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Hojman's theorem of the third-order ordinary differential equation*

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This paper extends Hojman's conservation law to the third-order differential equation. A new conserved quantity is constructed based on the Lie group of transformation generators of the equations of motion. The generators contain variations of the time and generalized coordinates. Two independent non-trivial conserved quantities of the third-order ordinary differential equation are obtained. A simple example is presented to illustrate the applications of the results.

Keywords: third-order ordinary differential equation, Lie symmetry, Hojman's conservation law
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1. Introduction

The symmetry problem of dynamical systems is one of the most important subjects in physics, and has been studied for a long time. The symmetry of a dynamical system will be useful for integrating the equations of motion, since it is closely related to invariants (or first integrals) of the system. The modern approaches of finding invariants are mainly in terms of Noether symmetry, Lie symmetry and Mei's form invariance. The Noether symmetry is an invariance of the Hamiltonian action under the infinitesimal transformation of a continuous group.[1] The Noether's theorem associates one conserved quantity with each Noether symmetry. The Mei's form invariance is an invariance of dynamical functions of a system, such as Lagrangian, non-potential generalized force and generalized constrained force, etc., under the infinitesimal transformation of a continuous group.[2−6] A new conserved quantity can be obtained by using Mei's form invariance.[7] Lie's method consists in finding the continuous symmetry transformations that leave a system of differential equations invariant. These symmetry transformations constitute a Lie group. Once one has obtained the symmetry group, then one can has some ways to find the first integrals.

In 1976, Katzin and Levine[8] derived a formula for constants of motion associated with Lie point symmetries for a Lagrangian system. 1979 Lutzky[9] showed that constants of motion for a Lagrangian system are determined even by point symmetries which do not preserve the action, and demonstrated that the explicit form of these conserved quantities can be specified from a knowledge of the symmetry group generator. In 2003, Fu and Chen[10] studied non-Noether symmetries and conserved quantities of non-conservative dynamical systems. In 1992, Hojman[11] presented a new conservation law without using either Lagrangian or Hamiltonian, solely based on the existence of symmetry. This direct method is generalized by González–Gascón,[12] by using geometric techniques. Lutzky[13] used this direct method to derive an invariant for the case where a Lagrangian does exist. Although Pillay and Leach[14] demonstrated that this invariant is trivial, if the symmetry used is a Noether symmetry, it may be meaningful to apply this direct method to find the non-Noether conserved quantity of dynamical systems. Mei studied respectively the non-Noether conserved quantity for the differential equation of motion in phase space[15] and the non-Noether conserved quantity for the generalized Hamiltonian system.[16] Zhang and Chen presented a unified form of Hojman's conservation law and Lutzky's conservation law.[17]

The study of third-order ordinary differential equations has already appeared in much of the literature of applied mathematics.[18−21] The Falkner–Skan equation is a third-order ordinary differential

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equation, which arises from Peandtl boundary layer equations.[22] In this paper, one should study the Lie symmetries of the third-order ordinary differential equation, and construct a conservation law only by using Lie symmetry generators, the method being essentially a natural generalisation of what was done previously.[23]

Throughout the paper, Einstein's summation convention is used. The indices s, k and ρ run from 1 to n.

2. The infinitesimal transformations and determining equations

Consider a set of third-order differential equations

$$\dddot{q}_s = F_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}), \quad (s = 1, 2, \cdots, n) \tag{1}$$

where $F_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ are arbitrary differential functions.

Introduce the generalized infinitesimal transformations for Eq.(1):

$$t^* = t + \Delta t, \tag{2}$$

$$q_s^* = q_s + \Delta q_s, \tag{3}$$

and their expanded form

$$t^* = t + \varepsilon\tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}), \tag{4}$$

$$q_s^* = q_s + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}), \tag{5}$$

where ε is an infinitesimal parameter, $\tau\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}\right)$ and $\xi_s\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}\right)$ are the coefficient functions.

Let

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} \tag{6}$$

be a generator vector of the infinitesimal transformation Eqs.(4) and (5), then we have its first-order prolongation and the second-order prolongation as follows:

$$X^{(1)} = X^{(0)} + \left(\dot{\xi}_s - \dot{q}_s\dot{\tau}\right)\frac{\partial}{\partial \dot{q}_s}, \tag{7}$$

$$X^{(2)} = X^{(1)} + \left(\ddot{\xi}_s - 2\ddot{q}_s\dot{\tau} - \dot{q}_s\ddot{\tau}\right)\frac{\partial}{\partial \ddot{q}_s}, \tag{8}$$

where the overdot denotes the total time derivative along a trajectory, i.e.,

$$\begin{aligned}\dot{I} &\equiv \mathrm{d}I/\mathrm{d}t \\ &= \partial I/\partial t + \dot{q}_s\partial I/\partial q_s + \ddot{q}_s\partial I/\partial \dot{q}_s \\ &\quad + F_s\partial I/\partial \ddot{q}_s.\end{aligned} \tag{9}$$

Proposition The infinitesimal transformations (4) and (5) are Lie symmetry transformations of Eq.(1), if and only if there exist functions $\tau\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}\right)$ and $\xi_s\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}\right)$ that satisfy the following determining equation

$$\begin{aligned}&\dddot{\xi}_s - 3F_s\dot{\tau} - 3\ddot{q}_s\ddot{\tau} - \dot{q}_s\dddot{\tau} = X^{(2)}\left(F_s\right), \\ &\quad (s = 1, 2, \cdots, n).\end{aligned} \tag{10}$$

3. The conserved quantity of motion for a third-order differential equation

In terms of the coefficient functions $\tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ and $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ which determine a symmetry generator vector $\tau\partial/\partial t + \xi_s\partial/\partial q_s + \left(\dot{\xi}_s - \dot{q}_s\dot{\tau}\right)\partial/\partial\dot{q}_s + \left(\ddot{\xi}_s - 2\ddot{q}_s\dot{\tau} - \dot{q}_s\ddot{\tau}\right)\partial/\partial\ddot{q}_s$ for Eq.(1), the concerning conserved quantity can be constructed. One has a theorem as follows:

Theorem The system represented by Eq.(1) possesses the following conserved quantity

$$\begin{aligned}I = &\frac{\partial\tau}{\partial t} + \frac{\partial\xi_s}{\partial q_s} + \frac{\partial}{\partial\dot{q}_s}\left(\dot{\xi}_s - \dot{q}_s\dot{\tau}\right) \\ &+ \frac{\partial}{\partial\ddot{q}_s}\left(\ddot{\xi}_s - 2\ddot{q}_s\dot{\tau} - \dot{q}_s\ddot{\tau}\right) \\ &+ X^{(2)}\left(\ln\lambda\right) - \dot{\tau},\end{aligned} \tag{11}$$

if the functions $\tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ and $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ satisfy Eq.(10), and the function $\lambda = \lambda(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ admits the following equation

$$\frac{\partial F_s}{\partial\ddot{q}_s} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda = 0. \tag{12}$$

Proof From Eq.(11), one has

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} = &\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\frac{\partial\tau}{\partial t} + \frac{\partial\xi_s}{\partial q_s} + \frac{\partial}{\partial\dot{q}_s}\left(\dot{\xi}_s - \dot{q}_s\dot{\tau}\right) \\ &+ \frac{\partial}{\partial\ddot{q}_s}\left(\ddot{\xi}_s - 2\ddot{q}_s\dot{\tau} - \dot{q}_s\ddot{\tau}\right)\Bigg) \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}X^{(2)}\left(\ln\lambda\right) - \dddot{\tau}.\end{aligned} \tag{13}$$

It is straightforward to show that for arbitrary differential function $A\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}\right)$,[14]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial A}{\partial t}\right) = \frac{\partial}{\partial t}\frac{\mathrm{d}A}{\mathrm{d}t} - \frac{\partial F_k}{\partial t}\frac{\partial A}{\partial\ddot{q}_k}, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial A}{\partial q_s}\right) = \frac{\partial}{\partial q_s}\frac{\mathrm{d}A}{\mathrm{d}t} - \frac{\partial F_k}{\partial q_s}\frac{\partial A}{\partial\ddot{q}_k}, \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial A}{\partial \dot{q}_s}\right)=\frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}A}{\mathrm{d}t}-\frac{\partial A}{\partial q_s}-\frac{\partial F_k}{\partial \dot{q}_s}\frac{\partial A}{\partial \ddot{q}_k}, \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial A}{\partial \ddot{q}_s}\right)=\frac{\partial}{\partial \ddot{q}_s}\frac{\mathrm{d}A}{\mathrm{d}t}-\frac{\partial A}{\partial \dot{q}_s}-\frac{\partial F_k}{\partial \ddot{q}_s}\frac{\partial A}{\partial \ddot{q}_k}, \tag{17}$$

$$\begin{aligned}X^{(2)}\left(\frac{\partial A}{\partial \ddot{q}_s}\right)=&\frac{\partial}{\partial \ddot{q}_s}X^{(2)}(A)-\frac{\partial \tau}{\partial \dot{q}_s}\frac{\partial A}{\partial t}-\frac{\partial \xi_k}{\partial \dot{q}_s}\frac{\partial A}{\partial q_k}\\&-\frac{\partial}{\partial \dot{q}_s}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)\frac{\partial A}{\partial \dot{q}_k}\\&-\frac{\partial}{\partial \ddot{q}_s}\left(\ddot{\xi}_k-2\ddot{q}_k\dot{\tau}-\ddot{q}_k\dot{\tau}\right)\frac{\partial A}{\partial \ddot{q}_k}.\end{aligned} \tag{18}$$

The condition that the $\tau(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ and $\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ define a symmetry for $\dddot{\boldsymbol{q}}_s=F_s(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ is given by Eq.(10). If $X^{(2)}$ is a symmetry, one can show the following relation:

$$\frac{\mathrm{d}}{\mathrm{d}t}X^{(2)}(A)=X^{(2)}\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)+\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}A}{\mathrm{d}t}, \tag{19}$$

where $A(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ is an arbitrary function, by using Eqs.(14)–(18), one obtains

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tau}{\partial t}+\frac{\partial \xi_s}{\partial q_s}+\frac{\partial}{\partial \dot{q}_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\tau}\right)+\frac{\partial}{\partial \ddot{q}_s}\left(\ddot{\xi}_s-2\ddot{q}_s\dot{\tau}-\dot{q}_s\ddot{\tau}\right)\right)\\&=\ddot{\tau}+\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\partial F_s}{\partial \ddot{q}_s}+\frac{\partial}{\partial \ddot{q}_s}\left(\dddot{\xi}_s-3\dddot{q}_s\dot{\tau}-3\ddot{q}_s\ddot{\tau}-\dot{q}_s\dddot{\tau}\right)-\frac{\partial F_k}{\partial t}\frac{\partial \tau}{\partial \ddot{q}_k}-\frac{\partial F_k}{\partial q_s}\frac{\partial \xi_s}{\partial \ddot{q}_k}\\&\quad-\frac{\partial F_k}{\partial \dot{q}_s}\frac{\partial}{\partial \ddot{q}_k}\left(\dot{\xi}_s-\dot{q}_s\dot{\tau}\right)-\frac{\partial F_k}{\partial \ddot{q}_s}\frac{\partial}{\partial \ddot{q}_k}\left(\ddot{\xi}_s-2\ddot{q}\dot{\tau}-\dot{q}_s\ddot{\tau}\right),\end{aligned} \tag{20}$$

furthermore, from Eqs.(10) and (19), one has

$$X^{(2)}\left(\frac{\partial F_k}{\partial \ddot{q}_k}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tau}{\partial t}+\frac{\partial \xi_s}{\partial q_s}+\frac{\partial}{\partial \dot{q}_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\tau}\right)+\frac{\partial}{\partial \ddot{q}_s}\left(\ddot{\xi}_s-2\ddot{q}_s\dot{\tau}-\dot{q}_s\ddot{\tau}\right)\right)-\dddot{\tau}. \tag{21}$$

Substituting Eqs.(10) and (19)–(21) into Eq.(13), one can conclude

$$\frac{\mathrm{d}I}{\mathrm{d}t}=0. \tag{22}$$

In terms of the above theorem, one can easily derive the following two corollaries.

Corollary 1 The system represented by Eq.(1) possesses the following conserved quantity

$$\begin{aligned}I=&\frac{\partial \xi_s}{\partial q_s}+\frac{\partial \dot{\xi}_s}{\partial \dot{q}_s}+\frac{\partial \ddot{\xi}_s}{\partial \ddot{q}_s}+\tilde{X}^{(2)}(\ln\lambda),\\&(s=1,2,\cdots,n),\end{aligned} \tag{23}$$

if the infintesimal generators $\tau(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})=0$, and $\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ satisfies the determining equation

$$\begin{aligned}&\dddot{\xi}_s-\xi_k\frac{\partial F_s}{\partial q_k}-\dot{\xi}_k\frac{\partial F_s}{\partial \dot{q}_k}-\ddot{\xi}_k\frac{\partial F_s}{\partial \ddot{q}_k}=0,\\&(s,k=1,2,\cdots,n),\end{aligned} \tag{24}$$

where $\tilde{X}^{(2)}=\xi_s\dfrac{\partial}{\partial q_s}+\dot{\xi}_s\dfrac{\partial}{\partial \dot{q}_s}+\ddot{\xi}_s\dfrac{\partial}{\partial \ddot{q}_s}$, and function $\lambda=\lambda(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ admits Eq.(12).

Corollary 2 The system represented by Eq.(1) possesses the following conserved quantity

$$\begin{aligned}I=&\frac{\partial \tau}{\partial t}-\frac{\partial}{\partial \dot{q}_s}\left(\dot{q}_s\dot{\tau}\right)-\frac{\partial}{\partial \ddot{q}_s}\left(2\ddot{q}_s\dot{\tau}+\dot{q}_s\ddot{\tau}\right)\\&+\bar{X}^{(2)}(\ln\lambda)-\dot{\tau},\end{aligned} \tag{25}$$

if the infintesimal generators $\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})=0$, and $\tau(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ satisfies the determining equation

$$-3F_s\dot{\tau}-3\ddot{q}_s\ddot{\tau}-\dot{q}_s\dddot{\tau}=\bar{X}^{(2)}(F_s), \tag{26}$$

where $\bar{X}^{(2)}=\tau\dfrac{\partial}{\partial t}-\dot{q}_s\dot{\tau}\dfrac{\partial}{\partial \dot{q}_s}-\left(2\ddot{q}_s\dot{\tau}+\dot{q}_s\ddot{\tau}\right)\dfrac{\partial}{\partial \ddot{q}_s}$, and function $\lambda=\lambda(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ admits Eq.(12).

4. Illustrative examples

To illustrate various aspects of the theory developed in the preceding sections, one considers a third-order differential equation as follows:

$$\dddot{q}=\frac{3\dot{q}\ddot{q}^2}{1+\dot{q}^2}. \tag{27}$$

Let

$$\tau=\tau(t,q,\dot{q},\ddot{q}), \tag{28}$$

$$\xi=\xi(t,q,\dot{q},\ddot{q}). \tag{29}$$

The determining equation of the generalized Lie symmetry of Eq.(27) under the infinitesimal transformations (28) and (29) is

$$\dddot{\xi} - \frac{9\dot{q}\ddot{q}^2}{1+\dot{q}^2}\dot{\tau} - 3\ddot{q}\ddot{\tau} - \dot{q}\dddot{\tau} = X^{(2)}\left(\frac{3\dot{q}\ddot{q}^2}{1+\dot{q}^2}\right). \quad (30)$$

Equation (30) has solutions

$$\tau = \frac{\dot{q}}{\sqrt{1+\dot{q}^2}}, \quad (31)$$

$$\xi = -\frac{1}{\sqrt{1+\dot{q}^2}}. \quad (32)$$

From Eq.(12), one obtains

$$\frac{6\dot{q}\ddot{q}}{1+\dot{q}^2} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda = 0. \quad (33)$$

Equation (33) exists a solution

$$\lambda = \frac{1}{(1+\dot{q}^2)^3}. \quad (34)$$

Inserting Eqs.(31), (32) and (34) into Eq.(11) leads to a conserved quantity

$$I = -\frac{3\ddot{q}}{(1+\dot{q}^2)^{3/2}}. \quad (35)$$

For Eq.(35), it is straightforward to verify $\mathrm{d}I/\mathrm{d}t = 0$ along the trajectory of the dynamical system (27).

5. Conclusion

In this paper we present a Hojman's conservation law for third-order ordinary differential equations. The conserved quantity is also constructed only in terms of a symmetry transformation vector of the equations of motion. Two independent non-trivial conserved quantities of a third-order ordinary differential equation can be obtained by using this new conservation law. Extending this method to higher-order differential equations is straightforward.

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