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Poisson theory of generalized Bikhoff equations*

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This paper presents a Poisson theory of the generalized Birkhoff equations, including the algebraic structure of the equations, the sufficient and necessary condition on the integral and the conditions under which a new integral can be deduced by a known integral as well as the form of the new integral.

Keywords: generalized Birkhoff equations, Poisson theory, integrals
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1. Introduction

The Birkhoff system, established in Refs.[1] and [2] has become a new research field of analytic mechanics. The most recent literature offers various results concerning the stability of the Birkhoff system, symmetries and their conserved quantities of the system (See e.g. Refs.[3]–[11]). In 1993, the generalized Birkhoff equation was proposed by Mei Fengxiang and the Noether symmetry and the Noether conserved quantities were studied in his work.[12] Theoretically speaking, the generalized Birkhoff equation has universality in comparison with the Birkhoff equation, from the practical point of view, the generalized Birkhoff equation is easier to be constructed. The Poisson theory of generalized Birkhoff equation is discussed in this article, and the generalized Poisson condition as well as the method of deriving a new integral from a known integral is presented.

2. Generalized Birkhoff's equations

The generalized Birkhoff equation has the form[12]

$$\Omega_{\mu\nu}\dot{a}^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}=-\Lambda_{\mu},\quad(\mu,\nu=1,\cdots,2n). \tag{1}$$

And $B=B(t,\boldsymbol{a})$ is the Birkhoffian, $R_{\mu}=R_{\mu}(t,\boldsymbol{a})$ are the Bikhoff's functions, $\Lambda_{\mu}=\Lambda_{\mu}(t,\boldsymbol{a})$ is the additive term of Bikhoff's equation,

$$\Omega_{\mu\nu}=\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}} \tag{2}$$

is the Birkhoff's tensor. Suppose that system (1) is non-singular, namely

$$\det(\Omega_{\mu\nu})\neq 0. \tag{3}$$

Thus, the equation (1) has the form of

$$\dot{a}^{\mu}-\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^{\nu}}+\frac{\partial R_{\nu}}{\partial t}-\Lambda_{\nu}\right)=0. \tag{4}$$

If there is not additional term Λ_{μ} in Eq.(1), then it is called Birkhoff's equation

$$\Omega_{\mu\nu}\dot{a}^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}=0. \tag{5}$$

3. The algebraic structure of the generalized Birkhoff's equation

The generalized Birkhoff's equation (4) can be expressed as the following form

$$\dot{a}^{\mu}-S^{\mu\nu}\frac{\partial B}{\partial a^{\nu}}=0, \tag{6}$$

where

$$S^{\mu\nu}=\Omega^{\mu\nu}+T^{\mu\nu}, \tag{7}$$

$$\Omega^{\mu\nu}\left(\frac{\partial R_{\nu}}{\partial t}-\Lambda_{\nu}\right)=T^{\mu\nu}\frac{\partial B}{\partial a^{\nu}}. \tag{8}$$

After making the derivative of function $A(\boldsymbol{a})$ with respect to time along with Eq.(5), we have a definition of product

$$\dot{A}=\frac{\partial A}{\partial a^{\mu}}S^{\mu\nu}\frac{\partial B}{\partial a^{\nu}}\overset{\text{def}}{=}A\circ B. \tag{9}$$

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This product has the property of admissible algebraic structure. Then another product can be defined as

$$[A, B] \stackrel{\text{def}}{=} A \circ B - B \circ A. \tag{10}$$

This product has the property of Lie algebraic structure. Hence, this product has the property of Lie admissible algebraic structure.

4. Poisson theory of the generalized Birkhoff's equation

Based on the above-mentioned algebraic structure, we have the following Poisson theory of the generalized Birkhoff's equation.

1) The necessary and sufficient condition for $I = I(t, \boldsymbol{a})$ whether it is an integral of Eq.(1) is that the following generalized Poisson condition is satisfied:

$$\frac{\partial I}{\partial t} + I \circ B = 0. \tag{11}$$

2) If the variable t is included in the function $I = I(t, \boldsymbol{a})$ which is an integral of Eq.(1), whereas it is not involved explicitly in the expressions $S^{\mu\nu}$ and B, then $\partial I/\partial t$, $\partial^2 I/\partial t^2, \ldots$ are also integrals of Eq.(1).

3) If the variable a^ρ is included in the function $I = I(t, \boldsymbol{a})$ which is an integral of Eq.(1), whereas it is not involved explicitly in the expressions $S^{\mu\nu}$ and B, then $\partial I/\partial a^\rho$, $\partial^2 I/\partial a^{\rho 2}, \ldots$ are also integrals of Eq.(1).

These three items compose the Poisson theory of the generalized Birkhoff's equation. The generalized Poisson condition (11) can be easily derived from Eqs.(6) and (9). The two latter items of the generalized Poisson theory can be proved by Eq.(11). The generalized Poisson condition (11) is usually applied to verify a function whether it is an integral of a differential equation. The two latter items of the generalized Poisson theory are applied to derive the new integral by the known integral. In general, if I_1 and I_2 are integrals of Eq.(1), then $I_1 \circ I_2$ is not an integral of Eq.(1).

5. Illustrating examples

Example 1 If the following condition is satisfied

$$\frac{\partial B}{\partial t} = 0, \quad \left(\frac{\partial R_\mu}{\partial t} - \Lambda_\mu\right)\dot{a}^\mu = 0, \tag{12}$$

then there is an integral of Eq.(1)

$$I = B = \text{const}. \tag{13}$$

Proof The generalized Poisson condition (11) gives

$$\begin{aligned} 0 &= \frac{\partial B}{\partial t} + \frac{\partial B}{\partial a^\mu} S^{\mu\nu} \frac{\partial B}{\partial a^\nu} \\ &= \frac{\partial B}{\partial t} + \frac{\partial B}{\partial a^\mu} \Omega^{\mu\nu} \left(\frac{\partial R_\nu}{\partial t} - \Lambda_\nu\right) \\ &= \frac{\partial B}{\partial t} + \left(\frac{\partial R_\nu}{\partial t} - \Lambda_\nu\right)\dot{a}^\nu. \end{aligned} \tag{14}$$

Applying the results of Eq.(12) to Eq.(14), obviously, we find that equation (14) is true.

For the Birkhoff's system (5), because of $\Lambda_\mu = 0$ $(\mu = 1, \ldots, 2n)$, equation (12) is true, when

$$\frac{\partial B}{\partial t} = \frac{\partial R_\mu}{\partial t} = 0. \tag{15}$$

Therefore, Birkhoffian B is an integral of the autonomous Birkhoff's system.

Example 2 Whittaker equations are

$$\ddot{q}_1 = q_1, \quad \ddot{q}_2 = \dot{q}_1. \tag{16}$$

Let

$$a^1 = q_1, \ a^2 = q_2, \ a^3 = \dot{q}_1, \ a^4 = \dot{q}_2,$$

then the equation (16) can be represented as

$$\dot{a}^1 = a^3, \ \dot{a}^2 = a^4, \ \dot{a}^3 = a^1, \ \dot{a}^4 = a^3. \tag{17}$$

Alternatively, equation (17) can be transformed into the form of a generalized Birkhoff's equation

$$\begin{aligned} &R_1 = a^3, \ R_2 = a^4, \ R_3 = R_4 = 0, \\ &B = \frac{1}{2}\left(a^3\right)^2 + \frac{1}{2}\left(a^4\right)^2, \\ &\Lambda_1 = a^1, \ \Lambda_2 = a^3, \ \Lambda_3 = \Lambda_4 = 0. \end{aligned} \tag{18}$$

According to Eqs.(7) and (8), we have

$$(S^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & a^1/a^3 & 0 \\ 0 & -1 & 0 & a^3/a^4 \end{pmatrix}. \tag{19}$$

There is an integral of Eqs.(17)

$$I_1 = a^3 + a^4 t - a^2 - a^1 t = \text{const}. \tag{20}$$

It can be verified by Eq.(11). The variable t is included in the expression I_1, whereas it is not involved in the expressions $S^{\mu\nu}$ and B. According to the second item of the Poisson theory of the generalized Birkhoff's

equation, it follows that a new integral can be derived from Eq.(20)

$$I_2 = \frac{\partial I_1}{\partial t} = a^4 - a^1 = \text{const.} \tag{21}$$

There is an integral of Eq.(17)

$$I_3 = \left(a^3 + a^4 t - a^2 - a^1 t\right)^2 = \text{const.} \tag{22}$$

It can be verified by Eq.(11). The variable a^2 is included in the expression I_3, whereas it is not involved in the expressions $S^{\mu\nu}$ and B. According to the third item of Poisson theory of the generalized Birkhoff's equation, it follows that a new integral can be derived from Eq.(22)

$$I_4 = \frac{\partial I_3}{\partial a^2} = -2\left(a^3 + a^4 t - a^2 - a^1 t\right) = \text{const.} \tag{23}$$

Equation (23) is not independent of Eq.(22).

6. Conclusions

The Poisson integral theory of the generalized Birkhoff's equation is established in this paper. The necessary and sufficient condition for integrating the differential equation, that is the generalized Poisson condition (11), is introduced, the method for deriving new integrals from the known integrals under some circumstances is also discussed. The Poisson integral theory of the Birkhoff's equation can be viewed as a special case of the theory of the generalized Birkhoff's equation.

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