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Conformal invariance and conserved quantities of dynamical system of relative motion*

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This paper discusses in detail the conformal invariance by infinitesimal transformations of a dynamical system of relative motion. The necessary and sufficient conditions of conformal invariance and Lie symmetry are given simultaneously by the action of infinitesimal transformations. Then it obtains the conserved quantities of conformal invariance by the infinitesimal transformations. Finally an example is given to illustrate the application of the results.

Keywords: dynamical system of relative motion, infinitesimal transformation, conformal invariance, conserved quantities

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1. Introduction

People have paid more and more attention to the research of symmetries and conserved quantities of mechanical systems and many results have sprung up during recent years.[1−29] These results develop people's knowledge of the essence of mechanical systems. In 1997, Galiullin *et al* discussed the symmetry of Birkhoff equations and its relation with the conformal invariance.[30] Recently conformal Hamiltonian systems were discussed by geometric methods, in which the author researched the geometric structure of conformal systems with symmetry and pointed out the difference of conformal systems from arbitrary symmetric systems.[31]

With the development of modern science and technology, the study of the dynamics of complicated mechanical systems has become more and more important.[1,32−35] The motion of these complicated systems includes the motion of the carrier and the motion of the carried carrier relative to the carrier. The motion of complicated systems can be studied in the inertial coordinate system, also in the mobile coordinate system closely linked to the carrier. The study on the dynamics of the relative motion of mechanical systems by the method of analytical mechanics not only reaches unity in expression but also shows superiority for the complicated systems. In this paper, the conformal invariance of a dynamical system of relative motion by infinitesimal transformations is first defined, and we discussed the conformal invariance and their relations with Lie symmetry of the system in detail. Then we seek the conserved quantities of the system. Finally an example is given to illustrate the application of the results.

2. Dynamical equations of relative motion

Suppose that the system is composed of a carrier and a carried carrier. The velocity $\boldsymbol{v}_0$ of the base point O and the angular velocity $\boldsymbol{\omega}$ of the carrier are known functions of time. The carried carrier is composed of N particles, and the configuration of the system is determined by the n generalized coordinates q_s $(s = 1, \ldots, n)$. Suppose the motion of the carried carrier does not affect the law of the carrier's original motion. The dynamical equations of relative motion of the holonomic mechanical system can be written as[35]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r}{\partial \dot{q}_s} - \frac{\partial T_r}{\partial q_s} = Q_s - \frac{\partial}{\partial q_s}(V^0 + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s, \quad (s = 1, \ldots, n), \tag{1}$$

where T_r is the kinetic energy of relative motion of the system, Q_s generalized forces, and V^0 is the uniform

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force field potential energy:

$$V^0 = M(\boldsymbol{v}_0^* + \boldsymbol{\omega} \times \boldsymbol{v}_0) \cdot \boldsymbol{r}_{\mathrm{c}}', \tag{2}$$

where M is the total mass of the system, $\boldsymbol{r}_{\mathrm{c}}'$ the radius vector of the mass center in the reference system of motion, and V^{ω} is the centrifugal force potential energy:

$$V^{\omega} = -\frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\theta}^0 \cdot \boldsymbol{\omega}, \tag{3}$$

where $\boldsymbol{\theta}^0$ is the inertia tensor in the point O of the system, $Q_s^{\dot{\omega}}$ is the generalized inertial force of gyration:

$$Q_s^{\dot{\omega}} = -(\dot{\boldsymbol{\omega}} \times m_i \boldsymbol{r}_{\mathrm{i}}') \cdot \frac{\partial \boldsymbol{r}_{\mathrm{i}}'}{\partial q_s}. \tag{4}$$

Γ_s is the generalized gyroscopic force:

$$\Gamma_s = \gamma_{sk}\dot{q}_k, \quad \gamma_{sk} = 2\boldsymbol{\omega} \cdot \left(m_i \frac{\partial \boldsymbol{r}_{\mathrm{i}}'}{\partial q_s} \times \frac{\partial \boldsymbol{r}_{\mathrm{i}}'}{\partial q_k} \right). \tag{5}$$

The generalized force Q_s is divided into the potential part Q_s' and the nonpotential part Q_s'', i.e.

$$Q_s = Q_s' + Q_s'', \quad Q_s' = -\frac{\partial V}{\partial q_s}. \tag{6}$$

Let

$$L_r = T_r - V - V^0 - V^{\omega}, \tag{7}$$

then equation (1) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial \dot{q}_s} - \frac{\partial L_r}{\partial q_s} = Q_s'' + Q_s^{\dot{\omega}} + \Gamma_s, \ \ (s = 1, \dots, n). \tag{8}$$

And more, equation (8) can be described by

$$A_{sk}\ddot{q}_k + B_s - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s = 0, \quad (s, k = 1, \dots, n), \tag{9}$$

where

$$A_{sk} = \frac{\partial^2 L_r}{\partial \dot{q}_s \partial \dot{q}_k},$$
$$B_s = \frac{\partial^2 L_r}{\partial \dot{q}_s \partial q_k}\dot{q}_k + \frac{\partial^2 L_r}{\partial \dot{q}_s \partial t} - \frac{\partial L_r}{\partial q_s}. \tag{10}$$

Suppose that the system is not singular, i.e.

$$\det(h_{sk}) = \det\left(\frac{\partial^2 L_r}{\partial \dot{q}_s \partial \dot{q}_k} \right) \neq 0, \tag{11}$$

then all the generalized accelerations can be solved from Eq.(8),

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, \dots, n). \tag{12}$$

3. Infinitesimal transformations and the determining equations of conformal invariance

Let

$$F_s = A_{sk}\ddot{q}_k + B_s - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s. \tag{13}$$

Introduce the infinitesimal transformations of groups of time and generalized coordinates

$$t^* = t + \Delta t, \quad q_s^*(t^*) = q_s(t) + \Delta q_s, \quad (s = 1, \dots, n), \tag{14}$$

and their expansions

$$\mathrm{t}^* = t + \varepsilon\xi_0\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right), \quad q_s^* = q_s + \varepsilon\xi_{\mathrm{s}}\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right), \tag{15}$$

where ε is an infinitesimal parameter, ξ_0 and ξ_s are infinitesimal generators. Introduce an infinitesimal generator vector

$$X^{(0)} = \xi_k \frac{\partial}{\partial q_k} + \xi_0 \frac{\partial}{\partial t},$$

and its extensions

$$X^{(1)} = X^{(0)} + \left(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0 \right) \frac{\partial}{\partial \dot{q}_k},$$

$$X^{(2)} = X^{(1)} + \left[\left(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0 \right)^{\cdot} - \ddot{q}_k\dot{\xi}_0 \right] \frac{\partial}{\partial \ddot{q}_k}.$$

Definition 1 In the infinitesimal transformations of infinitesimal generators $\xi_0\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right)$ and $\xi_s\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right)$, if F_s satisfy the following expressions

$$X^{(2)}(F_s) = \ell_s^l(F_l), \tag{16}$$

then the infinitesimal transformations have conformal invariance to the corresponding differential equations of relative motion, where the expressions of Eq.(16) are determining equations of Eq.(13), and ℓ_s^l is called the conformal factor.

4. The conformal factor of conformal invariance and Lie symmetry

In order to get the conformal factor, we need to suppose that the differential equations have conformal invariance and Lie symmetry simultaneously under the infinitesimal transformations.

When equation (13) is Lie symmetric, we have

$$X^{(2)}F_s\,|_{F_s=0} = 0 (s = 1, \dots, n). \tag{17}$$

To obtain the general expression of conformal invariance, we calculate the difference

$$X^{(2)}F_s - X^{(2)}F_s\,|_{F_s=0}\,. \tag{18}$$

Owing to

$$X^{(2)}F_s = X^{(2)}(A_{sk}\ddot{q}_k + B_s - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s)$$
$$= A_{sk}(\ddot{\xi}_k - 2\ddot{q}_k\dot{\xi}_0 - \dot{q}_k\ddot{\xi}_0) + X^{(1)}\left(B_s - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s\right) + X^{(0)}\left(A_{sk}\right)\ddot{q}_k\ , \tag{19}$$

where

$$\dot{\xi}_k = \frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial q_m}\dot{q}_m + \frac{\partial \xi_k}{\partial \dot{q}_m}\ddot{q}_m, \tag{20}$$

$$\ddot{\xi}_k = \frac{\partial^2 \xi_k}{\partial t \partial \dot{q}_m}\ddot{q}_m + \frac{\partial^2 \xi_k}{\partial t \partial q_m}\dot{q}_m + \frac{\partial^2 \xi_k}{\partial t^2} + \frac{\partial^2 \xi_k}{\partial q_l \partial \dot{q}_m}\ddot{q}_m\dot{q}_l + \frac{\partial^2 \xi_k}{\partial q_l \partial t}\dot{q}_l + \frac{\partial^2 \xi_k}{\partial q_l \partial q_m}\dot{q}_m\dot{q}_l$$
$$+ \frac{\partial \xi_k}{\partial q_l}\ddot{q}_l + \left(\frac{\partial \xi_k}{\partial \dot{q}_l}\right)^{\cdot}\ddot{q}_l + \frac{\partial \xi_k}{\partial \dot{q}_l}\dddot{q}_l\ , \tag{21}$$

$$\ddot{\xi}_0 = \frac{\partial^2 \xi_0}{\partial t \partial \dot{q}_m}\ddot{q}_m + \frac{\partial^2 \xi_0}{\partial t \partial q_m}\dot{q}_m + \frac{\partial^2 \xi_0}{\partial t^2} + \frac{\partial^2 \xi_0}{\partial q_l \partial \dot{q}_m}\ddot{q}_m\dot{q}_l + \frac{\partial^2 \xi_0}{\partial q_l \partial t}\dot{q}_l + \frac{\partial^2 \xi_0}{\partial q_l \partial q_m}\dot{q}_m\dot{q}_l$$
$$+ \frac{\partial \xi_0}{\partial q_l}\ddot{q}_l + \left(\frac{\partial \xi_0}{\partial \dot{q}_l}\right)^{\cdot}\ddot{q}_l + \frac{\partial \xi_0}{\partial \dot{q}_l}\dddot{q}_l\ , \tag{22}$$

and

$$X^{(1)}\left(B_s - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s\right) = X^{(0)}\left(B_s - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s\right)$$
$$+ \left[\frac{\partial \xi_k}{\partial q_l}\ \dot{q}_l + \frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial \dot{q}_l}\ddot{q}_l - \dot{q}_k\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_l}\dot{q}_l + \frac{\partial \xi_0}{\partial \dot{q}_l}\ddot{q}_l\right)\right]$$
$$\times \frac{\partial\left(B_s - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s\right)}{\partial \dot{q}_k}\ , \tag{23}$$

substituting Eqs.(20)–(23) into Eq.(19) and considering Eq.(12), we have

$$X^{(2)}F_s - X^{(2)}\,|_{F_s=0} = A_{sk}\left[\frac{\partial^2 \xi_k}{\partial t \partial \dot{q}_m}\ (\ddot{q}_m - \alpha_m) + \frac{\partial^2 \xi_k}{\partial q_l \partial \dot{q}_m}\,(\ddot{q}_m - \alpha_m)\,\dot{q}_l + \frac{\partial \xi_k}{\partial q_l}\,(\ddot{q}_l - \alpha_l)\right.$$
$$+ \left(\frac{\partial \xi_k}{\partial \dot{q}_l}\right)^{\cdot}(\ddot{q}_l - \alpha_l) - 2\,(\ddot{q}_k - \alpha_k)\,\dot{\xi}_0 - \dot{q}_k\left(\frac{\partial^2 \xi_0}{\partial t \partial \dot{q}_m}\,(\ddot{q}_m - \alpha_m)\right.$$
$$\left.\left. + \frac{\partial^2 \xi_0}{\partial q_l \partial \dot{q}_m}\,(\ddot{q}_m - \alpha_m)\,\dot{q}_l + \frac{\partial \xi_0}{\partial q_l}\,(\ddot{q}_l - \alpha_l) + \left(\frac{\partial \xi_0}{\partial \dot{q}_l}\right)^{\cdot}(\ddot{q}_l - \alpha_l)\right)\right]$$
$$+ \left[\frac{\partial \xi_k}{\partial q_l}\,(\ddot{q}_l - \alpha_l) - \dot{q}_k\,\frac{\partial \xi_0}{\partial q_l}\,(\ddot{q}_l - \alpha_l)\right]\frac{\partial\left(B_s - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s\right)}{\partial \dot{q}_k}$$
$$+ X^{(0)}\left(A_{sk}\right)(\ddot{q}_k - \alpha_k)\ . \tag{24}$$

Owing to

$$\ddot{q}_m - \alpha_m = \ddot{q}_m + A^{mr}(B_r - Q_r'' - Q_r^{\dot{\omega}} - \Gamma_r) = A^{mr}(A_{rm}\ddot{q}_m + B_r - Q_r'' - Q_r^{\dot{\omega}} - \Gamma_r) = A^{mr}F_r, \tag{25}$$

therefore

$$X^{(2)}F_s - X^{(2)}\,|_{F_s=0} = \left\{A_{sk}\left[\frac{\partial^2 \xi_k}{\partial t \partial \dot{q}_m}\ + \frac{\partial^2 \xi_k}{\partial q_l \partial \dot{q}_m}\dot{q}_l + \frac{\partial \xi_k}{\partial q_m} + \left(\frac{\partial \xi_k}{\partial \dot{q}_m}\right)^{\cdot}\right.\right.$$
$$\left. - \dot{q}_k\left(\frac{\partial^2 \xi_0}{\partial t \partial \dot{q}_m}\ + \frac{\partial^2 \xi_0}{\partial q_l \partial \dot{q}_m}\dot{q}_l + \frac{\partial \xi_0}{\partial q_m} + \left(\frac{\partial \xi_0}{\partial \dot{q}_m}\right)^{\cdot}\right)\right]A^{mr} - 2\delta_s^r\dot{\xi}_0$$
$$\left. + \left(\frac{\partial \xi_k}{\partial \dot{q}_m}\ - \dot{q}_k\,\frac{\partial \xi_0}{\partial q_m}\right)A^{mr}\,\frac{\partial\left(B_s - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s\right)}{\partial \dot{q}_k} + X^{(0)}\left(A_{sk}\right)A^{kr}\right\}F_r\ . \tag{26}$$

Let

$$B_s^r = A_{sk}\left[\frac{\partial^2 \xi_k}{\partial t \partial \dot{q}_m}\ + \frac{\partial^2 \xi_k}{\partial q_l \partial \dot{q}_m}\dot{q}_l + \frac{\partial \xi_k}{\partial q_m} + \left(\frac{\partial \xi_k}{\partial \dot{q}_m}\right)^{\cdot}\right.$$
$$\left. - \dot{q}_k\left(\frac{\partial^2 \xi_0}{\partial t \partial \dot{q}_m}\ + \frac{\partial^2 \xi_0}{\partial q_l \partial \dot{q}_m}\dot{q}_l + \frac{\partial \xi_0}{\partial q_m} + \left(\frac{\partial \xi_0}{\partial \dot{q}_m}\right)^{\cdot}\right)\right]A^{mr} - 2\delta_s^r\dot{\xi}_0$$
$$+ \left(\frac{\partial \xi_k}{\partial \dot{q}_m}\ - \dot{q}_k\,\frac{\partial \xi_0}{\partial q_m}\right)A^{mr}\frac{\partial\left(B_s - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s\right)}{\partial \dot{q}_k} + X^{(0)}\left(A_{sk}\right)A^{kr}\ , \tag{27}$$

then equation (26) becomes

$$X^{(\mathbf{2})}F_s - X^{(\mathbf{2})}F_s\,|_{F_s=\mathbf{0}} = B_s^r F_r. \tag{28}$$

When equation (13) has conformal invariance and Lie symmetry simultaneously by the action of the infinitesimal transformations of one-parameter group Eq.(15), we can obtain

$$\ell_s^r F_r - B_s^r F_r = X^{(\mathbf{2})}F_s\,|_{F_s=\mathbf{0}} = 0\ , \tag{29}$$

i.e.

$$(\ell_s^r - B_s^r)\,F_r = \mathbf{0}, \tag{30}$$

we have

$$\ell_s^r = B_s^r. \tag{31}$$

Thus, from the above discussion there is the following proposition.

Proposition 1 To Eq.(13), the necessary and sufficient conditions of its conformal invariance which is Lie symmetric simultaneously by the action of the infinitesimal one-parameter groups are that there exists the conformal factor

$$\begin{aligned}\ell_s^r = B_s^r = A_{sk}\Bigg[&\frac{\partial^2\xi_k}{\partial t\partial\dot{q}_m} + \frac{\partial^2\xi_k}{\partial q_l\partial\dot{q}_m}\dot{q}_l + \frac{\partial\xi_k}{\partial q_m} + \left(\frac{\partial\xi_k}{\partial\dot{q}_m}\right)^{\cdot}\\ &- \dot{q}_k\left(\frac{\partial^2\xi_0}{\partial t\partial\dot{q}_m} + \frac{\partial^2\xi_0}{\partial q_l\partial\dot{q}_m}\dot{q}_l + \frac{\partial\xi_0}{\partial q_m} + \left(\frac{\partial\xi_0}{\partial\dot{q}_m}\right)^{\cdot}\right)\Bigg]A^{mr} - 2\delta_s^r\dot{\xi}_0\\ &+ \left(\frac{\partial\xi_k}{\partial\dot{q}_m} - \dot{q}_k\,\frac{\partial\xi_0}{\partial q_m}\right)A^{mr}\frac{\partial\left(B_s - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s\right)}{\partial\dot{q}_k} + X^{(0)}\left(A_{sk}\right)A^{kr}\ .\end{aligned} \tag{32}$$

5. Conformal invariance and conserved quantities

By the action of infinitesimal transformations, when there exists the conformal factor which makes the dynamical equations of relative motion have conformal invariance and Lie symmetry simultaneously, then the Noether conserved quantities can be deduced.

Proposition 2 If there exists conformal factor Eq.(32), and the infinitesimal generators and the gauge function G_N satisfy the following Noether equations

$$L_r\dot{\xi}_0 + X^{(1)}\left(L_r\right) + \left(Q_s'' + Q_s^{\dot{\omega}} + \Gamma_s\right)\left(\xi_s - \dot{q}_s\xi_0\right) + \dot{G}_N = 0, \tag{33}$$

then the conformal invariance leads the following Noether conserved quantities

$$I_N = L_r\xi_0 + \frac{\partial L_r}{\partial\dot{q}_s}\left(\xi_s - \dot{q}_s\xi_0\right) + G_N = \text{const.} \tag{34}$$

Proof

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} &= \dot{L}_r\xi_0 + L_r\dot{\xi}_0 + \frac{\partial L_r}{\partial\dot{q}_s}\left(\dot{\xi}_s - \ddot{q}_s\xi_0 - \dot{q}_s\dot{\xi}_0\right) + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial\dot{q}_s}\left(\xi_s - \dot{q}_s\xi_0\right)\\ &\quad - L_r\dot{\xi}_0 - X^{(1)}\left(L_r\right) - \left(Q_s'' + Q_s^{\dot{\omega}} + \Gamma_s\right)\left(\xi_s - \dot{q}_s\xi_0\right)\\ &= \left(\xi_s - \dot{q}_s\xi_0\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial\dot{q}_s} - \frac{\partial L_r}{\partial q_s} - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s\right) = 0.\end{aligned}$$

6. Illustrative example

The dynamical system of relative motion is defined as

$$\begin{aligned}&T_r = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2)\ ,\quad V = \frac{1}{2}k(q_1^2 + q_2^2)\ ,\\ &V^{\omega} = -\frac{1}{2}m(q_1^2 + q_2^2)\omega^2\ ,\quad V^0 = \Gamma_s = Q_s^{\dot{\omega}} = Q_s'' = 0\ ,\end{aligned} \tag{35}$$

where m, g, a, ω are constants. Try to study the conformal invariance and conserved quantities of the system.

The differential equations are

$$F = \begin{pmatrix} m\ddot{q}_1 + kq_1 - m\omega^2 q_1 \\ m\ddot{q}_2 + kq_2 - m\omega^2 q_2 \end{pmatrix} = 0. \tag{36}$$

When the infinitesimal generators are

$$\xi_0 = 0, \quad \xi_1 = -q_2, \quad \xi_2 = q_1, \tag{37}$$

we have

$$\begin{aligned} X^{(2)} &= \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \left(\dot{\xi}_s - \dot{q}_s \dot{\xi}_0\right) \frac{\partial}{\partial \dot{q}_s} + \left(\ddot{\xi}_s - 2\ddot{q}_s \dot{\xi}_0 - \dot{q}_s \ddot{\xi}_0\right) \frac{\partial}{\partial \ddot{q}_s} \\ &= \xi_s \frac{\partial}{\partial q_s} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s} + \ddot{\xi}_s \frac{\partial}{\partial \ddot{q}_s} \,. \end{aligned} \tag{38}$$

Furthermore, we have

$$\begin{aligned} X^{(2)} F &= \left(\xi_s \frac{\partial}{\partial q_s} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s} + \ddot{\xi}_s \frac{\partial}{\partial \ddot{q}_s}\right) \begin{pmatrix} m\ddot{q}_1 + kq_1 - m\omega^2 q_1 \\ m\ddot{q}_2 + kq_2 - m\omega^2 q_2 \end{pmatrix} \\ &= \begin{pmatrix} -m\ddot{q}_2 - kq_2 + m\omega^2 q_2 \\ m\ddot{q}_1 + kq_1 - m\omega^2 q_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m\ddot{q}_1 + kq_1 - m\omega^2 q_1 \\ m\ddot{q}_2 + kq_2 - m\omega^2 q_2 \end{pmatrix}. \end{aligned}$$

Therefore, the conformal factor is

$$\ell_s^r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{39}$$

Utilizing Eq.(32) of Proposition 1, we can also obtain

$$\begin{aligned} \ell_s^r &= \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \,. \end{aligned} \tag{40}$$

By all appearances, the result is the same as Eq.(39). Therefore the determining equations of conformal invariance of Eqs.(36) by the action of special infinitesimal transformations are

$$X^{(2)} F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m\ddot{q}_1 + kq_1 - m\omega^2 q_1 \\ m\ddot{q}_2 + kq_2 - m\omega^2 q_2 \end{pmatrix}. \tag{41}$$

Here equation (36) by the action of Eq.(37) has conformal invariance and Lie symmetry simultaneously. Substituting Eq.(37) into Eq.(33), we obtain

$$G_N = 0,$$

consequently we obtain the conserved quantity

$$I_N = m\left(q_1 \dot{q}_2 - \dot{q}_1 q_2\right) = \text{const.}$$

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