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Travelling solitary wave solutions for the generalized Burgers–Huxley equation with nonlinear terms of any order*

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In this paper, the travelling wave solutions for the generalized Burgers–Huxley equation with nonlinear terms of any order are studied. By using the first integral method, which is based on the divisor theorem, some exact explicit travelling solitary wave solutions for the above equation are obtained. As a result, some minor errors and some known results in the previousl literature are clarified and improved.

Keywords: travelling wave solutions, first integral method, generalized Burgers–Huxley equation with nonlinear terms of any order

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1. Introduction

It is well known that searching for explicit exact solutions of nonlinear partial differential equations (NPDE) plays an important role in the study of nonlinear physical phenomena. In recent years, a number of very powerful and direct methods to find the explicit solutions of NPDE have been proposed and developed, such as the Hirota bilinear method, the Bäcklund transformation method, the Painlevé expansion method, the homotopy perturbation method, the homotopy analysis method, the homogenous balance method, the Jacobi elliptic function expansion method, the bifurcation method of planar dynamical systems, the auxiliary equation method and so on.[1−12] Recently, a new and more powerful method, the first integral method which is based on the divisor theorem, has been proposed by Feng[13] and developed to study the travelling wave solutions of various nonlinear evolution equations.[14−16] The exact travelling solitary wave solutions for some nonlinear evolution equations can be solved more easily and more quickly by the first-integrable method than the traditional methods for exact solutions as well as numerical solutions. It has many advantages over other traditional techniques, and it mainly avoids a great deal of complicated and tedious calculation and provides more exact and explicit travelling solitary solutions with high accuracy.

In the present work, we shall consider the travelling wave solutions of the following generalized Burgers–Huxley equation with nonlinear terms of any order[17]

$$u_t + \alpha u^\delta u_x - \frac{m}{u}u_x^2 - u_{xx} = \beta u(1-u^\delta)(u^\delta - r), \quad (1)$$

where α, β, δ, m and r are parameters, $r \in (0,1), \alpha \geq 0, \beta \geq 0, \delta > 0, \rho = \alpha^2 + 4\beta(\delta + 1 + m) > 0$. Clearly, when $m = 0, \alpha = 0, \delta = 1$, equation (1) is reduced to the Huxley equation which describes nerve pulse propagation in nerve fibres and wall motion in liquid crystals;[18,19] when $m = 0, \beta = 0, \delta = 1$, equation (1) is reduced to the Burgers equation which describes the far field of wave propagation in nonlinear dissipative systems;[20] when $m = 0, \delta = 1, \alpha \neq 0, \beta \neq 0$, equation (1) becomes the Burgers–Huxley equation, which shows a prototype model for describing the interaction between reaction mechanisms, convection effects and diffusion transport;[21] when $m = 0$, equation (1) reduces to the extended form of the Burgers–Huxley equation.[22,23]

The remainder of this paper is organized as follows. In Section 2, using the first-integral method

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which is based on the divisor theorem, we establish several exact explicit travelling solitary wave solutions for Eq.(1). Moreover, some minor errors in the previous article are clarified. Finally, some conclusions are given in Section 3.

2. Exact traveling wave solutions of Eq.(1)

Making the transformation

$$\phi = u^{\delta},$$

then equation (1) becomes

$$\begin{aligned}&\phi_t - \phi_{xx} + \alpha\phi\phi_x + \left[1 - \frac{(m+1)}{\delta}\right]\frac{1}{\phi}(\phi_x)^2 \\ &= \beta\delta\phi(1-\phi)(\phi - r);\end{aligned} \tag{2}$$

Assume equation (2) has the travelling wave solution as follows:

$$\phi(x,t) = \phi(\xi), \quad \xi = x - ct, \tag{3}$$

where c is the wave velocity. Substituting Eq.(3) into Eq.(2) yields

$$\begin{aligned}&-c\phi' + \left[1 - \frac{(m+1)}{\delta}\right]\frac{1}{\phi}(\phi')^2 - \phi'' + \alpha\phi\phi' \\ &- \beta\delta\phi(1-\phi)(\phi - r) = 0,\end{aligned} \tag{4}$$

where ϕ' and ϕ'' denote $\mathrm{d}\phi/\mathrm{d}\xi$ and $\mathrm{d}^2\phi/\mathrm{d}\xi^2$, respectively.

If we let $\phi' = y$, then equation (4) can be written as the two-dimensional autonomous system

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}\xi} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} &= -cy + \left[1 - \frac{(m+1)}{\delta}\right]\frac{1}{\phi}y^2 + \alpha\phi y \\ &\quad - \beta\delta\phi(1-\phi)(\phi - r).\end{aligned} \tag{5}$$

Assume that

$$\mathrm{d}\tau = \frac{\mathrm{d}\xi}{\phi}$$

then equation (5) becomes

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= \phi y, \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= -c\phi y + \left[1 - \frac{(m+1)}{\delta}\right]y^2 + \alpha\phi^2 y \\ &\quad - \beta\delta\phi^2(1-\phi)(\phi - r).\end{aligned} \tag{6}$$

In order to find the travelling wave solutions of Eq.(1), we are now applying the first-integral method whose key idea is to utilize the so-called divisor theorem which is based on the ring theory of commutative algebra and to obtain first integrals to system (6) under various parameter conditions. Then, using these first integrals, the above two-dimensional autonomous system (6) can be reduced to some different first-order integrable differential equations. Finally, through solving these first-order differential equations directly, travelling wave solutions for Eq.(1) can be established easily.

Next, let us recall the divisor theorem for two variables in the complex domain $\mathcal{C}$, as pointed out by Feng,[24] which follows from the so-called Hilbert–Nullstellensatz theorem derived from the ring theory in commutative algebra.[25]

Divisor Theorem Suppose that $P(\omega, z)$ and $Q(\omega, z)$ are polynomials of two variables ω and z in $\mathcal{C}[\omega, z]$ and $P(\omega, z)$ is irreducible in $\mathcal{C}[\omega, z]$. If $Q(\omega, z)$ vanishes at all zero points of $P(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $\mathcal{C}(\omega, z)$ such that

$$Q(\omega, z) = P(\omega, z)G(\omega, z).$$

Now, we are applying the above divisor theorem to look for the first integral of system (6). Suppose that $\phi = \phi(\tau)$ and $z = z(\tau)$ are the nontrivial solutions to system (6), and $\Omega(\phi, y) = \sum_{i=0}^{m} a_i(\phi)y^i$ is an irreducible polynomial in $\mathcal{C}(\phi, y)$ such that

$$\Omega(\phi(\tau), y(\tau)) = \sum_{i=0}^{m} a_i(\phi)y^i = 0, \tag{7}$$

where $a_i(\phi)$ $(i = 1, 2, \ldots, m)$ are polynomials of ϕ and $a_m(\phi) \neq 0$. We start our study with $m = 1$. Note that $\mathrm{d}\Omega/\mathrm{d}\tau$ is a polynomial of ϕ and y, and $\Omega(\phi(\tau), y(\tau)) = 0$ implies that $\mathrm{d}\Omega/\mathrm{d}\xi\big|_{(6)} = 0$. According to the divisor theorem, there exists a polynomial $H(\phi, y) = p(\phi) + q(\phi)y$ in $\mathcal{C}(\phi, y)$ such that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\xi}\Big|_{(6)} = H(\phi, y)\Omega(\phi, y), \tag{8}$$

that is

$$\begin{aligned}&\sum_{i=0}^{1} a_i'(\phi)y^i \cdot \phi y + \sum_{i=0}^{1} a_i(\phi) i y^{i-1} \\ &\quad \left\{-c\phi y + \left[1 - \frac{(m+1)}{\delta}\right]y^2 \right. \\ &\quad \left. + \alpha\phi^2 y - \beta\delta\phi^2(1-\phi)(\phi - r)\right\} \\ &= [a_0(\phi) + a_1(\phi)y][p(\phi) + q(\phi)y].\end{aligned} \tag{9}$$

Equating the coefficients of y^i $(i = 2, 1, 0)$ on both sides of Eq.(9), we obtain that

$$\boldsymbol{e}'(\phi) = \boldsymbol{B}(\phi) \cdot \boldsymbol{e}(\phi), \tag{10}$$

and

$$\left[-\beta\delta\phi^2(\phi-1)(\phi-r), p(\phi)\right] \cdot \boldsymbol{e}(\phi) = 0, \tag{11}$$

where $\boldsymbol{e}(\phi) = (a_1(\phi), a_0(\phi))^{\mathrm{T}}$ and

$$\boldsymbol{B}(\phi) = \begin{pmatrix} \dfrac{q(\phi) - 1 + \dfrac{(m+1)}{\delta}}{\phi} & 0 \\ \dfrac{p(\phi) - \alpha\phi^2 + c\phi}{\phi} & \dfrac{q(\phi)}{\phi} \end{pmatrix}$$

.

Since $a_i(\phi)$ $(i = 0, 1)$ are polynomials of ϕ, from Eq.(10), we can deduce that $q(\phi)$ must be a constant, and $a_1(\phi) = c_0 \mathrm{e}^{\int \frac{q(\phi)-1+\frac{m+1}{\delta}}{\phi} \mathrm{d}\phi}$, where c_0 is integral constant. Obviously, the simplest case for $a_0(\phi)$ and $a_1(\phi)$ is that $q(\phi) = 1 - (m+1)/\delta$. That is, $a_1(\phi)$ is a constant. For computational convenience, we assume that $a_1(\phi) = 1$. Meanwhile, from Eq.(10), we have

$$\begin{aligned} p(\phi) \;=\; & \phi a_0'(\phi) - c\phi + \alpha\phi^2 \\ & - \left[1 - \frac{(m+1)}{\delta}\right] a_0(\phi). \end{aligned} \tag{12}$$

From Eq.(11), we have also

$$p(\phi) \cdot a_0(\phi) = \beta\delta\phi^2(\phi-1)(\phi-r). \tag{13}$$

From Eqs.(12) and (13), we can deduce that $\deg(a_0(\phi)) = \deg(p(\phi)) = 2$. Then it means that when both $p(\phi)$ and $a_0(\phi)$ are all polynomials of ϕ, there exist only three possible cases for $p(\phi)$ and $a_0(\phi)$ as follows.

Case (i) Assume that $a_0(\phi) = A_1\phi(\phi - 1)$, $p(\phi) = B_1\phi(\phi - r)$, substituting them into Eqs.(12) and (13) and equating the corresponding coefficients of ϕ^2, ϕ and constant terms, we have

$$\begin{aligned} & A_1 B_1 = \beta\delta, \\ & B_1 r = c + \frac{(m+1)}{\delta} A_1, \\ & B_1 = \left[1 + \frac{(m+1)}{\delta}\right] A_1 + \alpha. \end{aligned} \tag{14}$$

Solving Eq.(14), we can obtain that

$$\begin{aligned} & A_1 = \frac{(-\alpha \pm \rho)\delta}{2(\delta + m + 1)}, \\ & B_1 = \frac{\alpha \pm \rho}{2}, \\ & c = \frac{(\alpha \pm \rho)r}{2} + \frac{(m+1)(\alpha \mp \rho)}{2(\delta + m + 1)}. \end{aligned} \tag{15}$$

In this case, equation (7) becomes

$$y = \frac{(\alpha \mp \rho)\delta}{2(\delta + m + 1)}\phi(\phi - 1). \tag{16}$$

Using this first integral, the second-order ordinary differential Eq.(5) reduces to

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \frac{(-\alpha \pm \rho)\delta}{2(\delta + m + 1)}\phi(1 - \phi). \tag{17}$$

By direct integration of Eq.(17), we get the following travelling solitary wave solutions to Eq.(1)

$$u(x,t) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{(-\alpha \pm \rho)\delta}{4(\delta + m + 1)}[x - ct + x_0]\right)\right]^{1/\delta}, \tag{18}$$

$$u(x,t) = \left[\frac{1}{2} + \frac{1}{2}\coth\left(\frac{(-\alpha \pm \rho)\delta}{4(\delta + m + 1)}[x - ct + x_0]\right)\right]^{1/\delta}, \tag{19}$$

where x_0 is an arbitrary constant and c is determined by Eqs.(15).

Case (ii) Assume that $a_0(\phi) = A_2\phi(\phi - r), p(\phi) = B_2\phi(\phi - 1)$. Substituting them into Eqs.(12) and (13) and equating the corresponding coefficients of ϕ^2, ϕ and constant terms, we have

$$\begin{aligned} & A_2 B_2 = \beta\delta, \\ & \frac{(m+1)}{\delta} A_2 r + c = B_2, \\ & B_2 = \alpha + \left(1 + \frac{m+1}{\delta}\right) A_2. \end{aligned} \tag{20}$$

Solving Eq.(20), we can obtain that

$$\begin{aligned} & A_2 = \frac{-\alpha \pm \rho}{2(\delta + m + 1)}\delta, \\ & B_2 = \frac{\alpha \pm \rho}{2}, \end{aligned}$$

$$c = \frac{\alpha \pm \rho}{2} + \frac{(\alpha \mp \rho)(m+1)}{2(\delta + m + 1)} r. \tag{21}$$

In this case, equation (7) becomes

$$y = \frac{-\alpha \pm \rho}{2(\delta + m + 1)} \delta\phi(r - \phi). \tag{22}$$

Then equation (5) reduces to

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \frac{-\alpha \pm \rho}{2(\delta + m + 1)} \delta\phi(r - \phi). \tag{23}$$

Similar to case (i), we can obtain an explicit travelling solitary wave solutions to Eq.(1)

$$u(x,t) = \left[\frac{r}{2} + \frac{r}{2}\tanh\left(\frac{(-\alpha \pm \rho)\delta r}{4(\delta + m + 1)}[x - ct + x_0]\right)\right]^{1/\delta}, \tag{24}$$

$$u(x,t) = \left[\frac{r}{2} + \frac{r}{2}\coth\left(\frac{(-\alpha \pm \rho)\delta r}{4(\delta + m + 1)}[x - ct + x_0]\right)\right]^{1/\delta}, \tag{25}$$

where x_0 is an arbitrary constant and c is determined by Eqs.(21).

Case (iii) Assume that $a_0(\phi) = A_3(\phi - 1)(\phi - r), p(\phi) = B_3\phi^2$. Substituting them into Eqs.(12) and (13) and equating the corresponding coefficients of ϕ^2, ϕ and constant terms, we have

$$\begin{aligned}
&A_3 B_3 = \beta\delta,\\
&\left(1 + \frac{m+1}{\delta}\right) A_3 + \alpha = B_3,\\
&(r+1)\left(1 - \frac{m+1}{\delta}\right) - c - A_3(r+1) = 0,\\
&\left(1 - \frac{m+1}{\delta}\right) r A_3 = 0.
\end{aligned} \tag{26}$$

Solving Eq.(26), we can obtain that

$$\begin{aligned}
&A_3 = \frac{-\alpha \pm \sqrt{\alpha^2 + 8\beta\delta}}{4},\\
&B_3 = \frac{\alpha \pm \sqrt{\alpha^2 + 8\beta\delta}}{2},\\
&\delta = m + 1,\\
&c = \frac{\alpha \mp \sqrt{\alpha^2 + 8\beta\delta}}{4}(r+1).
\end{aligned} \tag{27}$$

In this case, equation (7) becomes

$$y = \frac{-\alpha \pm \sqrt{\alpha^2 + 8\beta\delta}}{4}(\phi - 1)(r - \phi). \tag{28}$$

So euqation (5) reduces to

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \frac{-\alpha \pm \sqrt{\alpha^2 + 8\beta\delta}}{4}(\phi - 1)(r - \phi). \tag{29}$$

Similar to the previous arguments, we can obtain that when $\delta = m + 1$, equation (1) has explicit travelling solitary wave solutions as follows:

$$u(x,t) = \left[\frac{r+1}{2} + \frac{1-r}{2}\tanh\left(\frac{(-\alpha \pm \sqrt{\alpha^2 + 8\beta\delta})(1-r)}{8}(x - ct + x_0)\right)\right]^{1/\delta}, \tag{30}$$

$$u(x,t) = \left[\frac{r+1}{2} + \frac{1-r}{2}\coth\left(\frac{(-\alpha \pm \sqrt{\alpha^2 + 8\beta\delta})(1-r)}{8}(x - ct + x_0)\right)\right]^{1/\delta}, \tag{31}$$

where x_0 is an arbitrary constant and c is determined by Eqs.(27).

It is easy to see that the exact solutions (20) and (21) described in Ref.[17] by using nonlinear transformations are merely identical to our results (18) and (24), respectively. However, there are two minor errors in Ref.[17], that is, $x + \frac{\alpha[(m+1+\delta)r + m + 1]}{2(\delta + 1 + m)} \pm \frac{[(m+1+\delta)r - m - 1]\sqrt{\alpha^2 + 4\beta(\delta + 1 + m)}}{2(\delta + m + 1)} t$ in formula (20) should be $x - \left\{\frac{\alpha[(m+1+\delta)r + m + 1]}{2(\delta + 1 + m)} + \frac{[(m+1+\delta)r - m - 1]\sqrt{\alpha^2 + 4\beta(\delta + 1 + m)}}{2(\delta + m + 1)}\right\} t$, while $x - \frac{\alpha[(m+1)r + m + 1 + \delta]}{2(\delta + 1 + m)} \mp \frac{[(m+1)r - m - 1 - \delta]\sqrt{\alpha^2 + 4\beta(\delta + 1 + m)}}{2(\delta + m + 1)} t$ in formula (21) should be $x -$

$\left\{\dfrac{\alpha[(m+1)r+m+1+\delta]}{2(\delta+1+m)} - \dfrac{[(m+1)r-m-1-\delta]\sqrt{\alpha^2+4\beta(\delta+1+m)}}{2(\delta+m+1)}\right\}t$. Moreover, to the best of our knowledge, it seems that results (30) and (31) have not been presented before.

Besides, note that when $m = 0$, equation (1) reduces to the generalized Burgers–Huxley equation. Apparently, it means that all the results described in Refs.[22] and [23] are also corrected and generalized, respectively.

3. Conclusions

In summary, the exact travelling solitary waves for the generalized Burgers–Huxley equation with nonlinear terms of any order are established, by using the first integral method. It is shown that this method, to some extent, appears to give a rigorous basis for the use of traditional methods, especially the Riccati equation ansatze method,[26,27] for solving the nonlinear wave equation. We believe that this method can be applied widely to many other similar nonlinear evolution equations.

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