbol{r}-\boldsymbol{r}'|$ is easily derived from $|\boldsymbol{r}''-\boldsymbol{r}'|=\sqrt{\overline{BD}^2+\overline{CD}^2}$ and $|\boldsymbol{r}-\boldsymbol{r}'|=\sqrt{\overline{AD}^2+\overline{CD}^2}$. Hence the tangent point B is the point required by the lemma.

From the randomness of $\boldsymbol{r}$, the lemma is then proved.

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Furthermore, Proposition 1 is proved as follows on the basis of Lemma 2.

Proof The proof has two parts.

1) In the case of convex scatterer

For any $\boldsymbol{r}\in(V-V')$, i.e., $\boldsymbol{r}\in V'$, Lemma 2 indicates the existence of a $\boldsymbol{r}''\in\partial V'$ to satisfy $|\boldsymbol{r}''-\boldsymbol{r}'|<|\boldsymbol{r}-\boldsymbol{r}'|$ for all $\boldsymbol{r}'\in V'$, then

$$\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'<\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}''-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}' \quad \text{(A1)}$$

This relation holds for all $\boldsymbol{r}\in(V-V')$, so we have

$$\begin{aligned}&\max_{r\in(V-V')}\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\\&<\max_{r''\in\partial V'}\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}''-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\end{aligned} \quad \text{(A2)}$$

At the same time $\partial V'$ is the boundary of V, so there must be

$$\begin{aligned}&\max_{r\in\partial V'}\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\\&<\max_{r\in V'}\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\end{aligned} \quad \text{(A3)}$$

Consequently, the combination of expressions (A2) and (A3) leads to

$$\begin{aligned}&\max_{r\in(V-V')}\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\\ <&\max_{r\in V'}\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}''-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\end{aligned}\tag{A4}$$

Because $V=V'+(V-V')$, there is

$$\begin{aligned}||I||=&\max_{r\in V}\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\\ =&\max\left(\begin{array}{c}\max\limits_{r\in(V-V')}\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\\ \max\limits_{r\in V'}\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\end{array}\right)\end{aligned}\tag{A5}$$

Given inequality (A4), equation (A5) becomes

$$||I||=\max_{r\in V'}\iiint_{V'}\frac{|\Delta\varepsilon_r(r')|}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'$$

The first part of Proposition 1 is then proved.

2) In the case of non-convex scatterer

Figure 10 depicts a scatterer with a nonconvex boundary.

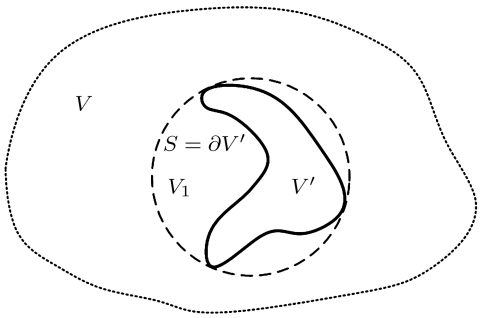


Fig.10. Scatterer with a nonconvex boundary.

In this case, we assume a convex surface ∂V_1 envelops the scatterer (V'). Based on the foregoing proof, it is easy to obtain $||I|| = \max_{r\in V1}\iiint_{V'}(|\Delta\varepsilon_r(r')|/|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}^3\boldsymbol{r}'$, then the second part of the proposition is proved.

Q. E. D..

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