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Symmetry of Lagrangians of holonomic systems in terms of quasi-coordinates*

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This paper discusses the symmetry of Lagrangians of holonomic systems in terms of quasi-coordinates. Firstly, the definition and the criterion of the symmetry are given. Secondly, the condition under which there exists a conserved quantity and the form of the conserved quantity are obtained. Finally, an example is shown to illustrate the application of the results.

Keywords: quasi-coordinate, holonomic system, symmetry of Lagrangians, conserved quantity
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1. Introduction

The differential equations of motion of a mechanical system can be expressed in terms of either generalized coordinates or quasi-coordinates. The latter have better universality, so their application gets more attention.[1] Reference [2] studied the symmetries and the conserved quantities of holonomic systems in terms of quasi-coordinates, including the Noether symmetry, the Lie symmetry, the form invariance and the conserved quantities that arise from them: the Noether conserved quantity, the Hojman conserved quantity and the new conserved quantity. References [3]–[11] discussed different kinds of symmetries and conserved quantities. But all the above belong to the invariance obtained under infinitesimal transformation of time and coordinates. In 1966, the authors in Ref.[12] studied the problem of equivalent Lagrangians for mechanical systems with a single degree of freedom and pointed out that there exist conserved quantities in this case. In 1981, the authors in Ref.[13] extended the result to general multi-degree-of-freedom systems. The results of Refs.[12] and [13] can only be applied to conservative holonomic systems. Reference [14] discussed the problem of equivalent systems of functions: (L, Q_s) and $(\bar{L}, \bar{Q}_s)$ for nonconservative holonomic systems, and pointed out the condition under which there exist conserved quantities and the formation of a conserved quantity. Reference [15] addressed the symmetry of Lagrangians of nonholonomic systems. This paper extends the result of Ref.[14] to holonomic systems in terms of quasi-coordinates.

2. Differential equations of motion of systems

Suppose that the position of a mechanical system is determined by the n generalized coordinates $q_s\,(s = 1, \ldots, n)$. Introduce from elsewhere n quasi-velocities ω_s which are independent and compatible with each other:

$$\omega_s = a_{sk}(\boldsymbol{q})\dot{q}_k, \qquad (s, k = 1, \ldots, n). \tag{1}$$

And assume that from which all generalized velocities $\dot{q}_s$ can be solved as:

$$\dot{q}_s = b_{sk}(\boldsymbol{q})\omega_k. \tag{2}$$

Then the differential equations of motion of the system have the form[16]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \omega_s} + \frac{\partial L^*}{\partial \omega_k}\gamma^k_{rs}\omega_r - \frac{\partial L^*}{\partial \pi_s} = P^*_s, \tag{3}$$

where

$$\gamma^k_{rs} = \left(\frac{\partial a_{km}}{\partial q_l} - \frac{\partial a_{kl}}{\partial q_m}\right) b_{lr} b_{ms} \tag{4}$$

are Boltzmann three-index symbols, satisfying

$$\gamma^k_{rs} = -\gamma^k_{sr}, \tag{5}$$

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and

$$L^*(t,q_s,\omega_s)=L(t,q_s,b_{sk}\omega_k) \tag{6}$$

is the Lagrangian in terms of quasi-velocities. The partial derivative with respect to quasi-coordinate π_s is

$$\frac{\partial}{\partial\pi_s}=\frac{\partial}{\partial q_k}b_{ks}, \tag{7}$$

and P_s^* are non-potential generalized forces in terms of quasi-velocities

$$P_s^*(t,q_k,\omega_k)=Q_r(t,q_s,b_{sk}\omega_k)b_{rs}. \tag{8}$$

3. Definition and criterion of symmetry of Lagrangians

Discuss two groups of dynamical functions: L^*,P_s^* and $\bar{L}^*,\bar{P}_s^*$. Let

$$\begin{aligned}\mathcal{L}_s^*&=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial\omega_s}+\frac{\partial L^*}{\partial\omega_k}\gamma_{rs}^k\omega_r-\frac{\partial L^*}{\partial\pi_s}-P_s^*\\&=W_{sk}\dot{\omega}_k+\frac{\partial^2L^*}{\partial\omega_s\partial\pi_r}\omega_r+\frac{\partial^2L^*}{\partial\omega_s\partial t}+\frac{\partial L^*}{\partial\omega_k}\gamma_{rs}^k\omega_r\\&\quad-\frac{\partial L^*}{\partial\pi_s}-P_s^*,\end{aligned} \tag{9}$$

$$\begin{aligned}\bar{\mathcal{L}}_s^*&=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\bar{L}^*}{\partial\omega_s}+\frac{\partial\bar{L}^*}{\partial\omega_k}\gamma_{rs}^k\omega_r-\frac{\partial\bar{L}^*}{\partial\pi_s}-\bar{P}_s^*\\&=\bar{W}_{sk}\dot{\omega}_k+\frac{\partial^2\bar{L}^*}{\partial\omega_s\partial\pi_r}\omega_r+\frac{\partial^2\bar{L}^*}{\partial\omega_s\partial t}+\frac{\partial\bar{L}^*}{\partial\omega_k}\gamma_{rs}^k\omega_r\\&\quad-\frac{\partial\bar{L}^*}{\partial\pi_s}-\bar{P}_s^*,\end{aligned} \tag{10}$$

where

$$W_{sk}=\frac{\partial^2L^*}{\partial\omega_s\partial\omega_k},\quad \bar{W}_{sk}=\frac{\partial^2\bar{L}^*}{\partial\omega_s\partial\omega_k}. \tag{11}$$

Definition For the system (3), if one can obtain $\bar{\mathcal{L}}_k^*=0\,(k=1,\ldots,n)$ from $\mathcal{L}_s^*=0\,(s=1,\ldots,n)$ and *vice versa*, then the symmetry is called Lagrangian symmetry.

From $\bar{\mathcal{L}}_s^*=0$, solving all $\dot{\omega}_s$, we have

$$\begin{aligned}\dot{\omega}_s&=\bar{U}^{sk}\bigg(\bar{P}_k^*+\frac{\partial\bar{L}^*}{\partial\pi_k}-\frac{\partial^2\bar{L}^*}{\partial\omega_k\partial t}\\&\quad-\frac{\partial^2\bar{L}^*}{\partial\omega_k\partial\pi_r}\omega_r-\frac{\partial\bar{L}^*}{\partial\omega_r}\gamma_{lk}^r\omega_l\bigg),\end{aligned} \tag{12}$$

where

$$\bar{W}_{sk}\bar{U}^{kr}=\delta_s^r. \tag{13}$$

Substituting Eq.(12) into $\mathcal{L}_s^*=0$, we obtain

$$\begin{aligned}&W_{sk}\bar{U}^{kl}\left(\bar{P}_l^*+\frac{\partial\bar{L}^*}{\partial\pi_l}-\frac{\partial^2\bar{L}^*}{\partial\omega_l\partial t}-\frac{\partial^2\bar{L}^*}{\partial\omega_l\partial\pi_r}\omega_r-\frac{\partial\bar{L}^*}{\partial\omega_r}\gamma_{ml}^r\omega_m\right)\\&=P_s^*+\frac{\partial L^*}{\partial\pi_s}-\frac{\partial^2L^*}{\partial\omega_s\partial t}-\frac{\partial^2L^*}{\partial\omega_s\partial\pi_r}\omega_r-\frac{\partial L^*}{\partial\omega_k}\gamma_{rs}^k\omega_r\quad(s,k,r,l,m=1,\ldots,n).\end{aligned} \tag{14}$$

Then we have the following criterion.

Criterion If two groups of dynamical functions L^*,P_s^* and $\bar{L}^*,\bar{P}_s^*$ satisfy Eq.(14), then the corresponding symmetry is a symmetry of Lagrangians of holonomic system (3) in terms of quasi-coordinates.

Equations (14) are called the criterion equations for symmetry of Lagrangians of holonomic systems in terms of quasi-coordinates.

4. Conserved quantity deduced by symmetry of Lagrangians

We now deduce the conserved quantity by using the symmetry of Lagrangians.

Substituting Eq.(14) into Eq.(9), and using Eqs.(10), (13), we obtain

$$\begin{aligned}\mathcal{L}_s^*&=W_{sr}\dot{\omega}_r-W_{sk}\bar{U}^{kl}\left(\bar{P}_l^*+\frac{\partial\bar{L}^*}{\partial\pi_l}-\frac{\partial^2\bar{L}^*}{\partial\omega_l\partial t}-\frac{\partial^2\bar{L}^*}{\partial\omega_l\partial\pi_r}\omega_r-\frac{\partial\bar{L}^*}{\partial\omega_r}\gamma_{ml}^r\omega_m\right)\\&=W_{sr}\bar{U}^{rk}\left(\bar{W}_{kl}\dot{\omega}_l+\frac{\partial^2\bar{L}^*}{\partial\omega_k\partial\pi_l}\omega_l+\frac{\partial^2\bar{L}^*}{\partial\omega_k\partial t}+\frac{\partial\bar{L}^*}{\partial\omega_l}\gamma_{mk}^l\omega_m-\frac{\partial\bar{L}^*}{\partial\pi_k}-\bar{P}_k^*\right)\\&=W_{sr}\bar{U}^{rk}\bar{\mathcal{L}}_k^*.\end{aligned} \tag{15}$$

Let

$$\bar{W}_{sr}U^{rk} = \Lambda^k_s, \tag{16}$$

then equation (15) can be written in the form

$$\bar{\mathcal{L}}^*_s = \Lambda^r_s \mathcal{L}^*_r. \tag{17}$$

By virtue of Eq.(16), we have

$$\bar{W}_{sk} = \Lambda^r_s W_{rk}. \tag{18}$$

Taking the partial derivative of Eq.(18) with respect to ω_l, we obtain

$$\begin{aligned}\frac{\partial \Lambda^r_s}{\partial \omega_l}W_{rk} &= \frac{\partial^3 \bar{L}^*}{\partial\omega_s\partial\omega_k\partial\omega_l} - \Lambda^r_s\frac{\partial^3 L^*}{\partial\omega_r\partial\omega_k\partial\omega_l}\\ &= \frac{\partial \bar{W}_{sl}}{\partial\omega_k} - \Lambda^r_s\frac{\partial W_{lr}}{\partial\omega_k}\\ &= \frac{\partial \Lambda^r_s}{\partial\omega_k}W_{lr}.\end{aligned} \tag{19}$$

Taking the partial derivative of Eq.(18) with respect to π_l, multiplying the result by ω_l and summing it over l, we have

$$\Lambda^r_s\frac{\partial W_{rk}}{\partial\pi_l}\omega_l - \frac{\partial \bar{W}_{sk}}{\partial\pi_l}\omega_l = -\frac{\partial \Lambda^r_s}{\partial\pi_l}W_{rk}\omega_l. \tag{20}$$

Also, criterion equations (14) can be written in the form

$$\frac{\partial^2 \bar{L}^*}{\partial\omega_s\partial\pi_r}\omega_r + \frac{\partial^2 \bar{L}^*}{\partial\omega_s\partial t} + \frac{\partial \bar{L}^*}{\partial\omega_r}\gamma^r_{ks}\omega_k - \frac{\partial \bar{L}^*}{\partial\pi_s} - \bar{P}^*_s = \Lambda^r_s\left(\frac{\partial^2 L^*}{\partial\omega_r\partial\pi_l}\omega_l + \frac{\partial^2 L^*}{\partial\omega_r\partial t} + \frac{\partial L^*}{\partial\omega_l}\gamma^l_{kr}\omega_k - \frac{\partial L^*}{\partial\pi_r} - P^*_r\right). \tag{21}$$

Taking the partial derivative of Eq.(21) with respect to ω_m, we obtain

$$\begin{aligned}&\frac{\partial^3 \bar{L}^*}{\partial\omega_s\partial\omega_m\partial\pi_r}\omega_r + \frac{\partial^2 \bar{L}^*}{\partial\omega_s\partial\pi_m} + \frac{\partial^3 \bar{L}^*}{\partial\omega_s\partial\omega_m\partial t} + \frac{\partial^2 \bar{L}^*}{\partial\omega_r\partial\omega_m}\gamma^r_{ks}\omega_k + \frac{\partial \bar{L}^*}{\partial\omega_r}\gamma^r_{ms} - \frac{\partial^2 \bar{L}^*}{\partial\pi_s\partial\omega_m} - \frac{\partial \bar{P}^*_s}{\partial\omega_m}\\ =\ &\frac{\partial \Lambda^r_s}{\partial\omega_m}\left(\frac{\partial^2 L^*}{\partial\omega_r\partial\pi_l}\omega_l + \frac{\partial^2 L^*}{\partial\omega_r\partial t} + \frac{\partial L^*}{\partial\omega_l}\gamma^l_{kr}\omega_k - \frac{\partial L^*}{\partial\pi_r} - P^*_r\right)\\ &+ \Lambda^r_s\left(\frac{\partial^3 L^*}{\partial\omega_r\partial\pi_l\partial\omega_m}\omega_l + \frac{\partial^2 L^*}{\partial\omega_r\partial\pi_m} + \frac{\partial^3 L^*}{\partial\omega_r\partial t\partial\omega_m} + \frac{\partial^2 L^*}{\partial\omega_l\partial\omega_m}\gamma^l_{kr}\omega_k + \frac{\partial L^*}{\partial\omega_l}\gamma^l_{mr} - \frac{\partial^2 L^*}{\partial\pi_r\partial\omega_m} - \frac{\partial P^*_r}{\partial\omega_m}\right),\end{aligned}$$

i.e.

$$\begin{aligned}&\frac{\partial \bar{W}_{sm}}{\partial\pi_r}\omega_r + \frac{\partial \bar{W}_{sm}}{\partial t} + \frac{\partial^2 \bar{L}^*}{\partial\omega_s\partial\pi_m} - \frac{\partial^2 \bar{L}^*}{\partial\omega_m\partial\pi_s} + \frac{\partial}{\partial\omega_m}\left(\frac{\partial \bar{L}^*}{\partial\omega_r}\gamma^r_{ks}\omega_k - \bar{P}^*_s\right)\\ =\ &\frac{\partial \Lambda^r_s}{\partial\omega_m}(-W_{rk}\dot{\omega}_k) + \Lambda^r_s\left(\frac{\partial W_{rm}}{\partial\pi_l}\omega_l + \frac{\partial^2 L^*}{\partial\omega_r\partial\pi_m} - \frac{\partial^2 L^*}{\partial\pi_r\partial\omega_m} + \frac{\partial W_{rm}}{\partial t} + W_{lm}\gamma^l_{kr}\omega_k + \frac{\partial L^*}{\partial\omega_l}\gamma^l_{mr} - \frac{\partial P^*_r}{\partial\omega_m}\right).\end{aligned} \tag{22}$$

Considering Eq.(19), we have

$$\frac{\partial \Lambda^r_s}{\partial\omega_m}W_{rk}\dot{\omega}_k = \frac{\partial \Lambda^r_s}{\partial\omega_k}W_{mr}\dot{\omega}_k = \left(\frac{\mathrm{d}\Lambda^r_s}{\mathrm{d}t} - \frac{\partial \Lambda^r_s}{\partial t} - \frac{\partial \Lambda^r_s}{\partial\pi_l}\omega_l\right)W_{mr}. \tag{23}$$

Noticing Eq.(20), we get

$$\Lambda^r_s\frac{\partial W_{rm}}{\partial\pi_l}\omega_l = \frac{\partial \bar{W}_{sm}}{\partial\pi_l}\omega_l - \frac{\partial \Lambda^r_s}{\partial\pi_l}W_{rm}\omega_l. \tag{24}$$

In addition, we have

$$\frac{\partial \Lambda^r_s}{\partial t}W_{mr} = \frac{\partial \bar{W}_{sr}}{\partial t} - \Lambda^r_s\frac{\partial W_{mr}}{\partial t}. \tag{25}$$

Substituting Eqs.(23), (24) and (25) into Eq.(22), we obtain

$$\begin{aligned}&\frac{\partial^2 \bar{L}^*}{\partial\omega_s\partial\pi_m} - \frac{\partial^2 \bar{L}^*}{\partial\omega_m\partial\pi_s} - \frac{\partial}{\partial\omega_m}\left(\frac{\partial \bar{L}^*}{\partial\omega_r}\gamma^r_{ks}\omega_k - \bar{P}^*_s\right)\\ =\ &-\frac{\mathrm{d}\Lambda^r_s}{\mathrm{d}t}W_{mr} + \Lambda^r_s\left[\frac{\partial^2 L^*}{\partial\omega_r\partial\pi_m} - \frac{\partial^2 L^*}{\partial\pi_r\partial\omega_m} - \frac{\partial}{\partial\omega_m}\left(\frac{\partial L^*}{\partial\omega_l}\gamma^l_{rk}\omega_k - P^*_r\right)\right].\end{aligned} \tag{26}$$

Suppose that two groups of dynamical functions satisfy the following condition:

$$\frac{\partial}{\partial\omega_m}\left(\frac{\partial \bar{L}^*}{\partial\omega_r}\gamma^r_{ks}\omega_k - \bar{P}^*_s\right) = \Lambda^r_s\frac{\partial}{\partial\omega_m}\left(\frac{\partial L^*}{\partial\omega_l}\gamma^l_{rk}\omega_k - P^*_r\right), \tag{27}$$

then from Eq.(26) we get

$$\frac{\partial^2 \bar{L}^*}{\partial \omega_s \partial \pi_m} - \frac{\partial^2 \bar{L}^*}{\partial \omega_m \partial \pi_s} = -\frac{\mathrm{d}\Lambda_s^r}{\mathrm{d}t} W_{mr} + \Lambda_s^r \left(\frac{\partial^2 L^*}{\partial \omega_r \partial \pi_m} - \frac{\partial^2 L^*}{\partial \omega_m \partial \pi_r} \right). \quad (28)$$

Introducing two matrices T and $\bar{T}$ whose elements are respectively

$$\begin{aligned} T_{sm} &= \frac{\partial^2 L^*}{\partial \omega_s \partial \pi_m} - \frac{\partial^2 L^*}{\partial \omega_m \partial \pi_s}, \\ \bar{T}_{sm} &= \frac{\partial^2 \bar{L}^*}{\partial \omega_s \partial \pi_m} - \frac{\partial^2 \bar{L}^*}{\partial \omega_m \partial \pi_s}, \end{aligned} \quad (29)$$

we can express Eq.(28) in the matrix form

$$\dot{\Lambda} = -\bar{T}U + \Lambda T U, \quad (30)$$

where $\Lambda = (\Lambda_s^r)$ and $U = (U^{sk})$ are $n \times n$ matrices. Then for any positive integer m, considering the fact that T and $\bar{T}$ are antisymmetric, U and $\bar{W}$ are symmetric, and $\Lambda = \bar{W}U$, and using the properties of the trace of matrices, we get

$$\begin{aligned} \dot{\Lambda}\Lambda^{m-1} &= (-\bar{T}U + \bar{W}UTU)\Lambda^{m-1} \\ &= -\bar{T}U(\bar{W}U)^{m-1} + \bar{W}UTU(\bar{W}U)^{m-1} \end{aligned}$$

$$\mathrm{tr}\left[\bar{T}U(\bar{W}U)^{m-1}\right] = 0$$

$$\mathrm{tr}\left[\bar{W}UTU(\bar{W}U)^{m-1}\right] = 0,$$

and

$$\mathrm{tr}\left(\dot{\Lambda}\Lambda^{m-1}\right) = 0.$$

Thus,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{tr}(\Lambda^m)\right] &= \mathrm{tr}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\Lambda^m)\right] \\ &= m\mathrm{tr}\left(\dot{\Lambda}\Lambda^{m-1}\right) = 0, \end{aligned}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{tr}(\Lambda^m)\right] = 0, \quad (31)$$

for any positive integer m, and we have the conserved quantities

$$\mathrm{tr}(\Lambda^m) = \mathrm{const.}, \quad (32)$$

namely, the traces of all positive integer powers of Λ are constants of the motion. Therefore, we have

Proposition For the system (3), if the condition (27) is satisfied, then the symmetry of Lagrangians of the system will lead to the conserved quantity (32).

Taking the generalized velocities as the quasi-velocities, i.e., $\omega_s = \dot{q}_s$, we have

$$\gamma_{sk}^l = 0, \quad P_s^* = Q_s, \quad L^* = L,$$
$$W_{sk} = \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}, \quad \bar{W}_{sk} = \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial \dot{q}_k}.$$

Then the criterion equations (14) can be expressed as

$$\begin{aligned} &W_{sk}\bar{U}^{kl}\left(\bar{Q}_l + \frac{\partial \bar{L}}{\partial q_l} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_l \partial t} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_l \partial q_r}\dot{q}_r\right) \\ &= Q_s + \frac{\partial L}{\partial q_s} - \frac{\partial^2 L}{\partial \dot{q}_s \partial t} - \frac{\partial^2 L}{\partial \dot{q}_s \partial q_r}\dot{q}_r, \end{aligned} \quad (33)$$

and the condition (27) changes into

$$\frac{\partial \bar{Q}_s}{\partial \dot{q}_k} = \Lambda_s^r \frac{\partial Q_r}{\partial \dot{q}_k}. \quad (34)$$

Thus, we have the following corollaries.

Corollary 1 For a holonomic system in terms of generalized coordinates, if two groups of dynamical functions: L, Q_s and $\bar{L}, \bar{Q}_s$ satisfy Eqs.(33), (34), then the symmetry of Lagrangians of the system will lead to the conserved quantity (32).

Corollary 2 For a Lagrange system, the symmetry of Lagrangians will lead to the conserved quantity (32).

Corollary 1 was obtained by Ref.[14].

5. Illustrative example

In order to illustrate the above results, we give an example.

Now, we study the symmetry of Lagrangians and a conserved quantity for the motion in a space of a free rigid body.

Take the Euler angles ψ, θ, φ and center-of-mass coordinates x_c, y_c, z_c as generalized coordinates

$$q_1 = \psi, q_2 = \theta, q_3 = \varphi, q_4 = x_c, q_5 = y_c, q_6 = z_c.$$

Let quasi-velocities

$$\begin{aligned} \omega_1 &= \dot{q}_1 \sin q_2 \sin q_3 + \dot{q}_2 \cos q_3, \\ \omega_2 &= \dot{q}_1 \sin q_2 \cos q_3 - \dot{q}_2 \sin q_3, \\ \omega_3 &= \dot{q}_1 \cos q_2 + \dot{q}_3, \\ \omega_4 &= \dot{q}_4, \quad \omega_5 = \dot{q}_5, \quad \omega_6 = \dot{q}_6, \end{aligned}$$

where $\omega_1, \omega_2, \omega_3$ are the projections of angular velocities on the central principal axes of inertia. Then the Boltzmann three-index symbols are as follows:

$$\begin{aligned} \gamma_{12}^3 &= -\gamma_{21}^3 = -1, \\ \gamma_{31}^2 &= -\gamma_{13}^2 = -1, \\ \gamma_{23}^1 &= -\gamma_{32}^1 = -1, \end{aligned}$$

the others equal zero. And the Lagrangian in terms of quasi-velocities has a simple form:

$$L^* = \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right) + \frac{1}{2}\left(\omega_4^2 + \omega_5^2 + \omega_6^2\right) - mgq_6,$$

where A, B, C are principal moments of inertia of the rigid body, m is its mass. The non-potential generalized forces

$$P_s^* = 0 \quad (s = 1, \ldots, 6).$$

Choose other group of dynamical functions as

$$\bar{L}^* = \frac{1}{6}m\omega_4^3 + \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right) + \frac{1}{2}m\left(\omega_5^2 + \omega_6^2\right) - mgq_6,$$
$$\bar{P}_s^* = 0,$$

then

$$\bar{L}_4^* = \omega_4 L_4, \quad \bar{L}_k^* = L_k, \quad (k = 1, 2, 3, 5, 6),$$
$$\Lambda_4^4 = \omega_4, \quad \Lambda_k^k = 1, \quad (k = 1, 2, 3, 5, 6),$$
$$\Lambda_r^k = 0 \quad (k \neq r),$$
$$\bar{P}_s^* = P_s^*, \quad (s = 1, \ldots, 6).$$

The conserved quantity (32) deduced by the symmetry of Lagrangians gives

$$I = \omega_4 + 5 = \text{const.}$$

Similarly, choose

$$\bar{L}^* = \frac{1}{6}m\omega_5^3 + \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right) + \frac{1}{2}m\left(\omega_4^2 + \omega_6^2\right) - mgq_6$$

and

$$\bar{L}^* = \frac{1}{6}m\left(\omega_6 + gt\right)^3 + \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right) + \frac{1}{2}m\left(\omega_4^2 + \omega_5^2\right) - mgq_6\omega_6,$$

the symmetry of Lagrangians can lead to the following conserved quantities respectively

$$I = \omega_5 + 5 = \text{const.},$$
$$I = \omega_6 + gt + 5 = \text{const.}$$

All the three integrals represent the conservation of generalized momenta.

6. Conclusions

This paper discusses the symmetry of Lagrangians and the conserved quantity deduced by using it for holonomic system in terms of quasi-coordinates. Quasi-coordinates are more general than generalized coordinates; the result of Ref.[14] is a special case of that of this paper.

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