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Conformal invariance and conserved quantities of general holonomic systems in phase space*

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This paper studies the conformal invariance and conserved quantities of general holonomic systems in phase space. The definition and the determining equation of conformal invariance for general holonomic systems in phase space are provided. The conformal factor expression is deduced from conformal invariance and Lie symmetry. The relationship between the conformal invariance and the Lie symmetry is discussed, and the necessary and sufficient condition that the conformal invariance would be the Lie symmetry of the system under the infinitesimal single-parameter transformation group is deduced. The conserved quantities of the system are given. An example is given to illustrate the application of the result.

Keywords: phase space, conformal invariance, conformal factor, conserved quantity
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1. Introduction

The symmetry properties of a physical system are important tools, they reduce the mathematical complexity of the system and also lead to physical consequences, such as conservation laws. The laws of conservation are powerful means of research, and permit us to obtain numerous important data on the problem of how mechanical phenomena proceed without having to solve equations of motion. There are mainly three kinds of symmetries: the Noether symmetry,[1] the Lie symmetry[2] and the Mei symmetry,[3] consequently, one has three kinds of conserved quantities: the Noether conserved quantity, the Hojman conserved quantity and the Mei conserved quantity. These theories developed rapidly and provided us some direct and useful methods of solving practical problems.[4−24]

The conformal invariance is a type of symmetry. It was introduced into physics by Conningham[25] and Bateman[24] shortly after Einstein's first paper on special relativity. Then it was applied widely to gauge theory[26] and in quantum field theory.[27] Recently, conformal invariance developed rapidly in dynamical mechanical systems,[28,29] and in 1997, Galiulin *et al*, the Russian scholars, studied conformal invariance (or conformal symmetry) for Birkhoff systems under special infinitesimal transformations and got the Noether conserved quantity.[30] The principle of conformal invariance demands invariance of all fundamental equations of physics under special transformations of units, i.e. conformal transformations, this means that the equation influencing the infinitesimal operators equals a conformal multiplier matrix of multipliers itself. Cai *et al* studied the conformal invariance and conserved quantities of mechanical systems.[31−37] In Ref.[31], Cai studied the conformal invariance and conserved quantities of a general holonomic system, in this paper, we study the conformal invariance and conserved quantities of general holonomic systems in phase space and obtain the Noether conserved quantity.

2. Conformal invariance of general holonomic systems in phase space

We consider a mechanical system composed of N particles, and its configuration is determined by the n generalized coordinates q_s $(s = 1, \ldots, n)$. The differential equations of motion for the mechanical system

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in phase space can be written in the form

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} + Q_s \quad (s = 1, \dots, n), \quad (1)$$

where $H = H(t, \boldsymbol{q}, \dot{\boldsymbol{p}})$ is the Hamiltonian of the system, $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{p}})$ are the non-potential generalized forces, q_s and p_s are canonical variables. Equation (1) can be expressed as follows:

$$\dot{q}_s = g_s(t, \boldsymbol{q}, \dot{\boldsymbol{p}}), \quad \dot{p}_s = h_s(t, \boldsymbol{q}, \dot{\boldsymbol{p}}), (s = 1, \dots, n) \quad (2)$$

or

$$F_s \equiv \begin{bmatrix} F_s^q \\ F_s^p \end{bmatrix} = \begin{bmatrix} \dot{q}_s - g_s(t, \boldsymbol{q}, \dot{\boldsymbol{p}}) \\ \dot{p}_s - h_s(t, \boldsymbol{q}, \dot{\boldsymbol{p}}) \end{bmatrix} = 0, (s = 1, \dots, n). \quad (3)$$

The equations of motion of the nonsingular general holonomic mechanical system can be written in canonical forms under the Legendre transformation. We introduce a single-parameter infinitesimal transformation with respect to time, generalized coordinates and generalized momenta

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \boldsymbol{p}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \boldsymbol{p}), \\ p_s^*(t^*) &= p_s(t) + \varepsilon\eta_s(t, \boldsymbol{q}, \boldsymbol{p}), \end{aligned} \quad (4)$$

where ε is an infinitesimal parameter, ξ_0, ξ_s and η_s are called the infinitesimal generators of t, q_s, p_s under the infinitesimal transformations (4), respectively. We introduce an infinitesimal generator vector

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \eta_s \frac{\partial}{\partial p_s}, \quad (5)$$

and its first extension

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s} + (\dot{\eta}_s - \dot{p}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{p}_s}. \quad (6)$$

Definition If there is a nonsingular matrix H_s^k satisfying the condition

$$X^{(1)}(F_s) = H_s^k(F_k), \quad (7)$$

then this kind of invariance is called the conformal invariance of Eq.(3) under single-parameter infinitesimal transformations (4). Equation (7) is the determining equation of conformal invariance of the general holonomic systems in phase space, and H_s^k is called the conformal factor. If the generators ξ_0, ξ_s and η_s of the infinitesimal transformations (4) satisfy the determining equation (7), then the above transformations are called conformal transformations.

3. The general expressions of conformal invariance

In order to get the expressions of the conformal factor of conformal invariance, we calculate the difference of

$$X^{(1)}(F_s) - X^{(1)}(F_s)\,|_{F_s=0}\,. \quad (8)$$

Since

$$\dot{\xi}_s = \frac{\partial \xi_s}{\partial t} + \frac{\partial \xi_s}{\partial q_k}\dot{q}_k + \frac{\partial \xi_s}{\partial p_k}\dot{p}_k, \quad (9)$$

$$\dot{\eta}_s = \frac{\partial \eta_s}{\partial t} + \frac{\partial \eta_s}{\partial q_k}\dot{q}_k + \frac{\partial \eta_s}{\partial p_k}\dot{p}_k \quad (s, k = 1, \dots, n), \quad (10)$$

we have

$$\begin{aligned} X^{(1)}(F_s^q) &= \dot{\xi}_s - \dot{q}_s\dot{\xi}_0 - \frac{\partial g_s}{\partial t}\xi_0 - \frac{\partial g_s}{\partial q_k}\xi_k - \frac{\partial g_s}{\partial p_k}\eta_k \\ &= \frac{\partial \xi_s}{\partial t} + \frac{\partial \xi_s}{\partial q_k}\dot{q}_k + \frac{\partial \xi_s}{\partial p_k}\dot{p}_k - \dot{q}_s\dot{\xi}_0 - \frac{\partial g_s}{\partial t}\xi_0 - \frac{\partial g_s}{\partial q_k}\xi_k - \frac{\partial g_s}{\partial p_k}\eta_k, \end{aligned} \quad (11)$$

$$\begin{aligned} X^{(1)}(F_s^p) &= \dot{\eta}_s - \dot{p}_s\dot{\xi}_0 - \frac{\partial h_s}{\partial t}\xi_0 - \frac{\partial h_s}{\partial q_k}\xi_k - \frac{\partial h_s}{\partial p_k}\eta_k \\ &= \frac{\partial \eta_s}{\partial t} + \frac{\partial \eta_s}{\partial q_k}\dot{q}_k + \frac{\partial \eta_s}{\partial p_k}\dot{p}_k - \dot{p}_s\dot{\xi}_0 - \frac{\partial h_s}{\partial t}\xi_0 - \frac{\partial h_s}{\partial q_k}\xi_k - \frac{\partial h_s}{\partial p_k}\eta_k, \end{aligned} \quad (12)$$

$$X^{(1)}(F_s^q)\Big|_{F_s^q=0} = \frac{\partial \xi_s}{\partial t} + \frac{\partial \xi_s}{\partial q_k}g_k + \frac{\partial \xi_s}{\partial p_k}\dot{p}_k - g_s\dot{\xi}_0 - \frac{\partial g_s}{\partial t}\xi_0 - \frac{\partial g_s}{\partial q_k}\xi_k - \frac{\partial g_s}{\partial p_k}\eta_k, \quad (13)$$

$$X^{(1)}(F_s^p)\Big|_{F_s^p=0} = \frac{\partial \eta_s}{\partial t} + \frac{\partial \eta_s}{\partial q_k}\dot{q}_k + \frac{\partial \eta_s}{\partial p_k}h_k - h_s\dot{\xi}_0 - \frac{\partial h_s}{\partial t}\xi_0 - \frac{\partial h_s}{\partial q_k}\xi_k - \frac{\partial h_s}{\partial p_k}\eta_k. \quad (14)$$

From Eqs.(11)–(14), we have

$$\begin{aligned}
&X^{(1)}(F_s^q) - X^{(1)}(F_s^q)\,|_{F_s^q=0} \\
&= \left(\frac{\partial \xi_s}{\partial q_k} - \delta_s^k \dot{\xi}_0\right)(\dot{q}_k - g_k) \\
&= \left(\frac{\partial \xi_s}{\partial q_k} - \delta_s^k \dot{\xi}_0\right)(F_k^q) \\
&= U_s^k(F_k^q), \\
&[X^{(1)}(F_s^p) - X^{(1)}(F_s^p)\,|_{F_s^p=0} \\
&= \left(\frac{\partial \eta_s}{\partial p_k} - \delta_s^k \dot{\xi}_0\right)(\dot{p}_k - h_k) \\
&= \left(\frac{\partial \eta_s}{\partial p_k} - \delta_s^k \dot{\xi}_0\right)(F_k^p) \\
&= V_s^k(F_k^q), \quad (s,k=1,\dots,n),
\end{aligned} \tag{15}$$

or

$$\begin{aligned}
&X^{(1)}(F_s) - X^{(1)}(F_s)\,|_{F_s=0} = \beta_s^k(F_k) \\
&(s,k=1,\dots,n),
\end{aligned} \tag{16}$$

where

$$F_s = \begin{pmatrix} F_s^q \\ F_s^p \end{pmatrix}, \quad \beta_s^k = \begin{pmatrix} U_s^k & 0 \\ 0 & V_s^k \end{pmatrix},$$

$$U_s^k = \left(\frac{\partial \xi_s}{\partial q_k} - \delta_s^k \dot{\xi}_0\right),$$

$$V_s^k = \left(\frac{\partial \eta_s}{\partial p_k} - \delta_s^k \dot{\xi}_0\right).$$

If equation (3) is of Lie symmetry under the infinitesimal point transformations (4), then

$$X^{(1)}(F_s)\,|_{F_s=0} = 0, \tag{17}$$

from Eqs.(16) and (7), we have $H_s^k = \beta_s^k$. If equation (3) is the conformal invariance under the infinitesimal point transformations (4), from Eq.(16), we get

$$(H_s^k - \beta_s^k)(F_k) = X^{(1)}(F_s)\,|_{F_s=0} \quad (s,k=1,\dots,n), \tag{18}$$

then the necessary and sufficient condition that the conformal invariance would be the Lie symmetry of the system is

$$H_s^k = \beta_s^k, \tag{19}$$

then we give the following proposition.

Proposition 1 Under the infinitesimal point transformations (4), the necessary and sufficient condition that the conformal invariance would be the Lie symmetry of Eq.(3) is the conformal factor

$$H_s^k = \begin{pmatrix} U_s^k & 0 \\ 0 & V_s^k \end{pmatrix}, \tag{20}$$

where

$$\begin{aligned}
U_s^k &= \left(\frac{\partial \xi_s}{\partial q_k} - \delta_s^k \dot{\xi}_0\right), \\
V_s^k &= \left(\frac{\partial \eta_s}{\partial p_k} - \delta_s^k \dot{\xi}_0\right) \quad (s,k=1,\dots,n).
\end{aligned} \tag{21}$$

4. Conformal invariance and conserved quantity of general holonomic systems in phase space

The following proposition will present a Noether conserved quantity led by the conformal invariance of general holonomic systems in phase space under certain conditions.

Proposition 2 For a general holonomic system in phase space, if the infinitesimal transformations (4) satisfy the conformal factor Eqs.(20) and (21), and there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{p}})$ satisfying the following structure equation of Lie symmetry

$$\begin{aligned}
&-H\dot{\xi}_0 - \frac{\partial H}{\partial t}\xi_0 - \frac{\partial H}{\partial q_s}\xi_s + p_s\dot{\xi}_s \\
&\qquad + Q_s(\xi_s - \dot{q}_s\xi_0) + \dot{G} = 0,
\end{aligned} \tag{22}$$

then the conformal invariance of the general holonomic system in phase space possesses the following conserved quantity:

$$I = -H\xi_0 + p_s\xi_s + G = \text{const.} \tag{23}$$

Proof

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= -\dot{H}\xi_0 - H\dot{\xi}_0 + \dot{p}_s\xi_s + p_s\dot{\xi}_s + H\dot{\xi}_0 \\
&\quad + \frac{\partial H}{\partial t}\xi_0 + \frac{\partial H}{\partial q_s}\xi_s - p_s\dot{\xi}_s - Q_s(\xi_s - \dot{q}_s\xi_0) \\
&= \left(\frac{\partial H}{\partial q_s} - Q_s\right)(\xi_s - \dot{q}_s\xi_0) + \dot{p}_s\left(\xi_s - \frac{\partial H}{\partial p_s}\xi_0\right) \\
&= 0.
\end{aligned}$$

5. An illustrative example

The Hamiltonian of a mechanical system is

$$H = \frac{1}{2}(p_1^2 + p_2^2). \tag{24}$$

The nonpotential generalized forces are

$$Q_1 = q_1, \quad Q_2 = q_2, \tag{25}$$

Let us study the conformal invariance and conserved quantity of this system.

The equations of motion are

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_1 = q_1, \quad \dot{p}_2 = q_2. \tag{26}$$

Taking

$$\xi_0 = 0, \quad \xi_1 = q_1, \quad \xi_2 = q_2, \quad \eta_1 = p_1, \quad \eta_2 = p_2, \tag{27}$$

we have

$$X^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \eta_s \frac{\partial}{\partial p_s} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s} + \dot{\eta}_s \frac{\partial}{\partial \dot{p}_s}, \tag{28}$$

$$\begin{aligned} X^{(1)} \begin{pmatrix} F_1^q \\ F_2^q \end{pmatrix} &= \left(\xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \eta_s \frac{\partial}{\partial p_s} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s} + \dot{\eta}_s \frac{\partial}{\partial \dot{p}_s} \right) \begin{pmatrix} \dot{q}_1 - p_1 \\ \dot{q}_2 - p_2 \end{pmatrix} \\ &= \begin{pmatrix} \dot{\xi}_1 - \eta_1 \\ \dot{\xi}_2 - \eta_2 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 - p_1 \\ \dot{q}_2 - p_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{q}_1 - p_1 \\ \dot{q}_2 - p_2 \end{pmatrix}, \end{aligned} \tag{29}$$

$$\begin{aligned} X^{(1)} \begin{pmatrix} F_1^p \\ F_2^p \end{pmatrix} &= \left(\xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \eta_s \frac{\partial}{\partial p_s} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s} + \dot{\eta}_s \frac{\partial}{\partial \dot{p}_s} \right) \begin{pmatrix} \dot{p}_1 - q_1 \\ \dot{p}_2 - q_2 \end{pmatrix} \\ &= \begin{pmatrix} \dot{\eta}_1 - \xi_1 \\ \dot{\eta}_2 - \xi_2 \end{pmatrix} = \begin{pmatrix} \dot{p}_1 - q_1 \\ \dot{p}_2 - q_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{p}_1 - q_1 \\ \dot{p}_2 - q_2 \end{pmatrix}. \end{aligned} \tag{30}$$

The conformal factor is

$$U_s^k = V_s^k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{31}$$

From Eq.(21), we also obtain the conformal factor

$$U_s^k = \left(\frac{\partial \xi_s}{\partial q_k} - \delta_s^k \dot{\xi}_0 \right) = \frac{\partial \xi_s}{\partial q_k}, \tag{32}$$

$$V_s^k = \left(\frac{\partial \eta_s}{\partial p_k} - \delta_s^k \dot{\xi}_0 \right) = \frac{\partial \eta_s}{\partial p_k}. \tag{33}$$

The determining equations of conformal invariance are

$$X^{(1)} \begin{pmatrix} F_s^q \\ F_s^p \end{pmatrix} = \begin{pmatrix} U_s^k & 0 \\ 0 & V_s^k \end{pmatrix} \begin{pmatrix} F_k^q \\ F_k^p \end{pmatrix}. \tag{34}$$

The conformal invariance would be the Lie symmetry of the system. From Eqs.(27) and (22), we obtain

$$G = -\frac{1}{3}(p_1^3 + p_2^3 + q_1^3 + q_2^3). \tag{35}$$

The conserved quantity (23) gives

$$I = p_1 q_1 + p_2 q_2 - \frac{1}{3}(p_1^3 + p_2^3 + q_1^3 + q_2^3) = \text{const.} \tag{36}$$

6. Conclusions

For general holonomic systems in phase space, by using the definition of conformal invariance and the Lie symmetrical determining equations of conformal invariance, we can obtain the conformal factor of determining equations of conformal invariance. The conformal factor is also the necessary and sufficient condition that the conformal invariance would be the Lie symmetry of the system. The conformal invariance can also lead to a conserved quantity under certain conditions. The main results of this paper are Eqs.(19), (20), (21) and (23).

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